Lagrangian manifolds and Hamiltonian systems, corresponding to asymptotic solutions of PDE's with singularities.

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Outline

1. Geometric asymptotics for equations with smooth coefficients (Maslov theory)
   - Spectral problems
   - Cauchy problems

2. Equations with singularities
   - Spectral problems for Schrödinger operator with δ-potential
     - Operator with δ-potential on the surface of revolution
     - Surface of revolution with conic point
   - Cauchy problem for Schrödinger equation with delta-potential
     - Reflection of Lagrangian manifolds
     - Reflection of vector bundles

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Spectral problem

Spectral problem for the Schrödinger operator.
Let $x \in \mathbb{R}^n$,

$$\hat{H} = H(x, -ih\frac{\partial}{\partial x})$$

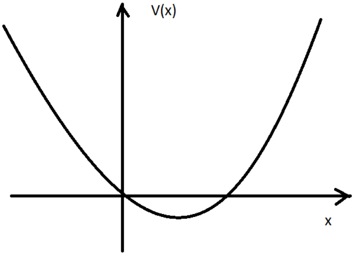
Problem: asymptotics of the spectrum as $h \to 0$.
Well-developed theory for smooth case. We consider examples with δ-potentials and with conic singularities.

1D example

Let $n = 1$,

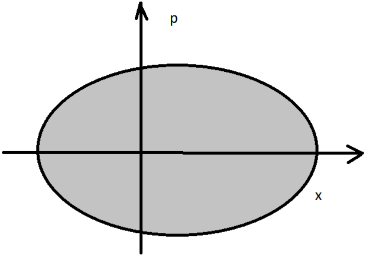
$$\hat{H} = -\frac{h^2}{2}\frac{d^2}{dx^2} + V(x),$$

$V(x) \to +\infty, \quad |x| \to \infty.$


V(x)
x

Λ — curve on the phase plane.

$$\frac{1}{2}p^2 + V(x) = E.$$



Theorem

Let E be solution of the Bohr — Sommerfeld equation

$$\frac{1}{2\pi h}\int_{\Lambda} p dx + \frac{1}{2} = m \in \mathbb{Z}.$$

Then there exists an eigenvalue λ of $\hat{H}$:

$$\lambda = E + o(h).$$

Maslov theory for smooth Hamiltonians

Maslov theory for smooth Hamiltonians.
$\hat{H} = H(x - ih\frac{\partial}{\partial x})$.
Let Λ be compact invariant manifold of the classical Hamilton system in $\mathbb{R}^{2n}$ with the Hamilton function $H(x, p)$.

Theorem (V.P. Maslov)

Let Λ satisfy quantization condition

$$\frac{1}{2\pi h}[\theta] + \frac{1}{4}[\mu] \in H^1(\Lambda, \mathbb{Z})$$

and let $\hat{H}$ be self-adjoint. Then there exists a point λ of the spectrum, such that

$$\lambda = H|_\Lambda + O(h^2).$$

$\theta = \sum_j p_j dx_j$.

$$\frac{1}{2\pi h}\int_{\gamma}\theta+\frac{1}{4}\mu(\gamma)=m\in\mathbb{Z}.$$

μ — Maslov index. $\pi:\mathbb{R}^{2n}_{(x,p)}\to\mathbb{R}^{n}_{x}$ — natural projection, Σ — cycle of singularities of π.

$$\mu(\gamma)=\gamma\circ\Sigma.$$

Example: integrable Hamiltonian system.
Λ — Liouville tori, I — action variables. Quantization conditions

$$\frac{1}{h}I_j + \frac{1}{4}\mu_j = m_j \in \mathbb{Z}.$$

$$\lambda = H(I(m)) + O(h^2).$$

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Cauchy problem for *h*-pseudodifferential evolutionary equation

$$ih\frac{\partial u}{\partial t} = H(x, -ih\frac{\partial}{\partial x})u, \quad x \in \mathbb{R}^n, h \to +0,$$

$H(x,p) : \mathbb{R}^{2n} \to \mathbb{R}$ is smooth.

$$u|_{t=0} = \varphi^0(x)e^{\frac{iS_0(x)}{h}}, \quad S_0 \in C^\infty, \varphi^0 \in C_0^\infty.$$

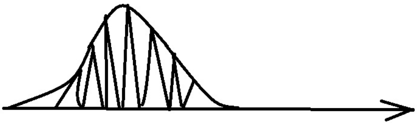

Figure: Wave packet

Solutions, corresponding to Lagrangian manifolds.

