Докл. Акад. Наук СССР
Том 268 (1983), № 1

Soviet Math. Dokl.
Vol. 27 (1983), No. 1

THE PRINCIPLE OF THROWING OUT CYCLES IN MORSE-NOVIKOV THEORY

UDC 513.835

I. A. TAĬMANOV

Let $\Omega^+(M^n)$ denote the space of closed parametrized curves on a connected manifold M^n, and let $\Omega_0^+(M^n)$ be its connected component of null-homotopic curves. By points of $\Omega^+(M^n)$ we understand continuous maps of the interval c: $[0, 1] \to M^n$, where $c(0) = c(1)$. Assigning to each curve in $\Omega^+(M^n)$ its base point $c(0)$, we obtain a fibration $\Omega^+(M^n) \overset{\pi}{\to} M^n$, where the fiber is the loop space on M^n beginning and ending at $c(0)$. There is a natural action of S^1 on $\Omega^+(M^n)$. Let $P_0^+(M^n)$ denote the quotient of $\Omega_0^+(M^n)$ by the action of S^1.

The objects of study of present-day theory, not covered by the classical Morse theory, are functionals which locally (that is for curves contained in any domain with trivial second homology group $U_\alpha \subset M^n$) have the form (see [1]–[3])

$$l_\alpha(c) = \int_c F_\alpha(x, \dot{x})\, dt, \tag{1}$$

where $(F_\alpha - F_\beta)\, dt = A_i^{(\alpha\beta)}\, dx^j = \omega^{(\alpha\beta)}$ in the intersection of domains $U_\alpha \cap U_\beta$, and $d\omega^{(\alpha\beta)} = 0$, $c \in \Omega_0^+(M^n)$ and $F_\alpha(x, k\dot{x}) = kF_\alpha(x, \dot{x})$, $k > 0$. We assume that the condition

$$\frac{\partial^2 F_\alpha}{\partial \dot{x}^i \partial \dot{x}^j}(x, v) > 0$$

is satisfied for $v \neq 0 \in TM_x^n$. Unlike standard Morse theory, we do not assume that the functional is globally positive and single valued. The functional naturally induces one on $P_0^+(M^n)$.

BASIC EXAMPLE (the Maupertuis functional for a charged particle in a magnetic field or for a solid body moving around a fixed point under the action of gyroscopic forces, when $M^3 = SO_3$).

$$l_\alpha^E(c) = \int_c \left(\sqrt{2Emg_{ij}\dot{x}^i\dot{x}^j} + A_i^{(\alpha)}\dot{x}^i \right) dt, \qquad E = \text{const}, \tag{2}$$

where g_{ij} is a complete riemannian metric on M^n and $\wedge A_i^{(\alpha)}$ is the vector potential of a magnetic field H_{ij} in a domain U_α, where $H_{ij}\, dx^i \wedge dx^j$ is exact, $dH = 0$ and, possibly, $[H] \neq 0$ in $H^2(M^n; R)$. If a functional l on $\Omega_0^+(M^n)$ is single-valued, nonnegative, and vanishes only on the manifold of one-point curves $M^n \subset \Omega_0^+(M^n)$, we obtain the usual "length" in Finsler geometry, where all tangent spaces are Banach spaces. This is the case studied by classical Morse theory. We do not consider it here.

There are 3 classes of examples of Novikov's theory, not reducible to classical Morse theory.

1980 *Mathematics Subject Classification*. Primary 57R70; Secondary 55P35.

1) The functionals may be "multi-valued", i.e. defining a closed 1-form $\delta l = \delta l_\alpha$ on the space $\Omega_0^+(M^n)$ (see [1]–[3]):

$$l_\alpha(c) = \int_c F(x, \dot{x})\, dt + \int_{\Pi_\alpha} H_{ij}\, dx^i \wedge dx^j, \tag{3}$$

where $[H] \neq 0$ in $H^2(M^n; R)$ and Π_α is a "membrane" pulled onto the contour c. Locally it has the form (1), multi-valuedness is connected with the choice of different equivalence classes of "membranes" Π_α, corresponding to elements of $H_2(M^n)/\text{Torsion}$ (see [3]). The analogue of Theorem 1, i.e. the "principle of throwing out of cycles" for essentially "many valued" functionals, was first applied in [1] and [3]. In this case it was established fairly simply.

