Докл. Акад. Наук СССР
Том 285 (1985), № 5

Soviet Math. Dokl.
Vol. 32 (1985), No. 3

EFFECTIVIZATION OF THETA FUNCTION FORMULAS FOR TWO-DIMENSIONAL POTENTIAL SCHRÖDINGER OPERATORS THAT ARE FINITE-ZONE AT ONE ENERGY LEVEL

UDC 513.835

I. A. TAĬMANOV

In [1]–[3] it was shown that finite-zone, two-dimensional potential Schrödinger operators are given in the form

$$(1)\qquad L = \partial\bar\partial + v(x,y), \qquad v(x,y) = 2\partial\bar\partial \ln\theta(Uz + V\bar z + \gamma_0) + C_0$$

where $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar\partial = \frac{1}{2}(\partial_x + i\partial_y)$, $z = x + iy$, $C_0 =$ const, and θ is the theta function corresponding to the Prymian of a Riemann surface Γ with involution σ having two fixed points, P_1 and P_2. In those notes a deformation was described of potential operators which is given by an equation of the type of an L-A-B-triple [4]:

$$(2)\qquad \begin{aligned} &dL/dt + [L, A + \overline{A}] = fL, \qquad L = \partial\bar\partial + v(x,y,t),\\ &A = \partial^3 + u\partial, \quad \overline{A} = \bar\partial^3 + w\bar\partial, \quad f = \partial u + \bar\partial w,\\ &3\partial v = \bar\partial u, \qquad 3\bar\partial v = \partial w,\\ &v_t = \partial^3 v + \bar\partial^3 v + \partial(uv) + \bar\partial(wv). \end{aligned}$$

In the present note, using an analogue of the method of [5], we carry out a partial effectivization of formula (1) with application of the nonlinear equation (2): in the case of general position the constant C_0 is recovered on the basis of the Riemann matrix giving the theta function on a g-dimensional Prymian and linearly independent vectors $U, V \in \mathbf{C}^g$. For $g = 2$ the effectivization is complete (the genus of Γ is equal to 4).

The Riemann theta function with characteristics $n \in \frac{1}{2}(\mathbf{Z}_2)^g$ is given by its Fourier series

$$\theta[n,0](z) = \sum_{N\in\mathbf{Z}^g} \exp\left\{\frac{1}{2}\langle B(N+n), N+n\rangle + \langle z, N+n\rangle\right\},$$
$$\theta(z|B) = \theta[0,0](z|B) = \theta(z),$$

where B is the Riemann matrix [5].

We define $\hat\theta[n](z) = \theta[n,0](z|2B)$. The values of the derivatives of $\hat\theta[n](z)$ at $z = 0$ we call the *theta constants*:

$$\hat\theta_{ij\cdots}[n] = \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\cdots\hat\theta[n](0).$$

From the theta constants we form a matrix $(\hat\theta_{ij}[n], \hat\theta[n])$ where theta constants corresponding to fixed characteristics n indexing the rows are situated along the rows, and $1 \le i, j \le g$.

DEFINITION. A Riemann matrix is called *nondegenerate* if the rank of the matrix $(\hat\theta_{ij}[n], \hat\theta[n])$ is maximal, i.e.,

$$\operatorname{rank}(\hat\theta_{ij}[n], \hat\theta[n]) = \frac{g(g+1)}{2} + 1.$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35J10, 14K25.

©1986 American Mathematical Society
0197-6788/86 \$1.00 + \$.25 per page

REMARK. If a Riemann matrix is nondegenerate, then it is indecomposable, i.e., the Abelian torus corresponding to it does not decompose into a direct product of Abelian tori of smaller dimension. The converse assertion is stated in [6]: if the Riemann matrix is indecomposable, then it is nondegenerate.

If the Riemann matrix is indecomposable, there always exists a collection of characteristics $n_1, \dots, n_r$ such that a minor of maximal rank $(\hat{\theta}_{ij}[n_k], \theta[n_k])$ is invertible, $1 \le k \le r = g(g+1)/2 + 1$. For an arbitrary vector U we denote by ∂_u the directional derivative: $\partial_U = \sum_i U_i \partial/\partial z_i$. For a nondegenerate Riemann matrix and the collection of characteristics corresponding to it as indicated above we consider the uniquely solvable system of equations for Q_{ij} and Q

$$\sum_{i,j} Q_{ij}\hat{\theta}_{ij}[n_k] + Q\hat{\theta}[n_k] = 4\partial_U^3 \partial_V \hat{\theta}[n_k]. \tag{3}$$

Q_{ij} and Q are then uniquely determined on the basis of U and V on whose components they depend in polynomial fashion. Thus, we assign to each nondegenerate Riemann matrix the collection of polynomials $Q_{ij}(U,V)$, $Q(U,V)$.

