Asymptotic data and Stokes data for the tt*-Toda equations, and some relations with physics

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INTRODUCTION

The tt* equations were introduced by Cecotti and Vafa in the context of supersymmetric quantum field theory.

S. Cecotti and C. Vafa, Topological — anti-topological fusion, Nuclear Phys. B 1991

Mathematical explanation: the tt* equations arise from Frobenius manifolds "with Hermitian metric" (or "with real structure"). (Example of a Frobenius manifold: quantum cohomology.)

The tt* equations are the equations for this metric — the tt* metric.

Dubrovin formulated the tt* equations (which are nonlinear equations) as an “integrable system” in two ways:

(i) the compatibility condition of a certain system of linear p.d.e. (a “zero curvature equation”)

(ii) the condition that a certain linear o.d.e. is isomonodromic.

B. Dubrovin, Geometry and integrability of topological-antitopological fusion, Comm. Math. Phys. 1993

Thus it is possible to apply various methods from the theory of integrable systems: conserved quantities, dressing actions (of infinite dimensional Lie groups), the Riemann-Hilbert Method,...

When the domain is 1-dimensional, the tt* equations are

$$\frac{\partial}{\partial \bar{t}}\left(g\frac{\partial}{\partial t}g^{-1}\right) - [C, gC^\dagger g^{-1}] = 0$$

where

- C = (holomorphic) chiral matrix of the theory
- $C^\dagger$ = conjugate-transpose of C,
- g^{-1} = Hermitian matrix representing the tt* metric

For quantum cohomology, C is the matrix of quantum multiplication by a generator $b \in H^2(M;\mathbb{C})$.

Eg: when $M = \mathbb{C}P^3$, $C = \begin{pmatrix} 0 & 0 & 0 & q \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

For the quantum cohomology of $M = \mathbb{C}P^n$, the tt* metric can be written

$$g = \operatorname{diag}(e^{-2w_0}, \ldots, e^{-2w_n})$$

where $w_0, \ldots, w_n$ are real valued functions of $q, \bar{q}$. The homogeneity property of quantum cohomology implies that w_i depends only on $|q|$. In view of this, it is convenient to make a change of variable of the form $q = \alpha t^\beta$, and then the tt* equations become

$$2(w_i)_{t\bar{t}} = -e^{2(w_{i+1}-w_i)} + e^{2(w_i - w_{i-1})},\ i = 0, 1, \ldots, n$$

together with the extra condition $w_i + w_{n-i} = 0$. This is a version of the periodic Toda equations. We call it the tt*-Toda equation.

(We interpret w_{n+1}, w_{-1} here as w_0, w_n respectively.)

Physically, a solution is a massive deformation of a conformal field theory, and the existence of such a deformation says something about that theory. Cecotti and Vafa made a series of conjectures about the solutions:

- there should exist (globally smooth) solutions $w = w(|t|)$ on $\mathbb{C}^*$
- these solutions should be characterized by asymptotic data at $t = 0$ (the "ultra-violet point"; here the data is the chiral charges, essentially the holomorphic matrix C)
- these solutions should equally be characterized by asymptotic data at $t = \infty$, (the "infra-red point"; here the data is the soliton multiplicities s_i).

Note: We will say more about solitons later.

RESULTS

Theorem 1: (M. Guest - A. Its - C.-S. Lin, arXiv 2010-2017, T. Mochizuki, arXiv 2013) For each $N > 0$, there is a $1 : 1$ correspondence between solutions of the tt*-Toda equations on $\mathbb{C}^*$ and 1-forms $\eta(z)\,dz$ on (the universal cover of) $\mathbb{C}^*$, where

$$\eta(z) = \left(\begin{array}{c|c|c|c} & & & z^{k_0} \\ \hline z^{k_1} & & & \\ \hline & \ddots & & \\ \hline & & z^{k_n} & \end{array} \right).$$

- $k_i \in [-1, \infty)$
- $n + 1 + \sum_{i=0}^{n} k_i = N$
- $k_i = k_{n-i+1}$ for $i = 1, \dots, n$

The variable z is related to the variable t of the tt*-Toda equations by $t = \frac{n+1}{N} z^{\frac{N}{n+1}}$.