Solutions, corresponding to Lagrangian manifolds.
Rapidly oscillating wave packet - S_0 is real. Asymptotic solution. Consider initial Lagrangian surface $\Lambda_0 \subset \mathbb{R}^{2n}$, $p = \frac{\partial S_0}{\partial x}$ and shift it by the flow g_t of the classical Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \Lambda_t = g_t \Lambda_0.$$

Volume form $\sigma_0 = dx$ on Λ_0, $\sigma_t = g_t^* dx$ on Λ_t

Theorem

(V.P. Maslov, $\sim$ 1965). Under certain technical conditions the solution $u(x,t,h)$ can be represented as asymptotic serie

$$u \sim K_{\Lambda_t,\sigma_t}\Big(\sum_{k=0} h^k \varphi_k\Big),$$

$K : C_0^\infty(\Lambda_t) \to C^\infty(\mathbb{R}^n_x)$ *is the Maslov canonical operator,* φ_k *are smooth functions on* Λ_t, $\varphi_0(\alpha) = \varphi^0(g_{-t}\alpha)$.

Figure: Squeezed state

Solutions, corresponding to complex vector bundles

Solutions, corresponding to complex vector bundles
Localized ("squeezed") initial state $S_0(x)$ is complex, $\Im S_0 \geq 0$, $\Im S_0 = 0$ on the smooth k-dimensional surface W_0, $d^2 \Im S_0|_{NL_0} > 0$. Consider k-dimensional isotropic surface $\Lambda_0 \subset \mathbb{R}^{2n}$: $x \in W_0, p = \frac{\partial S_0}{\partial x}$ and n-dimensional complex vector bundle ρ_0 over Λ_0 (Maslov complex germ): fiber $\rho(x, p)$ is the plane in ${}^{\mathbb{C}}T_{x,p}\mathbb{R}^{2n}$, $\xi_p = \frac{\partial^2 S_0}{\partial x^2}\xi_x$. Shifted bundle $\Lambda_t = g_t \Lambda_0$, $\rho_t = dg_t \rho_0$.

Theorem (V.P. Maslov)

Under certain technical conditions the solution $u(x, t, h)$ can be represented as asymptotic serie

$$u \sim \hat{K}_{\Lambda_t,\rho_t}(\sum_{k=0} h^k \varphi_k),$$

$\hat{K} : C_0^\infty(\Lambda_t) \to C^\infty(\mathbb{R}_x^n)$ *is the Maslov canonical operator on the complex germ,* φ_k *are smooth functions on* Λ_t*,*
$\varphi_0(\alpha) = \varphi^0(g_{-t}\alpha)$.

Simplest case:

$$S_0 = (p_0, x - x_0) + \frac{1}{2}(x - x_0, Q_0(x - x_0))), \quad p_0 \in \mathbb{R}^n, Q^t = Q, \Im Q > 0.$$

W_0 is the point x_0, $\rho_0 : \xi_p = Q_0 \xi_x$.

$$u(x, t, h) \sim e^{\frac{iS(x,t)}{h}} \sum_{k=0}^{\infty} (h^k \varphi_k(x, t)).$$

$$S = q(t) + (P(t), x - X(t)) + \frac{1}{2}(x - X(t), Q(t)(x - X(t))),$$

$$\dot{X} = \frac{\partial H}{\partial p}, \quad \dot{P} = -\frac{\partial H}{\partial x},$$

Q can be expressed explicitly in terms of solutions of the linearized system.

Problem

What happens if coefficients of initial equation contain singularities?

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1D example

Let $n = 1$,

$$\hat{H} = -\frac{h^2}{2}\frac{d^2}{dx^2} + V(x) + \alpha\delta(x - x_0).$$

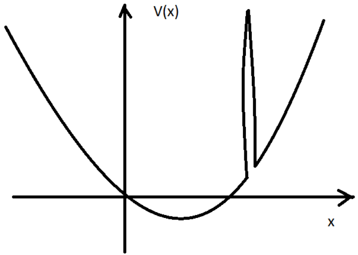
Formal definition:

$$\hat{H}_0 = -\frac{h^2}{2}\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{R}\backslash x_0.$$

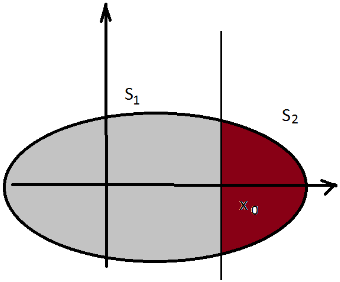
Boundary conditions

$$\psi(x_0 + 0) = \psi(x_0 - 0),$$

$$\psi'(x_0 + 0) - \psi'(x_0 - 0) = \frac{2\alpha}{h^2}\psi(x_0).$$


V(x)
x

$$\frac{1}{2}p^2 + V(x) = E.$$



Theorem

Let E be solution of the equation

$$\cos\Big(\frac{1}{2h}(S_1+S_2)\Big) =$$

$$= \frac{\alpha}{hp(x_0)}\Big(\sin\Big(\frac{1}{2h}(S_1+S_2)\Big) - \cos\Big(\frac{1}{2h}(S_1-S_2)\Big)\Big).$$