2) If in (3) $[H] \neq 0$ in $H^2(M^n; R)$ but $p^*H \sim 0$, where $\hat{M}^n \xrightarrow{p} M^n$ is the universal covering, we obtain a single valued functional on $\Omega_0^+(M^n)$, which on $\Omega_0^+(\hat{M}^n)$ takes the form (1) globally. In this case, in fact, $[H] \in H^2(\Pi_1(M); R)$. For some groups Π_1, for instance for free abelian ones, the functional l will not be positive for all values of the energy $0 < E < \infty$. This depends on homologico-algebraic properties of the pair $([H], \Pi_1)$ (see [1]).

3) If $[H] = 0$ in the homology of a compact manifold M^n, then there always exists a limiting energy E_0, $0 < E_0 < \infty$, such that the functional l is strictly positive for $E > E_0$ and is not always positive for $0 < E < E_0$.

Let W_l denote the set of curves in $\Omega_0^+(M^n)$, where $l < 0$. In view of the results of [1]–[3] we shall consider only single-valued functionals, where $W_l \neq \varnothing$. Let W_l^p denote W_l/S^1.

Recall that a periodic problem is said to be nondegenerate if the periodic extremals, with the exception of trivial ones, are nondegenerate. Nondegenerate cases are the cases of general position.

DEFINITION. We say that a set $U \subset M^n$ *admits "throwing out"* in W_l if there exists a homotopy φ: $U \times [0, 1] \to \Omega_0^+(M^n)$, where $\varphi|_{U \times 1}$: $U \to \Omega_0^+(M^n)$ is the inclusion of single-point curves and $\varphi(U \times 0) \subset W_l$.

This means that there exists a section $\varphi_1 = \varphi(U, 1)$ of the fibration $\Omega_0^+(M^n) \xrightarrow{\pi} M^n$, lying in W_l and homotopic to the one-point curve section. A homotopy that is actually constructed will be fiber preserving. In this case we speak of fiberwise "throwing out".

THEOREM 1. *If a functional l is single valued on $\Omega_0^+(M^n)$, where M^n is a closed connected manifold and $W_l \neq \varnothing$, then the whole manifold $U = M^n$ admits a fiberwise "throwing out" in W_l.*

COROLLARY 1. *If the periodic problem is nondegenerate, then, under the hypothesis of Theorem 1, we have the "Morse-Novikov inequalities" $R_k(l) \geq B_{k-1}(M^n)$, where $R_k(l)$ is the number of extremals of index k, where $l > 0$ and $B_{k-1}(M^n) = \dim H_{k-1}(M^n; R)$.*

THEOREM 2. *If l is a single valued and not everywhere positive (i.e. $W_l \neq \varnothing$) functional on $\Omega_0^+(M^n)$, where M^n is closed and connected, the periodic problem is nondegenerate and there exists a nonzero odd dimensional real cycle on M^n, then on M^n there exist at least two geometrically distinct nontrivial extremals on which $l > 0$.*

COROLLARY 2. *A solid body moving around a fixed point in a gyroscopic force field has, in the case of general position, at least two periodic trajectories for all energies E, $0 < E < E_0$, less than some limiting E_0, at which the functional becomes positive.*

In the above case $M^3 = SO_3$ and the functional is always single valued but not invariant with respect to time reversal $t \to -t$. It is interesting that here we are able to establish the existence of two periodic motions for a given energy only in the essentially nonpositive case. In situations of classical Morse theory, where the functionals are strictly positive, considerations of [6] are inapplicable due to the noninvertibility of the periodic problem.