This theorem has several interpretations:

(p.d.e.)

Asymptotics of the solution at $t = 0$:
$w_i \sim -m_i \log |t|$ as $t \to 0$, where the m_i are defined by $m_{i-1} - m_i + 1 = \frac{n+1}{N}(k_i + 1)$.

(harmonic maps)

$\frac{1}{\lambda}\eta(z)dz$ is a (normalized) DPW potential for the harmonic map corresponding to the solution ($\lambda = \hbar =$ loop parameter).

(DPW=Dorfmeister-Pedit-Wu)

(harmonic bundles)

$\eta(z)dz$ is a Higgs field for the parabolic harmonic bundle corresponding to the solution.

(chiral ring)

$\eta(z)$ is the matrix of quantum multiplication by a generator in the chiral ring corresponding to the solution,

E.g. $\mathbb{C}P^n$:

$k_0 = 0, k_1 = \cdots = k_n = -1$

$z = q$

$\eta(z)dz = C\,\dfrac{dq}{q}$

Using monodromy data (Stokes data), the solutions can also be characterized by their asymptotics at $t = \infty$:

Theorem 2: (M. Guest - A. Its - C.-S. Lin, arXiv 2010-2017)
There is a $1:1$ correspondence between solutions of the tt*-Toda equations on $\mathbb{C}^*$ with n-tuples of "Stokes parameters" $s = (s_1, \dots, s_n)$ where s_i is the i-th symmetric function of $e^{(2m_0+n)\frac{\pi\sqrt{-1}}{n+1}}$, $e^{(2m_1+n-2)\frac{\pi\sqrt{-1}}{n+1}}$, ... $e^{(2m_n-n)\frac{\pi\sqrt{-1}}{n+1}}$.

As $t \to \infty$ we have, for each $k = 1, ..., n$,

$$-\frac{4}{n+1} \sum_{p=0}^{[\frac{1}{2}(n-1)]} w_p \sin \frac{(2p+1)k\pi}{n+1} \sim s_k \, F(L_k|t|)$$

where $F(|t|) = \frac{1}{2}(\pi|t|)^{-\frac{1}{2}} e^{-2|t|}$, $L_k = 2 \sin \frac{k}{n+1}\pi$.

Remark: The Stokes parameters determine a matrix $M^{(0)}$ (which will be described later). In turn, the matrix $M^{(0)}$ determines all "Stokes matrices" in the classical sense.

STRATEGY OF THE PROOFS (OF THEOREMS 1-2)

STEP 1: Construct solutions on $0 < |t| < \epsilon$ such that $w_i \sim -m_i \log |t|$ as $t \to 0$.

(Banach Lie group factorization — can be done locally)

STEP 2: Construct solutions on $R < |t| < \infty$ (such that $w_i \sim 0$ as $t \to \infty$).

(Riemann-Hilbert problem — can be solved locally)

STEP 3: Characterize all global solutions on $0 < |t| < \infty$ such that $w_i \sim -m_i \log |t|$ as $t \to 0$ and $w_i \sim 0$ as $t \to \infty$.

(p.d.e. existence/uniqueness theorem — method of subsolutions and supersolutions)

STEP 4: Match up the results of Steps 1-3.

(Steps 1 and 3 give Theorem 1; Step 2 is needed in order to compute Stokes data in Theorem 2, and will be needed for more precise asymptotics of solutions)

STEP 1: Given $k_0, \dots, k_n \geq -1$, construct some solutions on intervals of the form $0 < |t| < \epsilon$ such that $w_i \sim -m_i \log |t|$ as $t \to 0$.

Start from

$$\eta(z) = \left(\begin{array}{c|c|c|c} & & & z^{k_0} \\ \hline z^{k_1} & & & \\ \hline & \ddots & & \\ \hline & & z^{k_n} & \end{array} \right).$$

Solve the (complex) o.d.e.

$$\frac{dL}{dz} = \frac{1}{\lambda} L\eta, \quad L(0) = I$$

for matrix valued $L = L(z, \lambda)$ (near $z = 0$).