Then there exists an eigenvalue λ of $\hat{H}$:

$$\lambda = E + o(h).$$

Limit cases

$\frac{\alpha}{h} \to 0$,

$$\frac{S_1 + S_2}{2\pi h} + \frac{1}{2} = m \in \mathbb{Z},$$

$\frac{\alpha}{h} \to \infty$,

$$\frac{S_1}{2\pi h} + \frac{1}{4} = m_1 \in \mathbb{Z}, \quad \frac{S_2}{2\pi h} + \frac{3}{4} = m_2 \in \mathbb{Z}.$$

M — Riemannian manifold, $\dim M \leq 3$,

$$\hat{H} = -\frac{h^2}{2}\Delta + \alpha\delta_P$$

Definition of the operator with delta-potential δ_P (Berezin, Faddeev). 2 properties

- $\hat{H}$ is self-adjoint;
- If $\psi(P) = 0$, then $\hat{H}\psi = -\frac{h^2}{2}\Delta\psi$.

Formal definition. $\hat{H}_0 = -\frac{h^2}{2}\Delta|_{\psi\in H^2(M),\psi(P)=0}$.
$\hat{H}$ is a self-adjoint extension of $\hat{H}_0$.

Explicit description of the domain.
For $\psi \in D(\hat{H})$ we have a decomposition

$$\psi = aF(x) + b + o(1),$$

$$F = -\frac{1}{4\pi d(x,P)}, \quad \mathrm{dim} M = 3, \quad F = \frac{1}{2\pi} \log d(x,P), \quad \mathrm{dim} M = 2.$$

Boundary condition

$$a = \frac{2\alpha}{h^2} b.$$

Symmetric manifold

Let M be 2D surface of revolution or 3D spherically symmetric manifold, $M \cong S^2$ or $M \cong S^3$.

$$M \subset \mathbb{R}^3, \quad y = (f(z)\cos\varphi, f(z)\sin\varphi f(z), z)$$

or

$$M \subset \mathbb{R}^4, \quad y = (f(z)\cos\theta\cos\varphi, f(z)\cos\theta\sin\varphi, f(z)\sin\theta, z)$$

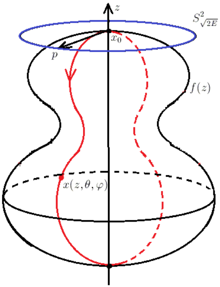
$z \in [z_1, z_2]$,
$f = \sqrt{(z - z_1)(z_2 - z)}w(z)$, w — analytic. Let δ-potential be localized in a pole.

Result: Lagrangian manifold

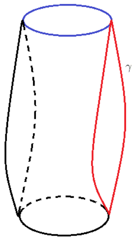
$\Lambda_0 : p \in T^*_P M, \quad |p| = 2E$, $\Lambda = \bigcup_t g_t \Lambda_0$, g_t — geodesic flow.

$$\Lambda \cong T^2, \quad \mathrm{dim} M = 2, \quad \Lambda \cong S^2 \times S^1, \quad \mathrm{dim} M = 3.$$

Trajectories



Lagrangian manifold



Result: eigenvalues

Theorem (Asilya Suleimanova, Tudor Ratiu, A.S.)

Let E be solution of the equation

$$\tan\left(\frac{1}{2h}\oint_\gamma (p, dx)\right) = \frac{2}{\pi}\left(\log\left(\frac{\sqrt{2E}}{h}\right) + \frac{\pi h^2}{\alpha} + c\right), \quad n = 2,$$

c is Euler constant,

$$\tan\left(\frac{1}{2h}\oint_\gamma (p, dx)\right) = \frac{2h^3}{\sqrt{2E}\alpha}, \quad n = 3.$$

Theorem (Asilya Suleimanova, Tudor Ratiu, A.S.)

Here γ is closed geodesic.
There exists an eigenvalue λ of $\hat{H}$, such that

$$\lambda = E + o(h).$$

Critical values of α.