THEOREM 1. *If a Riemann matrix of order $g \times g$ is nondegenerate and the vectors $U, V \in \mathbf{C}^g$ are linearly independent, then the systems of equations*

$$(*)\qquad \begin{aligned} &V_1 W_1^+ - 3C_0 U_1^2 = Q_{11}(U,V), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &V_j W_j^+ - 3C_0 U_j^2 = Q_{jj}(U,V), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &V_i W_j^+ + V_j W_i^+ - 6C_0 U_i U_j = Q_{ij}(U,V), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &1 \le i \ne j \le g; \end{aligned}$$

$$(**)\qquad \begin{aligned} &U_1 W_1^- - 3C_0 V_1^2 = Q_{11}(V,U), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &U_j W_j^- - 3C_0 V_j^2 = Q_{jj}(V,U), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &U_1 W_j^- + U_j W_i^- - 6C_0 V_i V_j = Q_{ij}(V,U), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &1 \le i \ne j \le g, \end{aligned}$$

for the unknowns W_k^+, W_l^-, and C_0 are uniquely solvable, and the function

$$v(x,y,t) = 2\partial\bar{\partial} \ln \theta(Uz + V\bar{z} + (W^+ + W^-)t + \gamma_0) + C_0, \tag{4}$$

where $v(x,y,0)$ is a finite-zone potential of the form (1), *gives finite-zone solutions of equation* (2). *Other solutions are obtained by the transformations*

$$\begin{aligned} &U \to \lambda U, \quad V \to \mu V, \quad W^+ \to \Lambda^3 W^+ + \lambda^2 \alpha U, \\ &W^- \to \mu^3 W^- + \mu^2 \beta V, \qquad C_0 \to \lambda\mu C_0, \end{aligned}$$

where $\alpha, \beta \in \mathbf{C}$ and $\lambda, \mu \in \mathbf{C}\backslash\{0\}$, and they have the form (4). *If $g = 2$, then the linearly independent vectors U and V in* (4) *can be chosen arbitrarily.*

COROLLARY. *Let $v(x,y)$ be a finite-zone potential of the form* (1). *If the Riemann matrix B is nondegenerate and the vectors $U, V \in \mathbf{C}^g$ are linearly independent, then the constant $C_0 = C_0(U,V,B)$ is uniquely determined by them. In particular, the operator*

$$L_1 = \partial\bar{\partial} + 2\partial\bar{\partial} \ln \theta(Uz + V\bar{z} + \gamma_0)$$

is finite-zone relative to one energy level $\varepsilon = -C_0(U,V,B)$. *For* $g = 2$ *we obtain a procedure for constructing finite-zone potentials on the basis of a nondegenerate Riemann matrix and an arbitrary pair of linearly independent vectors* $U, V \in \mathbf{C}^2$.

PROOF. According to [1]–[3], deformations of the two-point Baker-Akhiezer function on a Riemann surface Γ of finite genus, admitting an involution with fixed points P_1 and P_2, with essential singularities at P_1 and P_2 given by deformations of their asymptotics

$$\psi(x,y,t_1,t_2) \sim \exp(k_1 z + k_1^3 t_1) \quad \text{at } P_1,$$
$$\psi(x,y,t_1,t_2) \sim \pm\exp(k_1 \bar{z} + k_2^3 t_2) \quad \text{at } P_2$$

($1/k_i$ is the local parameter in a neighborhood of $P_i, i = 1,2$), generate deformations of the Schrödinger operators preserving the class of potential operators described in [1]–[3], for which they have the form

$$\begin{aligned} \partial v/\partial t_1 &= \partial^3 v + \partial(uv), \qquad 3\partial v = \bar{\partial} u; \\ \partial v/\partial t_2 &= \bar{\partial}^3 v + \bar{\partial}(wv), \qquad 3\bar{\partial} v = \partial w. \end{aligned} \tag{5}$$