PROOF OF THEOREM 1. We triangulate M^n and consider the skeleton filtration $K^0 \subset K^1 \subset \cdots \subset K^n = M^n$. Clearly, K^0 admits "throwing out" in W_l. Suppose that on K^{m-1} we have constructed a "throwing out": $g_t = g|_{K^{m-1}\times t}$: $K^{m-1} \to \Omega_0^+(M^n)$. We realize, in the usual way, an m-dimensional simplex σ^m of K in R^m. Let z_0 be an interior point of σ^m and let the $(m-1)$-dimensional faces be $\sigma_1^{m-1}, \ldots, \sigma_{m+1}^{m-1}$.

We define the following maps:

1) π: $\sigma^m \setminus z_0 \to \partial\sigma^m$, the retraction of σ^m onto the boundary $\partial\sigma^m$ along straight lines in R^m.

2) r: $\sigma^m \to [0, 1]$, defined by the conditions $r(z_0) = 0$ and $r(x) = |xz_0|/|\pi(x)z_0|$, $x \in \sigma^m \setminus z_0$.

3) j_α: $\sigma^m \to \sigma_\alpha^{m-1}$, projection onto a face.

4) $\gamma(\alpha, x)$: $[0, 1] \to \sigma^m$, the path given by the perpendicular from x to σ_α^{m-1}.

5) $\xi(x)$: $[0, 1] \to \sigma^m$, the path given by the interval from x to $\pi(x)$, where $x \in \sigma^m \setminus z_0$.

6) $\xi(x, t)$: $[0, 1] \to \sigma^m$, defined for $t \in [0, 1]$ by the condition $\xi(x, t)(s) = \xi(x)(st)$.

On all these paths we use the natural parameter.

Let σ_a^m denote the set of points in σ^m where $r \leqslant a$. All maps and paths will be considered in M^n. We construct g_0: $\sigma^m \to \Omega_0^+(M^n)$ for $x \in \sigma_{1/3}^m$:

$$g_0(x) = \prod_\alpha \left(\gamma(x, \alpha)^{-1} g_0(j_\alpha(x))\gamma(x, \alpha)\right); \qquad x \in \sigma_{1/2}^m \setminus \sigma_{1/3}^m,$$

$$g_0(x) = \left(\prod_\alpha \left(\gamma(x, \alpha)^{-1} g_0(j_\alpha(x))\gamma(x, \alpha)\right)\right) \xi(x, 6r(x) - 2)^{-1} \xi(x, 6r(x) - 2);$$
$$x \in \sigma_{2/3}^m \setminus \sigma_{1/2}^m,$$

$$g_0(x) = \left(\prod_\alpha \left(\gamma(x, \alpha)^{-1} g_0(j_\alpha(x))\gamma(x, \alpha)\right)\right) \xi(x)^{-1} g_{4-6r(x)}(\pi(x))\xi(x);$$
$$x \in \sigma_{3/4}^m \setminus \sigma_{2/3}^m,$$

$$g_0(x) = \left(\prod_\alpha \left(\gamma(x, \alpha)^{-1} g_{12r(x)-8}(j_\alpha(x))\gamma(x, \alpha)\right)\right) \xi(x)^{-1} g_0(\pi(x))\xi(x);$$

on $\sigma_{4/5}^m \setminus \sigma_{3/4}^m$ we contract the loop $\gamma(x, \alpha)^{-1}\gamma(x, \alpha)$ to x; $x \in \sigma^m \setminus \sigma_{4/5}^m$, $g_0(x) = \xi(x)^{-1} g_0(\pi(x))\xi(x)$. It is easy to show that $g_0(K^m) \subset W_l$ if $L + T_0 + T_1 < 0$, where

$$L = \max_{x \in K^m, t \in [0,1]} \left(\sum_\alpha l\left(\gamma(x, \alpha)^{-1}\gamma(x, \alpha)\right) + l\left(\xi(x, t)^{-1}\xi(x, t)\right)\right),$$

$$T_0 = \left(\max_{x \in K^{m-1}, t \in [0,1]} l(g_t(x))\right)(m + 1) > 0, \qquad T_1 = \max_{x \in K^{m-1}} l(g_0(x)) < 0.$$