Jump of the Maslov index
2D-case. Let

$$\frac{\alpha \log 1/h}{h^2} \to 0 \quad or \quad \frac{\alpha \log 1/h}{h^2} \to \infty.$$

Then E up to small terms satisfies

$$\frac{1}{2\pi h} \int_\gamma (p, dx) + \frac{1}{2} = m \in \mathbb{Z}.$$

Critical value

$$\alpha \sim \frac{h^2}{\log(1/h)}.$$

Critical values of α.

3D case.
Let $\alpha/h^3 \to 0$. Then E satisfies

$$\frac{1}{2\pi h}\int_\gamma (p, dx) + \frac{1}{2} = m \in \mathbb{Z}.$$

Let $\alpha/h^3 \to \infty$. Then E satisfies

$$\frac{1}{2\pi h}\int_\gamma (p, dx) = m \in \mathbb{Z}.$$

Critical value $\alpha \sim h^3$.

Jump of the Maslov index

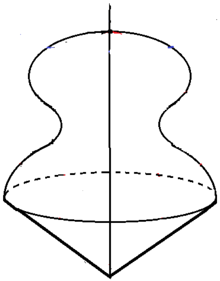
In 3D case the analog of the Maslov index jumps as α passes through the critical value. $\Lambda_0 : p \in T^*_P M, |p| = 2E$,
$F : \Lambda_0 \to \Lambda_0, \quad F(p) = -p$
General formula for big α

$$\frac{1}{2\pi h} \int_\gamma (p, dx) + \frac{1}{4}(\mu(\gamma) + (\deg F - 1)) = m \in \mathbb{Z}.$$

Surface of revolution with conic point.

$$ds^2 = dz^2 + u^2(z)d\varphi^2, \quad z \in [0, L/2]$$

1. $u(z) > 0 \;\; z \in (0, L/2)$, $u(0) = u(L/2) = 0$.
2. $z = 0$ is a conic point with total angle $2\pi\beta$ ($\beta > 0$). Near the point $z = 0$ $u(z) = \beta z u_0(z)$, near the point $z = L/2$ $u(z) = (\frac{L}{2} - z)u_1(\frac{L}{2} - z)$, u_0, u_1 — analytic functions, $u_j(0) = 1$.



Spectral problem

$$-\frac{h^2}{2}\Delta\psi = \lambda\psi$$

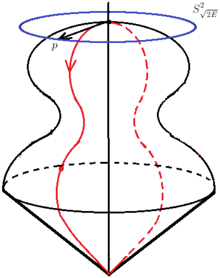
Domain of the Laplacian.

$$F_0^+ = 1, \quad F_0^- = \log z,$$
$$F_k^\pm = (\frac{|k|}{\beta})^{-1/2} z^{\pm(\frac{|k|}{\beta})} e^{ik\varphi}, \quad k \in \mathbb{Z}, 0 < |k| < \beta.$$

$$\psi = \sum_k (\alpha_k^+ F_k^+ + \alpha_k^- F_k^-) + \psi_0, \quad \psi_0 = O(z).$$

$$i(I+U)\alpha^- + (I-U)\alpha^+ = 0.$$

Lagrangian manifold.
$\Lambda_0 : p \in T^*_{x_1}M, \quad |p| = 2E$, x_1 — antipodal of the conic point.
$\Lambda = \bigcup_t g_t\Lambda_0$, g_t — geodesic flow.
$\Lambda \cong T^2$.
γ is closed geodesic.


$S^2_{\sqrt{2E}}$
p

Large harmonics. Fix integer l, $l \geq \beta$.

Theorem (A.S.)

Let E be solution of the equation

$$\frac{1}{2\pi h}\int_{\gamma}\theta = \frac{l+\beta(l+1)}{2\beta} + m, \quad m \in \mathbb{Z}, \quad m = O(\frac{1}{h}),$$

$$\theta = (p, dx).$$

Then there exist an eigenvalue $\lambda = E + o(h)$.

Small harmonics. U does not depend on h.

Theorem (A.S.)

Let E be solution of the equation

$$\frac{1}{2\pi h}\int_\gamma \theta = \frac{|k| + \beta(|k|+1)}{2\beta} + m_k \in \mathbb{Z}, \quad |k| \leq \beta; \quad k \neq 0,$$

or

$$\frac{1}{2\pi h}\int_\gamma \theta + \frac{1}{2} = m_0 \in \mathbb{Z};$$

Then there exist an eigenvalue $\lambda = E + o(h)$.