Factorize: $L = L_{\mathbb{R}} L_+$ near I (Iwasawa factorization, regarding L as a Banach loop group valued function).

Then $\alpha = (L_{\mathbb{R}})^{-1} dL_{\mathbb{R}}$ satisfies $d\alpha + \alpha \wedge \alpha = 0$, and this is the zero curvature form of the Toda equation.

STEP 2: Given $s_1, \dots, s_n \in \mathbb{R}$, construct some solutions on intervals of the form $R < |t| < \infty$ such that $-\frac{4}{n+1} \sum_{p=0}^{[\frac{1}{2}(n-1)]} w_p \sin \frac{(2p+1)k\pi}{n+1} \sim s_k \, F(L_k|t|)$ as $t \to \infty$.

Start with the isomonodromic form of the Toda equation. This is a complex o.d.e. in λ (with poles of order 2 at $\lambda = 0, \infty$).

Calculate the monodromy data (Stokes matrices, connection matrices) of the solutions constructed in Step 1.

Pose a Riemann-Hilbert problem based on the above monodromy data. This is equivalent to a linear singular integral equation. It can be solved near $t = \infty$.

STEP 3: Characterize all global solutions on $0 < |t| < \infty$ such that $w_i \sim -m_i \log |t|$ as $t \to 0$ and $w_i \sim 0$ as $t \to \infty$.

P.d.e. theorem: Assume $m_{i-1} - m_i + 1 \geq 0$ (i.e. $k_i \geq -1$). Then $\exists!$ solution on $0 < |t| < \infty$ such that $w_i \sim -m_i \log |t|$ as $t \to 0$ and $w_i \sim 0$ as $t \to \infty$.

STEP 4: Match up the results of Steps 1-3.

This uses the actual values of the Stokes matrices and connection matrices which were computed in Step 2.

As a "bonus" (going beyond Theorems 1 and 2), we mention the more precise asymptotics of solutions (near $t = 0$) which follow from that computation:

SAMPLE FOR THE GENERIC CASE
$(m_{i-1} - m_i + 1 > 0)$:

$n = 3 \quad w_0, w_1 \quad (w_2 = -w_1, w_3 = -w_0)$

$w_0 \sim -m_0 \log |t| + \frac{1}{2}\rho_0 + o(1)$

where

$$\rho_0 = -\log\ 2^{-4m_0} \frac{\Gamma(\frac{-m_0}{2}+\frac{1}{4})\Gamma(\frac{-m_0-m_1}{4}+\frac{1}{2})\Gamma(\frac{-m_0+m_1}{4}+\frac{3}{4})}{\Gamma(\frac{-m_1+m_0}{4}+\frac{1}{4})\Gamma(\frac{m_0+m_1}{4}+\frac{1}{2})\Gamma(\frac{m_0}{2}+\frac{3}{4})}$$

$w_1 \sim -m_1 \log |t| + \frac{1}{2}\rho_1 + o(1)$

where

$$\rho_1 = -\log\ 2^{-4m_1} \frac{\Gamma(\frac{-m_1+m_0}{4}+\frac{1}{4})\Gamma(\frac{-m_0-m_1}{4}+\frac{1}{2})\Gamma(\frac{-m_1}{2}+\frac{3}{4})}{\Gamma(\frac{m_1}{2}+\frac{1}{4})\Gamma(\frac{m_0+m_1}{4}+\frac{1}{2})\Gamma(\frac{-m_0+m_1}{4}+\frac{3}{4})}$$

SAMPLE FOR THE NON-GENERIC CASE:

$n = 3 \quad w_0, w_1 \quad (w_2 = -w_1, w_3 = -w_0)$

$m_0 = -\frac{3}{2}, m_1 = -\frac{1}{2} \quad$ (here $m_0 - m_1 + 1 = 0$)

$$w_0(t) = \frac{3}{2}\log|t| + \frac{1}{2}\log\Big(-\frac{1}{24}\zeta(3) - \frac{4}{3}\gamma_{\mathrm{eu}}^3 - 4\gamma_{\mathrm{eu}}^2 \log\frac{|t|}{4} -4\gamma_{\mathrm{eu}}\log^2\frac{|t|}{4} - \frac{4}{3}\log^3\frac{|t|}{4}\Big) + O(|t|^4\log^6|t|)$$

$$w_0(t) + w_1(t) = 2\log|t| + \frac{1}{2}\log\Big(-\frac{1}{12}\gamma_{\mathrm{eu}}\zeta(3) + \frac{4}{3}\gamma_{\mathrm{eu}}^4 + (-\frac{1}{12}\zeta(3) + \frac{16}{3}\gamma_{\mathrm{eu}}^3)\log\frac{|t|}{4} + 8\gamma_{\mathrm{eu}}^2\log^2\frac{|t|}{4} + \frac{16}{3}\gamma_{\mathrm{eu}}\log^3\frac{|t|}{4} + \frac{4}{3}\log^4\frac{|t|}{4}\Big) + O(|t|^4\log^6|t|)$$

($\gamma_{\mathrm{eu}} =$ Euler constant, $\zeta(3) = \sum_{k=1}^{\infty} k^{-3}$)

SOME APPLICATIONS

Application 1 (Lie theory): The Coxeter Plane

Let $\mathfrak{g}$ be a complex simple Lie algebra, with corresponding simply-connected Lie group G.

Let $\alpha_1, \ldots, \alpha_l \in \mathfrak{h}^*$ be a choice of simple roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. Let Δ be the set of all roots.

(in this talk $\mathfrak{g} = \mathfrak{sl}_{n+1}\mathbb{C}$)

The Weyl group W is the finite group generated by the reflections r_α in all root planes $\ker \alpha$, $\alpha \in \Delta$.

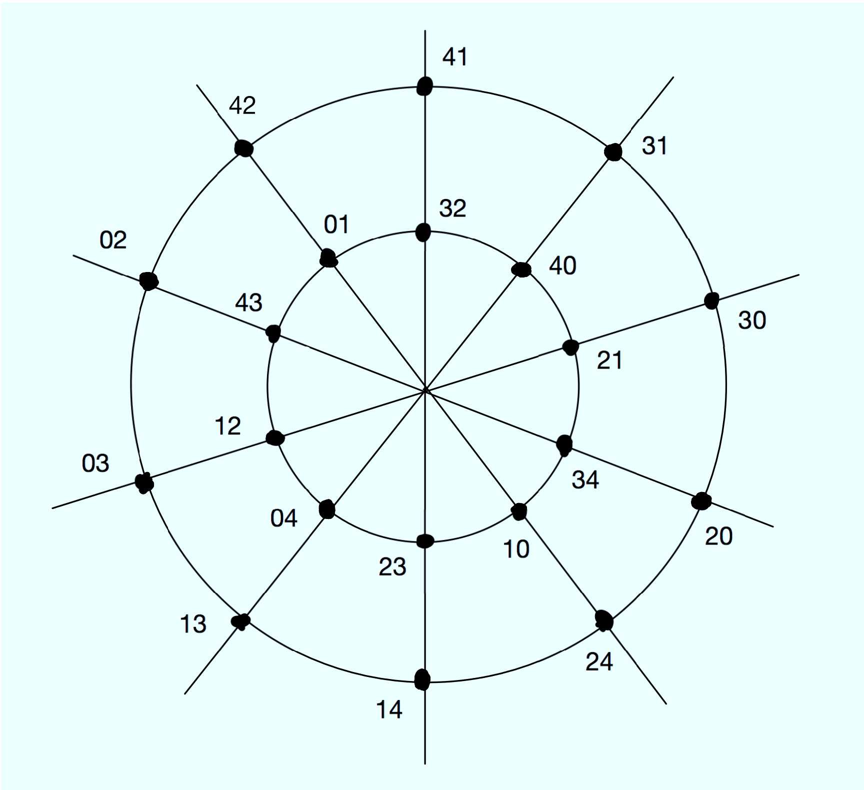
The Coxeter element is the element $\gamma = r_{\alpha_1} \dots r_{\alpha_l}$ of W. Its order is called the Coxeter number of $\mathfrak{g}$, and we denote it by s.

