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# ON AN ANALOGUE OF NOVIKOV'S CONJECTURE IN A PROBLEM OF RIEMANN-SCHOTTKY TYPE FOR PRYM VARIETIES

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**1.** We consider two-sheeted branched coverings $\hat{\Gamma} \to \Gamma$ with $2h$, $h > 0$, branch points, where the Riemann surfaces $\hat{\Gamma}$ and $\Gamma$ are nonsingular. A holomorphic involution $\sigma$ acts on $\hat{\Gamma}$ which interchanges the sheets of the covering and has $2h$ fixed points. It generates an involution $\sigma_*\colon J(\hat{\Gamma}) \to J(\hat{\Gamma})$ which acts in a natural way on the Jacobian of the surface $\hat{\Gamma}$. The Prym variety ("Prymian") $\mathrm{Pr}_\sigma(\hat{\Gamma})$ is distinguished in $J(\hat{\Gamma})$ by the equation $\sigma_*(x) = -x$. In the case when $\hat{\Gamma}$ is hyperelliptic, we have $\Gamma = \mathbf{C}P^1$ and $\sigma\colon \hat{\Gamma} \to \hat{\Gamma}$ is the hyperelliptic involution of $\mathrm{Pr}_\sigma(\hat{\Gamma}) = J(\hat{\Gamma})$, and the Prymian is a principally polarized abelian variety. In the remaining cases, the Prym varieties are not principally polarized and only when $h = 1$ are their polarizations multiples of a principal one, which is obtained by division by two (an analogous situation holds also for suitably defined Prymians of unramified coverings whose polarization is also divided by two to get a principal one). Thus, in the case of branched coverings, only for coverings with two branch points does there exist a natural procedure (the Prym mapping) which associates principally polarized abelian varieties to them [1], [2].

Let $R_g$ be the space of moduli of Riemann surfaces of genus $g \geq 2$, and $A_g$ the space of moduli of principally polarized abelian varieties of dimension (which is always understood to be complex) $g$. We consider $L_g$, the space of moduli of two-sheeted branched coverings of surfaces of genus $g \geq 2$ with two branch points. There exists a natural bundle $L_g \to R_g$ which associates the base $\Gamma \in R_g$ to a covering. The fiber of this bundle over a point $\Gamma \in R_g$ is a connected $2^{2g}$-sheeted covering over the variety of effective divisors of degree 2 on $\Gamma$, since a covering over a fixed curve is given uniquely by the ramification divisor and half of it, defined up to $2^{2g}$ possibilities. As indicated above, there exists the so-called Prym mapping $F_0\colon L_g \to A_g$. The technique used in [3] for other classes of coverings allows us to compactify $L_g$ to an algebraic variety, and $F_0$ becomes algebraic. An analogue of the Riemann-Schottky problem is the problem of giving an effective description of the image of the Prym mapping $F_0(L_g) \subset A_g$. In order to study it we shall use the families of solutions of the Veselov-Novikov equations constructed with respect to such coverings. These equations are two-dimensional generalizations of the Korteweg-de Vries equation that are different from the Kadomtsev-Petviashvili equation [4], [5].

According to [6], for any nonsingular Riemann surface $\Gamma$ of genus $\tilde{g}$, points $Q_+$, $Q_- \in \Gamma$ $(Q_+ \neq Q_-)$, local parameters $k_+$ and $k_-$ in neighborhoods of these points, and a nonspecial divisor $D = P_1 + \cdots + P_{\tilde{g}}$, $P_i \in \Gamma$, $i = 1, \dots, \tilde{g}$, one can construct a unique differential operator $L = \partial\bar{\partial} + A\bar{\partial} + v$ with quasiperiodic coefficients, where $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. In case the involution $\sigma$ acts on $\Gamma$ in such a way that $Q_+$ and $Q_-$ are exactly all its fixed points, $\sigma(k_\pm) = -k_\pm$ and $D + \sigma(D) \sim Q_+ + Q_- + K$, where

