Closed geodesics on non-simply-connected manifolds

I.A. Taimanov

We consider the Hilbert manifold ΛM consisting of the H^1-maps of the parametrized circle S^1 into a closed Riemannian manifold M^n [1]. Let $h \in \pi_1(M^n, x_0)$, $[h]$ be the corresponding free homotopy class of closed paths, $\Lambda M([h])$ the connected component of ΛM containing the curves in the class $[h]$, and h_i the automorphism of $\pi_i(M^n, x_0)$ corresponding to the element h under the standard action of π_1 on π_i.

Any point $c \in \Lambda M$ can be realized by a map $\overline{c}: [0, 1] \to M^n$, where $\overline{c}(0) = \overline{c}(1)$. We define a map $p: \Lambda M \to M^n$ by $p(c) = \overline{c}(0)$, that is, we assign distinguished points to closed parameterized curves. This is the Serre fibration, where a fibre is a space of closed curves with a fixed distinguished point. Let $p(c) = x_0$ and let c realize the element $h \in \pi_1(M^n, x_0)$. We restrict the fibration to $\Lambda M([h])$ and consider an exact homotopy sequence. Since $\pi_i(p^{-1}(x_0), c) = \pi_{i+1}(M^n, x_0)$ this takes the form

$$\to \pi_i(\Lambda M([h]), c) \xrightarrow{p_i} \pi_i(M^n, x_0) \xrightarrow{f_i} \pi_i(M^n, x_0) \to .$$

Theorem 1. 1) $p_i(\pi_i(\Lambda M([h]), c)) = \mathrm{St}(h_i)$, *where* $\mathrm{St}(h_i)$ *is the subgroup of* $\pi_i(M^n, x_0)$ *consisting of the elements fixed under* h_i.

2) $f_i = I - h_i$, $i \geqslant 2$, *where* I *is the identity automorphism.*

(The original study of $\pi_i(\Lambda M([h]))$ was carried out by the author by direct geometrical constructions; D.V. Anosov drew his attention to the inclusion of these in the fibration structure and the consequent form of f_i.)

Theorem 2. *Suppose that all closed geodesics on a closed manifold are non-singular and that there is an* $h \in \pi_1(M^n, x_0)$ *such that for an infinite sequence of natural numbers* k_j ($j = 1, 2, \ldots$) *all the classes* $[h^{k_j}]$ *are distinct. If at least one of the following conditions holds:* 1) $(I - h_N^{k_j})$ *are not automorphisms for all* j, *for some* $N \geqslant 2$; 2) $\mathrm{St}(h_1) \neq \mathbf{Z}$, *then* M^n *has infinitely many geometrically distinct closed geodesics.*

Proof. By Λ_j and Λ_j^a we denote $\Lambda M([h^{k_j}])$ and its subspace consisting of curves of length at most a. Each Λ_j has a minimal extremal of index 0 and length r_j. We remark that $\Lambda_j^{r_j}$ is homeomorphic to a circle. If $\mathrm{St}(h_1) \neq \mathbf{Z}$ or $\pi_2(M^n)/(I - h_2^{k_j})\pi_2(M^n) \neq 0$, then $\pi_1(\Lambda_j)$ has at least two generators. If $\mathrm{Ker}(I - h_i^{k_j}) \neq 0$ or $\pi_{i+1}(M^n)/(I - h_{i+1}^{k_j})\pi_{i+1}(M^n) \neq 0$ for $i \geqslant 2$, then $\pi_i(\Lambda_j) \neq 0$. Considering by means of Morse theory the rearrangements of the Λ_j^a and their homotopy groups as a passes through a critical level of the length functional, we find that under the conditions of the theorem each Λ_j contains an extremal of index λ, where $1 \leqslant \lambda \leqslant N$. The following fact is standard: let g be a closed geodesic and $\mathrm{ind}\, g^m > 0$ for some $m \geqslant 1$ (ind is the Morse index of an extremal). Then there are an $\alpha > 0$ and a $\beta \geqslant 0$ such that $\mathrm{ind}\, g^{l+1} \geqslant \mathrm{ind}\, g + \alpha l + \beta$ for all integers $l \geqslant 1$ ([1], [2]). Hence it follows that the resulting infinite set of closed geodesics of positive bounded index cannot be formed by multiples of finitely many simple ones, which establishes the theorem.

Using the fact that if A is an automorphism of a finitely generated Abelian group, then for infinitely many natural numbers k_j ($j = 1, 2, \ldots$) the homomorphisms $I - A^{k_j}$ (I is the identity automorphism) are not automorphisms, when all the $[h^n]$, $n \geqslant 1$, are distinct, one can simplify the statement of Theorem 2. In particular, we have the following theorem.

Theorem 3. *Suppose that all the closed geodesics on a closed manifold* M^n *are non-singular and that there is an* $h \in \pi_1(M^n, x_0)$ *for which all the* $[h^m]$, $m \geqslant 1$, *are distinct. If* $\pi_k(M^n)$ *is finitely generated for some* $k \geqslant 2$, *and* $\pi_k(M^n) \neq 0$, *then* M^n *has infinitely many geometrically distinct closed geodesics.*

When $\pi_1(M^n) = \mathbf{Z}$, Theorem 3 gives an answer to the question, posed in [3] and [4], about an estimate of the number of closed geodesics on such manifolds when there are non-trivial finitely generated higher homotopy groups.

The author is grateful to his supervisor S.P. Novikov, to D.V. Anosov, and to I.K. Babenko, for useful discussions.

After this note was sent to the printers D.V. Anosov informed the author in December 1984 of the appearance of a paper by V. Bangert and N. Hingston in the Journal of Differential Geometry (1984), 277-282, whose results overlap those of this note.

References

[1] W. Klingenberg, Lectures on closed geodesics, Springer-Verlag, Berlin-Heidelberg-New York 1978. MR 57 # 17563.
Translation: *Lektsii o zamknutykh geodezicheskikh*, Mir, Moscow 1982.

[2] D. Gromoll and W. Meyer, Periodic geodesics on compact Riemannian manifolds, J. Differential Geometry 3 (1969), 493-510. MR 41 # 9143.

[3] D. Sullivan, Infinitesimal computations in topology, IHES Publ. Math. 47 (1977), 269-332. MR 58 # 31119.

[4] W. Ballmann, G. Thorbergsson, and W. Ziller, Closed geodesics on the fundamental group, Duke Math. J. 48 (1981), 585-588. MR 82j:58041.

Moscow State University

Received by the Board of Governors 29 October 1984