In order to obtain this we alter the "throwing out" on K^{m-1}. Let h_t: $K^{m-1} \to \Omega_0^+(M^n)$ be given by $h_t = g_{t-(l-1)/N} g_0^{l-1}$, where $t \in [(l-1)/N, l/N]$, and replace g_t by h_t. This alteration keeps L and T_0 the same and diminishes T_1 N times. Clearly, it is possible to choose N so large that $g_0(K^m) \subset W_l$. To complete the proof we observe that it is easy to construct a "throwing out" h: $K^m \times [0, 1] \to \Omega_0^+(M^n)$, where $h|_{K^m \times 0} = h_0$.

For the proof of Corollary 1, observe that M^n is the manifold of local minima in $P_0^+(M^n)$ and hence

$$H_i(M^n \cup W_l^p; R) = H_i(M^n; R) \oplus H_i(W_l^p; R).$$

Let $[u] \in H_i(M^n; R)$. According to Theorem 1, $[u]$ is homologous to some cycle

$$[v] \in H_i(W_l^p; R).$$

Consider the exact homology sequence of the pair $(P_0^+(M^n), M^n \cup W_l^p)$:

$$\to H_{i+1}(P_0^+(M^n), M^n \cup W_l^p; R) \xrightarrow{\partial} H_i(M^n; R) \oplus H_i(W_l^p; R) \xrightarrow{j} H_i(P_0^+(M^n); R).$$

We have $j([u] - [v]) = 0$, and hence

$$\dim H_{i+1}(P_0^+(M^n), M^n \cup W_l^p; R) \geqslant \dim H_i(M^n; R) = B_i(M^n).$$

From Morse theory it follows that

$$R_k(l) \geqslant \dim H_k(P_0^+(M^n), M^n \cup W_l^p; R).$$

This completes the proof of Corollary 1.

According to a theorem of Bott [4], for each simple null-homotopic closed geodesic there exists an integer valued function Λ on the set of roots of unity in C such that the index of the given k-fold geodesic is equal to the sum of the values of Λ on kth roots of unity and $\Lambda(z) = \Lambda(\bar{z})$. According to a theorem of Shvarts [5], if $\lambda(g^2) - \lambda(g^1)$ is odd (λ —index of extremal), then the type numbers of even-fold extremals g^{2s} in $P_0^+(M^n)$, with the field of coefficients R, are equal to zero, and in the remaining cases the type numbers are determined by the index in the usual (as in the space of loops) way. Beginning with this, Fet [6] found that the dimensions of nonzero type numbers of multiple extremals have the same parity. Comparing Fet's result with those obtained above, we obtain Theorem 2. Corollary 2 follows from it easily

Moscow State University

Received 31/MAY/82

BIBLIOGRAPHY

1. S. P. Novikov, Dokl. Akad. Nauk SSSR **260** (1981), 31–35; English transl. in Soviet Math. Dokl. **24** (1981).

2. S. P. Novikov and I. Shmel'tser, Funktsional. Anal. i Prilozhen. **15** (1981), no. 3, 54–66; English transl. in Functional Anal. Appl. **15** (1981).

3. S. P. Novikov, Funktsional. Anal. i Prilozhen. **15** (1981), no. 4, 37–52; English transl. in Functional Anal. Appl. **15** (1981).

4. Raoul Bott, Comm. Pure Appl. Math. **9** (1956), 171–206.

5. A. S. Shvarts, Trudy Moskov. Mat. O'bshch. **9** (1960), 3–44. (Russian)

6. A. I. Fet, Dokl. Akad. Nauk SSS' **160** (1965), 287–289; English transl. in Soviet Math. Dokl. **6** (1965).

Translated by A. KOZLOWSKI