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$K$ is the canonical divisor of meromorphic forms on $\Gamma$, then we obtain $A \equiv 0$ and

$$v = 2\partial\bar{\partial} \ln \theta(Uz + V\bar{z} - e) + C, \tag{1}$$

where $z = x + iy$, $\theta$ is the theta-function of the Prym variety of the covering $\Gamma \to \Gamma/\sigma$, the vectors $U$ and $V$ and the constant $C$ are determined by the set $(\Gamma, \sigma, k_\pm)$ and do not depend on $x$ or $y$, and $e = e(\Gamma, \sigma, D)$. On $\Gamma\backslash(Q_+ \cup Q_-)$ there will exist a meromorphic function $\psi(x, y, P)$ such that $L\psi = 0$ [4]. The function $\psi$ is a generalization of the Bloch eigenfunction of the operator $L = \partial^2_{xx} + u(x)$ with a periodic potential $u(x)$, and in the case when the periodic potential $v$ depends only on $x$, $\psi$ actually becomes this eigenfunction. We note that for such operators the "spectral data" $\Gamma, Q_+, Q_-, k_+, k_-, D$ have a rather simple meaning and are recovered uniquely from the operators [7], [8]. According to [4] and [11] the potentials $v$ of the form (1) are included in a family of potentials of the form

$$v(x, y, t_1, t_2) = 2\partial\bar{\partial} \ln \theta(Uz + V\bar{z} + W^+ t_1 + W^- t_2 - e) + C, \tag{2}$$

which satisfy the equations

$$\partial v/\partial t_1 = \partial^3 v + \partial(uv); \tag{3}$$

$$\partial v/\partial t_2 = \bar{\partial}^3 v + \bar{\partial}(wv), \tag{4}$$

where

$$u = 6\partial^2 \ln \theta(Uz + V\bar{z} + W^+ t_1 + W^- t_2 - e) + d_1,$$
$$w = 6\bar{\partial}^2 \ln \theta(Uz + V\bar{z} + W^+ t_1 + W^- t_2 - e) + d_2.$$

Substituting the theta-functional expressions for $v$, $u$, and $w$ in (3) and (4) leads to compatibility conditions on the vectors $U$, $V$, $W^+$, and $W^-$, the constants $C$, $d_1$, and $d_2$, and the theta-function of the Prymian. There is a well-known conjecture of S. P. Novikov that such compatibility conditions, obtained from the consideration of finite-gap solutions of the Kadomtsev-Petviashvili equation, exactly distinguish the Jacobians in $A_g$. This conjecture is partially proved in [9] and [10]. It was shown that the relations obtained for the theta-constants of the Riemann matrices distinguish in $A_g$ an algebraic subvariety, one of whose irreducible components is the closure of the set of Jacobians in $A_g$ (the question of the existence of other components is open). In this paper, using the technique of [9] and [10], we consider analogues of Novikov's conjecture for the Prym varieties described above.

**2.** In [11] it is shown that if the Prymian is indecomposable, i.e., it does not split into a direct product of abelian varieties of smaller dimension, then the substitution of (2) in (3) leads to the compatibility conditions

$$(\partial_V \partial_{W^+} - d_1 \partial_V \partial_U - 3C\partial_U^2 - 4\partial_U^3 \partial_V + a)\hat{\theta}[n] = 0, \tag{5}$$

where $\hat{\theta}[n](z)$ is the theta-function with characteristics $[n, 0]$, constructed from the doubled Riemann matrix of the Prymian, $n \in \frac{1}{2}(\mathbf{Z}_2)^g$, $a \in \mathbf{C}$, $z \in \mathbf{C}^g$, and where an expression of the form $\partial_V \hat{\theta}[n]$, for example, means the derivative of $\hat{\theta}[n](z)$ in the direction $V$ for $z = 0$.