Fact (Kostant): The Coxeter element γ acts on the set of roots Δ with l orbits, each containing s elements.

(if $\mathfrak{g} = \mathfrak{sl}_{n+1}\mathbb{C}$ then $l = n$, $s = n + 1$, and W is the permutation group on $n + 1$ objects)

The Coxeter Plane is the result of projecting Δ orthogonally onto a certain real plane in $\mathfrak{h}^*$.

E.g. the Coxeter Plane for $\mathfrak{g} = \mathfrak{sl}_5\mathbb{C}$:



(there are 20 roots $x_i - x_j$, $0 \leq i \neq j \leq 4$, and the Coxeter element acts by the permutation (43210); there are $l = 4$ orbits, each containing $s = 5$ elements)

Theorem: (M. Guest - N.-K. Ho, arXiv 2018)

(i) The Coxeter Plane is a diagram of the Stokes sectors for the tt*-Toda equation.

(ii) The Stokes matrices can be computed Lie-theoretically in terms of a Lie group element

$$M^{(0)} = C(s_1, \dots, s_l) \in \mathrm{SL}_{n+1}\mathbb{C}$$

where C is a "Steinberg cross-section" of the set of regular conjugacy classes of $\mathrm{SL}_{n+1}\mathbb{C}$.

Remark: $A \in \mathrm{SL}_{n+1}\mathbb{C}$ is regular iff it satisfies "minimal poly. of A = characteristic poly. of A".

Moreover, the space of solutions also has a Lie-theoretic interpretation:

Recall that the solutions are parametrized by $(n+1)$-tuples $(m_0, \ldots, m_n)$ satisfying

(*) $m_{i-1} - m_i + 1 \geq 0$ (i.e. $k_i \geq -1$)

(**) $m_i + m_{n-i} = 0$ (i.e. $k_i = k_{n-i+1}$)

The inequalities (*) define a convex polytope.

Let us write

$$m = \mathrm{diag}(m_0, m_1, \ldots, m_n)$$

$$\rho = \mathrm{diag}(\tfrac{n}{2}, \tfrac{n}{2} - 1, \ldots, -\tfrac{n}{2})$$

Then the convex polytope given by the points

$$\frac{2\pi\sqrt{-1}}{n+1}(m + \rho)$$

is the Fundamental Weyl Alcove of the Lie algebra.

Application 2 (physics): Particles and polytopes

In this section we show how the Coxeter Plane and the tt*-Toda equations give a mathematical foundation for certain field theory models proposed by physicists in the 1990's.

The Coxeter Plane has appeared (implicitly) in articles on Toda field theory:

M. Freeman, On the mass spectrum of affine Toda field theory, Phys. Lett. B 1991

P. Dorey, Root systems and purely elastic S-matrices I,II, Nuclear Phys. B 1991,1992

In this “toy model” the authors proposed (amongst other things) the correspondence

$$\begin{aligned} \text{particle} &\leftrightarrow \text{orbit of root in Coxeter Plane} \\ \text{mass of particle} &\leftrightarrow \text{distance of root from origin} \end{aligned}$$

(if $\mathfrak{g} = \mathfrak{sl}_{n+1}\mathbb{C}$ the mass of the particle corresponding to the orbit of the root $x_i - x_j$ is $2\sin|i-j|\frac{\pi}{n+1}$)

They checked that these proposals (as well as the other things) were consistent with the expected properties of a field theory.

