

PERIODIC EXTREMALS OF MANY-VALUED OR NOT-EVERYWHERE-POSITIVE FUNCTIONALS

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I. We consider (possibly many-valued) functionals of the type of length in Finsler metric, which in local coordinates x_α^j have the form

$$(1) \quad l^{(\alpha)}(\gamma) = \oint F(x, \dot{x}) dt + \oint_\gamma A_j^{(\alpha)}(x) dx_\alpha^j.$$

Here it is assumed that $F(x, \lambda \dot{x}) = \lambda F(x, \dot{x})$, for $\lambda > 0$, $F > 0$, $\partial^2 F / \partial \dot{x}^i \partial \dot{x}^j$ is a positive form, and the expression $\Omega_{ij} = \partial_i A_j^{(\alpha)} - \partial_j A_i^{(\alpha)}$ (magnetic field) is uniquely defined on the whole manifold M^n as a closed but possibly nonexact 2-form. The first summand is defined as a positive Finsler metric, but the second may make the functional nonpositive or many-valued. This note is a continuation of [1]–[5].

REMARK. This note contains, in particular, corrections of certain errors in [2]–[4]. It is necessary to remove the term “nonselfintersecting” from all statements except those in §II of this article. Moreover, in §5 of [3] there is an assertion in which the term “flat section” should be replaced by the term “ T -invariant curve”. These curves, however, can be considered nonselfintersecting only for functionals invariant under reversal of direction.

II. Two-dimensional problems. Functionals of the form (1) occur, for example, as Maupertuis-Fermat functionals for a charged particle in an external magnetic field or (if $M^n = SO_3$) for the motion of a solid body under the action of gyroscopic forces. These functionals, as was shown in [2], occur on the two-dimensional sphere S^2 as the result of a reduction of the Hamiltonian formalism of the Kirchhoff-Thompson equations for the motion of a solid in a fluid or of a body around a fixed point, with the total flux of the “effective magnetic field” Ω_{12} through the sphere S^2 proportional to the area constant. Let $\hat{\Omega}^+$ denote the space of directed nonselfintersecting (piecewise smooth) curves. On this space a functional of the form (1) is single-valued. We normalize it so that it takes the value zero on one-point curves. In the many-valued case the functional is normalized so that it takes a positive value on the second component of the set of one-point curves, which is “split into two” by the construction of this single-valued branch of the functional (see [3], the beginning of §2).

THEOREM 1. *If on S^2 there exists at least one nonselfintersecting closed curve γ such that $l(\gamma) < 0$, then there exists a nonselfintersecting periodic extremal γ_0 such that $l(\gamma_0) < 0$.*

REMARK. This extremal may not be a minimum of the functional among nonselfintersecting curves. This is the point of an error in §1 of [3].

Theorem 1 follows from the following lemmas.

LEMMA 1. *If a minimum $\bar{\gamma}$ of the functional l lying in the closure of the set of nonselfintersecting curves is not a periodic extremal, then there exists a closed curve $\bar{\bar{\gamma}}$*

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consisting of a single nonselfintersecting closed piece of an extremal containing one turning point (zero angle) with $l(\bar{\gamma}) < 0$.

LEMMA 2. Given a functional of the form (1) on the plane and a convex extremal polygon $\bar{\gamma}$ with all the angles between the links not greater than π (the zero angle is allowed) and $l(\bar{\gamma}) < 0$, there exists within its interior a smooth nonselfintersecting periodic extremal γ such that $l(\gamma) < 0$.

Theorem 2 follows from the fact that, due to the convexity, the shortening deformations lead into the interior of $\bar{\gamma}$.

THEOREM 2. Suppose that on the torus T^2 there is given a functional of the form (1), where the total flux of the field Ω_{12} through T^2 (or a chamber in the plane R^2) is nonzero. Then in the general case there exist at least 4 nondegenerate periodic extremals on R^2 (up to a translation by a vector in the lattice $\mathbf{Z} + \mathbf{Z}$) with index 1, 2, 2, 3 for which the Whitney number is zero and $l > 0$.

COROLLARY 1. All these extremals are geometrically distinct, since the Whitney number of an iterated curve is nonzero.

The proof follows [3], §4.III (see the footnote), starting from the "principle of throwing out cycles", in this case first formulated in [3]; for its generalization see [5]. However, in the process of deformation "downwards" there appear selfintersecting curves, and only the Whitney number remains invariant.

III. We turn now to arbitrary dimensions.

LEMMA 3. If a functional of the form (1) is single-valued and positive on the space of curves nullhomotopic in $\Omega_1^+(M^n)$, then it is semibounded and single-valued on all spaces of curves $\Omega_g^+(M^n)$ of arbitrary homotopy classes g (with a fixed starting point).

If the functional is many-valued on some space $\Omega_g^+(M^n)$, $g \neq 1$, then in Ω_g^+ the group π_1 is nontrivial. A map $S^1 \xrightarrow{f} \Omega_g^+$ defines a map $T^2 \xrightarrow{f} M^n$, $f_*\pi_1(T^2) \neq 0$, such that the flux of the magnetic field Ω_{ij} through the cycle $f(T^2)$ is nonzero. We obtain a map of coverings $R^2 \xrightarrow{f} \hat{M}^n$. The image $\bar{\gamma}_R = f(\gamma_R)$ of a circle of radius R in R^2 as $R \rightarrow \infty$ is a curve for which $l(\bar{\gamma}_R) < 0$. From this Lemma 3 follows easily. Hence we obtain

THEOREM 3. On an arbitrary non-simply-connected manifold (closed) the functional (1) possesses at least one periodic extremal.

For the simply-connected case the theorem was obtained in [1].

CONDITION. a) $H_{2k+1}(M^n; R) \neq 0$ for some k .

b) For some g , $g \neq 1$, the number of generators of $\pi_1(\Omega_g^+(M^n))$ is not less than two and the class g^{2m+1} does not coincide with g^{2n} for any n , $m \geq 0$.

THEOREM 4. If the condition holds, then there exist at least two nondegenerate extremals of the functional (1) in general position on a closed manifold, which extremals are geometrically distinct and homotopic to each other.

The existence of at least two generators in $\pi_1(\Omega_g^+)$ gives the existence of two extremals in Ω_g^+ of index 0 and 1, if the functional is single-valued and semibounded on Ω_1^+ . Otherwise, the functional is nonpositive on Ω_1^+ , where, therefore, there exist geometrically distinct extremals of index 1 and $2k + 2$ [5]. The claim that the extremals in Ω_g^- are geometrically distinct is proved using Bott's theorem [6].

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