We introduce a variety $X_g$ whose points are collections of vectors $U$, $V \in \mathbf{C}^g\backslash\{0\}$, $W^+ \in \mathbf{C}^g$, matrices $B \in \mathbf{H}_g$ ($\mathbf{H}_g$, the Siegel half-plane, is the set of symmetric $g \times g$ matrices with negative definite real part) and constants $a$, $C$, $d_1 \in \mathbf{C}$, factored modulo the action of the group

$$\begin{aligned} &U \to \lambda U, \quad V \to \mu V, \quad W^+ \to \lambda^3 W^- + \lambda\alpha U, \quad a \to \lambda^3 \mu a, \\ &C \to \lambda\mu C, \quad d_1 \to \lambda^2 d_1 + \alpha, \quad B \to B, \quad \lambda, \mu \in \mathbf{C}\backslash\{0\}, \quad \alpha \in \mathbf{C}, \end{aligned} \tag{6}$$

and the group $\operatorname{Sp}(g, \mathbf{Z})$, the action of which is connected with change of the basis of the lattice $\Lambda$ of an abelian variety (in our case the Prymian) $\mathbf{C}^g/\Lambda$ and is easily calculated (see [9] and [10]).

The set of zeros $Y_g$ of the system (5) is invariant under the actions of the above groups. By the above there exists a mapping $F\colon L_g \to Y_g$ (note that the subset of indecomposable Prymians is open and everywhere dense in the set of all Prymians). Using the "perturbation method" of [9], we can show that $\dim Y_g^0 \leq 3g-1$, where $Y_g^0$ is the irreducible component of $Y_g$ containing $F(L_g)$, and we can take as local coordinates in a neighborhood of a generic point of $Y_g^0$ the sets $(U_1, \dots, U_g, V_1, \dots, V_g, \exp B_{11}, \dots, \exp B_{gg}, C)$, factored modulo the action of (6) (we shall show below that these coordinates are independent). In [9] the exact value of $\dim Y_g^0$ was determined using Torelli's theorem, whose analogue for Prymians is not known. But, using the results of [9] concerning the Riemann-Schottky problem for hyperelliptic curves and the above-indicated uniqueness of the recovery of "spectral data" for one-dimensional finite-gap periodic Schrödinger operators, we can show that there exist nonsingular points $\xi \in L_g$ and $\xi_0 \in Y_g^0$ such that $\xi$ is an isolated point of $F^{-1}(\xi_0)$. Using Osgood's theorem ([12], p. 518), we obtain that $\dim L_g \leq \dim Y_g^0$ (but $\dim L_g = 3g-1$) and in a neighborhood of $\xi$ the mapping $F$ is a finite-sheeted, possibly branched, covering over its image. From this we get

THEOREM 1. *In a neighborhood of a point in general position of $L_g$ the mapping $F\colon L_g \to Y_g^0$ is a local diffeomorphism.*

From Theorem 1 and the results of [11] we obtain

COROLLARY 1. *In a neighborhood of a point in general position of $L_g$ a covering is uniquely determined by the Prym variety and the vectors $U$ and $V$ (an elementary definition of them as vectors of b-periods of certain differential forms on the covering surface is given in* [4]).

We denote by $\Pi_g$ the projection of $Y_g^0$ onto $A_g$. As a consequence of the algebraicity of the Prym mapping we obtain

THEOREM 2. *$\Pi_g$ coincides with the closure of the image of the Prym mapping, i.e., $\Pi_g = \overline{F_0(L_g)}$.*

REMARKS. 1) It is known that $\overline{F_0(L_g)}$ also includes the Prymians of coverings with double points (Beauville pairs [3]) and direct products of them.

2) The algebraicity of the constructions in Theorem 1 is not essential. If we consider sets $(U, V, W^+, B, a, C, d_1)$, factored only by the action (6), then the solutions of (5) form an analytic set $\tilde{Y}_g$, and Theorem 1 will hold for the multivalued mapping $\tilde{F}\colon L_g \to \tilde{Y}_g$, where the images $\tilde{F}(\xi)$, $\xi \in L_g$, are in one-to-one correspondence with the various canonical homology bases of $H_1(\hat{\Gamma}; \mathbf{Z})$ ($\hat{\Gamma} \to \hat{\Gamma}/\sigma$ is the covering $\xi$). Under the projection of $\tilde{F}(L_g)$ onto $\mathbf{H}_g$ we obtain a set which after factoring by the action of the modular group $\operatorname{Sp}(g, \mathbf{Z})$ goes to $F_0(L_g) \subset \mathbf{H}_g/\operatorname{Sp}(g, \mathbf{Z}) = A_g$.