A variant of this proposal was made in

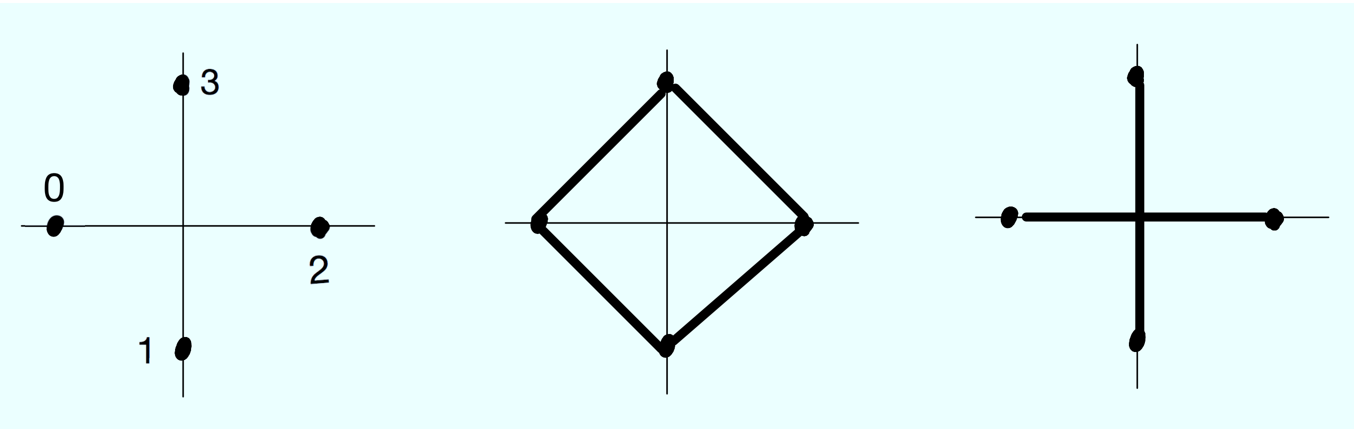
P. Fendley, W. Lerche, S. Mathur, and N. Warner, $N = 2$ supersymmetric integrable models from affine Toda theories, Nuclear Phys. B 1991

W. Lerche and N. Warner, Polytopes and solitons in integrable $N = 2$ supersymmetric Landau-Ginzburg theories, Nuclear Phys. B 1991

In these "polytopic models", a finite-dimensional representation θ of the Lie algebra $\mathfrak{g}$ on a vector space V is chosen, and the "polytope" is the polytope in $\mathfrak{h}^*$ spanned by the weights of the representation. The weight vectors (in V) are taken to be the vacua of the theory. In this theory, "solitonic particles" tunnel between vacua: a soliton connects two vacua v_i, v_j if and only if the corresponding weights λ_i, λ_j differ by a single root, i.e. $\lambda_i - \lambda_j \in \Delta$. The physical characteristics of the particle are those of that root.

This discussion is purely algebraic (there is no differential equation). However, the polytopic models include certain Landau-Ginzburg models. The quantum cohomology of $\mathbb{C}P^n$ is of this type, with: $\theta = \lambda_{n+1}$ (standard representation of $\mathfrak{sl}_{n+1}\mathbb{C}$). Thus we can expect a role for solitons in the quantum cohomology of $\mathbb{C}P^n$.

The solitons are illustrated below for $\mathfrak{sl}_4\mathbb{C}$.



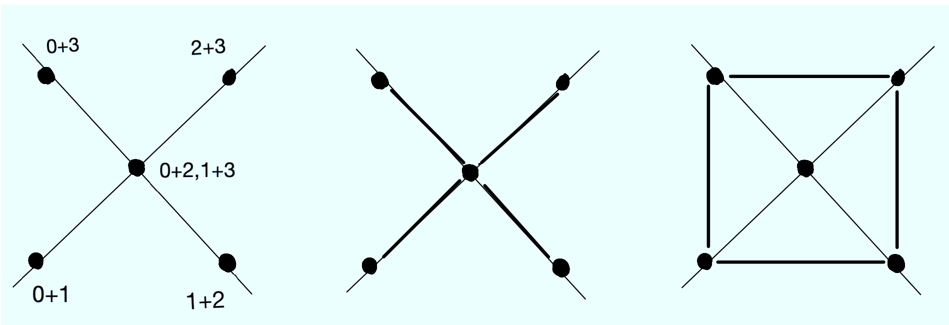
The first part shows the projections of the weights x_0, x_1, x_2, x_3. The second part shows (as heavy lines) the four solitons of type [01] (with mass $2\sin\frac{\pi}{4} = \sqrt{2}$). The third part shows the two solitons of type [02] (with mass $2\sin\frac{\pi}{2} = 2$). In this example, any two vacua are connected by a soliton.