- If $\beta < 1$ we have standard Bohr-Sommerfeld equation on Λ.
- Explicit formulae

$$E_k = \frac{4\pi^2 h^2}{L^2}(m_k - \frac{|k| + \beta(|k|+1)}{2\beta})^2, \quad k \neq 0,$$

$$E^{(0)} = \frac{4\pi^2 h^2}{L^2}(m_0 - \frac{1}{2})^2.$$

U depends on h. $B^{\pm}$ - diagonal matrices with elements

$$b_k^+ = \frac{\sqrt{\nu}\sin(\frac{\Phi}{2h} - \pi\frac{\nu+|k|}{2})}{(2h)^{\nu}\Gamma(\nu+1)},$$

$$b_k^- = -\frac{\sqrt{\nu}}{\pi}\cos(\frac{\Phi}{2h} - \pi\frac{\nu+|k|}{2})(2h)^{\nu}\Gamma(\nu)$$

$$b_0^+ = \sin\frac{\Phi}{2h} - \frac{2}{\pi}(\log\frac{\sqrt{2E}}{h} + c)\cos\frac{\Phi}{2h},$$

$$b_0^- = -\frac{2}{\pi}\cos\frac{\Phi}{2h}, \quad \nu = \frac{|k|}{\beta},$$

$\Phi = \int_{\gamma}(p, dx) = L\sqrt{E}$, c — Euler constant, $k = -[\beta], \dots [\beta]$.

$$W(E,h) = \det(i(I+U)B^- + (I-U)B^+) = 0.$$

Theorem

Let E be bounded solution of the equation

$$W(E,h)=0$$

Then there exist an eigenvalue $\lambda = E + o(h)$.

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$$ih\frac{\partial u}{\partial t} = -\frac{h^2}{2}\Delta u + V(x)u + q(x)\delta_M u, \quad x \in \mathbb{R}^n,$$

$$u|_{t=0} = \varphi^0 e^{\frac{iS_0}{h}}$$

M is a smooth oriented hypersurface, S_0 is real. Boundary conditions on M:

$$u_-|_M = u_+|_M, \quad h\frac{\partial u}{\partial m_-}|_M - h\frac{\partial u}{\partial m_+}|_M = qu|_M$$

Extended phase space $\mathbb{R}^{2n+2}_{(x,t,p,p_0)}$. Isotropic surface Λ_0: $t=0, p=\frac{\partial S_0}{\partial x}, H=0$, $H=p_0-\frac{1}{2}|p|^2-V(x)$, Lagrangian manifold $\Lambda^+=\bigcup_s g_s\Lambda_0$.
Hypersurface $\hat{M}\subset\mathbb{R}^{2n+2}$, $x\in M$. $N^+=\Lambda\bigcap\hat{M}$. For $x\in M$ let p_τ denote the projection of p to T_xM, p_n – normal component.
Map $Q:\hat{M}\to\hat{M}$, $Q(x,t,p_\tau,p_n,p_0)=(x,t,p_\tau,-p_n,p_0)$,
$N^-=Q(N^+)$. Reflected Lagrangian manifold $\Lambda^-=\bigcup_s g_sN^-$.

Volume form. On Λ_0 we have $\sigma_0 = dx$, construct invariant form on Λ^+: $\sigma^+(\alpha, s) = g_s^* \sigma_0 \wedge ds$. On N^+ consider $i_{p_n}\sigma^+$, map it to N^- and construct invariant form σ^-.

Consider formal series

$$u = K_{\Lambda^+}(\sum_{k=0}^{\infty} h^k \varphi_k^+) + K_{\Lambda^-}(\sum_{k=0}^{\infty} h^k \varphi_k^-)$$

on the negative side of M,

$$u = K_{\Lambda^-}(\sum_{k=0}^{\infty} h^k \varphi_k^*)$$

on the positive side.

$$\varphi_0^*|_{N^+} = \frac{2ip_n}{2ip_n + q}\varphi_0^+|_{N^+}, \quad \varphi_0^-|_{N^-} = \frac{-q}{q + 2ip_n}\varphi_0^+|_{N^+}$$

Theorem (Olga Shchegortsova, A.S.)

This series is asymptotic for the solution of the Cauchy problem for $t \in [0, T]$.

Remark

$$\tau = \frac{2ip_n}{2ip_n + q}, \quad r = \frac{-q}{q + 2ip_n}$$

are the analogs of the coefficients of transmission and reflection.

Reflection of vector bundles
Rules of reflection

The fibers are positive complex Lagrangian planes – quadratic forms on $T_P\mathbb{R}^n$. On T_PM it is shifted by $p_n b$, where b is the second fundamental form of M, on the pair (m, ξ) — by the value $p_n\partial_\xi(V)$, on the pair (m, m) – by $p_n^2\partial_m(V)$.

THANK YOU
FOR YOUR ATTENTION!