The finite-zone solutions have the form

$$v(x,y,t_1,t_2) = 2\partial\bar{\partial}\ln\theta(Uz + V\bar{z} + W^+t_1 + W^-t_2 + \gamma_0) + C_0.$$

Equation (2) describes a consistent deformation $t = t_1 = t_2$ preserving the class of real operators (in this case the condition $u = \bar{w}$ is added in (2)). To describe finite-zone solutions of (2) it therefore suffices to describe finite-zone solutions of (5). Functions u and w with the same character of quasiperiodicity in x and y as v can be recovered up to constants d_1 and d_2:

$$u = 6\partial^2 \ln\theta(Uz + V\bar{z} + W^+t_1 + W^-t_2 + \gamma_0) + d_1,$$
$$w = 6\bar{\partial}^2 \ln\theta(Uz + V\bar{z} + W^+t_1 + W^-t_2 + \gamma_0) + d_2.$$

For indecomposable Riemann matrices a technique for effectivization of theta function formulas was developed in [5]. Using it, we obtain a rather large collection of relations on the theta constants, effectivizing the formulas for the solutions (5):

$$\begin{aligned} (\partial_V\partial_{W^+} - d_1\partial_U\partial_V - 3C_0\partial_U^2 - 4\partial_U^3\partial_V - a_1)\hat{\theta}[n] &= 0, \\ (\partial_U\partial_{W^-} - d_2\partial_U\partial_V - 3C_0\partial_V^2 - 4\partial_V^3\partial_U - a_2)\hat{\theta}[n] &= 0, \end{aligned} \tag{6}$$

where $n \in \frac{1}{2}(\mathbf{Z}_2)^g$. This system is invariant relative to the transformations

$$\begin{aligned} &U \to \lambda U, \quad V \to \mu V, \quad W^+ \to \lambda^3 W^+ + \lambda^2 \alpha U, \\ &W^- \to \mu^3 W^- + \mu^2 \beta V, \qquad d_1 \to \lambda^2 d_1 + \lambda\alpha, \\ &d_2 \to \mu^2 d_2 + \mu\beta, \qquad C_0 \to \lambda\mu C_0, \\ &a_1 \to \lambda^3 \mu a_1, \qquad a_2 \to \mu^3 \lambda a_2, \end{aligned}$$

where $\lambda, \mu \in \mathbf{C}\backslash\{0\}$ and $\alpha, \beta \in \mathbf{C}$. Applying them, we reduce everything to the case $d_1 = d_2 = 0$. Since the systems (6) have the form (3), reducing them to systems of relations on the components U, V, W^+, W^- and the constant C_0 and applying the polynomials Q_{ij} and Q, we obtain the systems indicated in the theorem. From their form it is clear that they are uniquely solvable if and only if the vectors U and V are linearly independent. The theorem is proved.

We note that if the Riemann matrix is real and $U = \overline{V}$, then automatically $W^+ = \overline{W}^-$ and $C_0 \in \mathbf{R}$. This follows explicitly from the formulas. We have thus distinguished a class of real solutions.

THEOREM 2. *If in the hypotheses of Theorem* 1 $U = \overline{V}$ *and the Riemann matrix is real, then the finite-zone solutions constructed are real for* $\gamma_0 \in \mathbf{R}^g$.

The author thanks A. P. Veselov and B. A. Dubrovin for useful discussions.

Moscow State University Received 19/NOV/84

BIBLIOGRAPHY

1. A. P. Veselov and S. P. Novikov, Dokl. Akad. Nauk SSSR **279** (1984), 20–24; English transl. in Soviet Math. Dokl. **30** (1984).

2. ——, Dokl. Akad. Nauk SSSR **279** (1984), 784–788; English transl. in Soviet Math. Dokl. **30** (1984).

3. S. P. Novikov, Uspekhi Mat. Nauk **39** (1984), no. 4 (238), 97–98. (Russian)

4. S. V. Manakov, Uspekhi Mat. Nauk **31** (1976), no. 5 (191), 245–246. (Russian)

5. B. A. Dubrovin, Uspekhi Mat. Nauk **36** (1981), no. 2 (218), 11–80; English transl. in Russian Math. Surveys **36** (1981).

6. Bert van Geemen and Gerard van der Geer, *Kummer varieties and the moduli spaces of Abelian varieties*, Preprint No. 308, Univ. of Utrecht, Utrecht, 1983.

Translated by J. R. SCHULENBERGER