In the above example we chose $\theta = \lambda_{n+1}$. If we choose

$$\theta = \wedge^k \lambda_{n+1}$$

we obtain a different polytopic model. It turns out that the quantum cohomology of the Grassmannian $Gr_k(\mathbb{C}^{n+1})$ is of this type.

The solitons are illustrated below for $\mathfrak{sl}_4\mathbb{C}$. The first part shows the projections of the weights $x_i + x_j$ with $0 \le i \ne j \le 3$. The second part shows the four solitons of type [01] (with mass $\sqrt{2}$). The third part shows the four solitons of type [02] (with mass 2).



Each solution of the tt*-Toda equation (e.g. that with $m = -\rho$) is associated to a field theory. That theory fits into this framework as follows.

Corollary: (of the proof of Theorem 2 on the asymptotics at $t = \infty$)
The linear combination on the left hand side of

$$-\tfrac{4}{n+1} \sum_{p=0}^{[\frac{1}{2}(n-1)]} w_p \sin \tfrac{(2p+1)k\pi}{n+1} \sim s_k \, F(L_k|t|)$$

corresponds to a certain† basis vector of $\mathfrak{h}$ (or $\mathfrak{h}^*$) associated to an orbit of the Coxeter group.

Thus we can say that the Stokes parameter s_k is naturally associated to the k-th orbit, or particle. Physicists call s_k the soliton multiplicity.

†Reference: *M. Guest, arXiv 2020*

Cecotti and Vafa used this to give a physical argument for an “equivalence”

$$\wedge^k QH^*(\mathbb{C}P^n) \approx QH^*(Gr_k(\mathbb{C}^{n+1}))$$

(more precisely, an equivalence of underlying field theories). This tt* argument was explained in

M. Bourdeau, Grassmannian σ-models and topological–anti-topological fusion, Nuclear Phys. B 1995

Later, mathematicians gave proofs of versions of this isomorphism (which they regard as a special case of the quantum Satake isomorphism, or abelian-nonabelian correspondence). E.g.

V. Golyshev and L. Manivel, Quantum cohomology and the Satake isomorphism, arXiv:1106.3120

Our Lie-theoretic description of the solutions of the tt*-Toda equations supports the original physics argument, because

$$\text{solution with } m = -\rho \quad \overset{\theta=\lambda_{n+1}}{\longleftrightarrow} \quad QH^*(\mathbb{C}P^n)$$

$$\text{solution with } m = -\rho \quad \overset{\theta=\wedge^k\lambda_{n+1}}{\longleftrightarrow} QH^*(Gr_k(\mathbb{C}^{n+1}))$$

i.e. the same solution of the tt*-Toda equations gives both $QH^*(\mathbb{C}P^n)$ and $QH^*(Gr_k(\mathbb{C}^{n+1}))$

The Stokes matrices of the respective quantum differential equations are different (they can be read off from $M^{(0)}$ and $\wedge^k M^{(0)}$ respectively). But the Stokes parameters $s_k = \binom{n+1}{k}$ are the same for $QH^*(\mathbb{C}P^n)$ and $QH^*(Gr_k(\mathbb{C}^{n+1}))$.

Application 3 (physics): Minimal models

Recall that the "Higgs fields"

$$\eta(z) = \left(\begin{array}{c|c|c|c} & & & z^{k_0} \\ \hline z^{k_1} & & & \\ \hline & \ddots & & \\ \hline & & z^{k_n} & \end{array}\right)$$

(with $k_i \in [-1, \infty)$, $n + 1 + \sum_{i=0}^{n} k_i = N$, $k_i = k_{n-i+1}$ for $i = 1, \dots, n$) parametrize solutions of the tt*-Toda equations.

In this section we consider ηdz with $k_i \in \mathbb{Z}_{\geq 0}$ and assume that N is coprime to $k = \sum_{i=0}^{n} k_i$.

Thus we move away from the tt*-Toda equations (but we note that the Higgs fields with $k_i = k_{n-i+1}$ for $i = 1, \dots, n$ form a dense subset of solutions of the tt*-Toda equations).

The authors of

L. Fredrickson and A. Neitzke, From S^1-fixed points to W-algebra representations, arXiv:1709.06142

study a certain moduli space $M_{K,N}$ of Higgs fields with a $\mathbb{C}^*$-action whose fixed points are all the ηdz with K, N fixed (a finite number). Quoting from this article:

“We ... exhibit a curious 1-1 correspondence between these fixed points and certain representations of the vertex algebra W_K ; in particular we have $12\mu = K - 1 - c_{eff}$, where 12μ is a ... norm of the Higgs field, and c_{eff} is the effective Virasoro central charge.”

“The formula $12\mu = K - 1 - c_{eff}$ is puzzling. Why should W_K and $M_{K,N}$ have anything to do with one another?”

As an application of our Lie-theoretic Stokes formula

$$M^{(0)} = C(s_1, \dots, s_l) \in \mathrm{SL}_{n+1}\mathbb{C}$$

we shall give a mathematical explanation — a direct path from the Higgs field ηdz to the representation.

Reference: *M. Guest and T. Otofuji, arXiv 2021*

Recall that the irreducible positive energy representations of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$ of level k ($\in \mathbb{N}$) are parametrized by dominant weights (Λ, k), where Λ is a dominant weight of $\mathfrak{sl}_{n+1}\mathbb{C}$ of level k.

Let P_+ be the set of dominant weights of $\mathfrak{sl}_{n+1}\mathbb{C}$, and $P_k = \{$ dominant weights of level $k\}$.

It is well known that $P_k + \rho = P_+ \cap (k+n+1)\mathring{A}$ where $\mathring{A}$ denotes the interior of the Weyl alcove A.

Let $\mathring{A}_k = \left(\frac{1}{k+n+1}P_+\right) \cap \mathring{A}$. Let $\theta : \mathring{A}_k \to P_k + \rho$ be the identification given by
$\theta(v) = (k+n+1)v \in P_+ \cap (k+n+1)\mathring{A} = P_k + \rho$.

Recall that the Stokes data (of the Higgs field ηdz) is represented by the matrix $M^{(0)}$. It follows from the assumption $k_i \in \mathbb{Z}_{\geq 0}$ that $M^{(0)}$ is semisimple; in fact it is conjugate to the diagonal matrix

$$e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}$$

(recall that $\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)$ is in the Fundamental Weyl Alcove of $\mathfrak{sl}_{n+1}\mathbb{C}$).

Lemma: Let $2\pi_{\sqrt{-1}}\,\epsilon_1, \dots, 2\pi_{\sqrt{-1}}\,\epsilon_n$ denote the basic weights of $\mathfrak{sl}_{n+1}\mathbb{C}$. Then:
$\frac{N}{n+1}(m+\rho) = \rho + \sum_{i=1}^n k_i\epsilon_i$.

Proof: This is equivalent to the relation $m_{i-1} - m_i + 1 = \frac{n+1}{N}(k_i+1)$ which defines the m_i in terms of the k_i. □

It follows that $\theta(\frac{1}{n+1}(m+\rho)) = \rho + \sum_{i=1}^n k_i\epsilon_i$. Thus, from the Stokes data $M^{(0)}$ we obtain the positive energy representation with dominant weight $(\sum_{i=1}^n 2\pi_{\sqrt{-1}}\,k_i\epsilon_i, k)$.

It is well known (Bouwknegt and Schoutens) that the W-algebra W_{n+1} intertwines with any such representation, and that the effective central charge is given by the formula

$$c_{eff} = n - 12\tfrac{n+1}{N} \,|\textstyle\sum_{i=1}^{n} k_i\epsilon_i - \tfrac{k}{n+1}\rho|^2.$$

By the lemma we have $\sum_{i=1}^{n} k_i\epsilon_i - \frac{k}{n+1}\rho = \frac{N}{n+1}m$ so

$$c_{eff} = n - 12\tfrac{N}{n+1}\,|m|^2.$$

This is the formula of Fredrickson and Neitzke which Higgs fields and representations of W_{n+1}. Our construction shows that the Stokes data of the Higgs field is responsible for the relation.

Remark: For fixed $n+1$ and N the finite number of Higgs fields give a finite number of representations. These constitute the "$(n+1, N)$ W_{n+1} minimal model".

Thank you !