3) Theorem 2 proves Novikov's conjecture up to irreducible components and shows that the relations on the theta-constants obtained from (5) are sufficient for the description of the variety of Prymians in $A_g$ in a neighborhood of a given Prymian.

**3.** In this section we give formulas that completely effectivize the theta-functional solutions of the form (2) (i.e., finite-gap) of the equations (3), (4) in the case of a two-dimensional indecomposable Prymian. At the same time we shall give explicit formulas for the construction of two-dimensional finite-gap potentials of a Schrödinger operator of the form (1) from an arbitrary indecomposable $2 \times 2$ Riemann matrix $B \in \mathbf{H}_2$, and arbitrary linearly independent vectors $U, V \in \mathbf{C}^2$. The existence of such a procedure is indicated in [11], Corollary 1.

Since the Prymian is indecomposable, the matrix

$$(\hat\theta_{11}[n] \quad \hat\theta_{12}[n] \quad \hat\theta_{22}[n] \quad \hat\theta[n]), \qquad n \in \tfrac{1}{2}(\mathbf{Z}_2)^2,$$

is invertible (the conditions of invertibility and indecomposability of a Prymian are equivalent in this case [10]). We denote the inverse matrix by

$$(a_n^{11} \quad a_n^{12} \quad a_n^{22} \quad a_n),$$

where $a_n^{pq}\hat\theta_{kl}[n] = \delta_k^p\delta_l^q$, $a_n\hat\theta_{kl}[n] = a_n^{pq}\hat\theta[n] = 0$, and $a_n\hat\theta[n] = 1$ (everywhere there is an implicit summation over $n$).

We consider polynomials of the coefficients of the two-dimensional linearly independent vectors $X$, $Y \in \mathbf{C}^2$:

$$Q_{kl}(X,Y) = 4\sum_n a_n^{kl}\partial_X^3\partial_Y\hat\theta[n].$$

Using them we define functions $f_1, f_2, f_3$ with the same domain of definition:

$$\begin{aligned}
f_1(X,Y) = (Q_{11}(X,Y)(Y_1X_2^2 - 2X_1X_2Y_2) &+ Q_{12}(X,Y)X_1^2Y_2 \\
&- Q_{22}(X,Y)Y_1X_1^2)(X_1Y_2 - X_2Y_1)^{-2}, \\
f_2(X,Y) = (Q_{22}(X,Y)(Y_2X_1^2 - 2X_1X_2Y_1) &+ Q_{12}(X,Y)X_2^2Y_1 \\
&- Q_{11}(X,Y)Y_2X_2^2)(X_1Y_2 - X_2Y_1)^{-2}, \\
f_3(X,Y) = \tfrac{1}{3}(Q_{12}(X,Y)Y_1Y_2 - Q_{11}(X,Y)Y_2^2 &- Q_{22}(X,Y)Y_1^2)(X_1Y_2 - X_2Y_1)^{-2}.
\end{aligned}$$

THEOREM 3. *If a two-dimensional Riemann matrix is indecomposable, the vectors* $U$, $V \in \mathbf{C}^2$ *are linearly independent and a potential* $v(x,y,t_1,t_2)$ *of the form* (2) *is a solution of equations* (3) *and* (4), *then*

$$W_1^+ = f_1(U,V) + d_1U_1, \quad W_2^+ = f_2(U,V) + d_1U_2, \quad W_1^- = f_1(V,U) + d_2V_1,$$
$$W_2^- = f_2(V,U) + d_2V_2, \qquad C = f_3(U,V) = f_3(V,U).$$

THEOREM 4. *If a two-dimensional Riemann matrix is indecomposable and the vectors* $U$, $V \in \mathbf{C}^2$ *are linearly independent, then the operator* $L = \partial\bar\partial + v$, *where* $v$ *is of the form* (1), *will be finite-zone for* $C = f_3(U,V)$ *and* $e \in \mathbf{C}^2$.

Since the linearly independent vectors $U$ and $V$ can be arbitrary, the condition $f_3(U,V) = f_3(V,U)$ implies a set of relations on the coefficients of $f_3$:

COROLLARY 2.

$$\sum(a_n^{KM}\hat\theta_{MMMM}[n] + 2a_n^{KK}\hat\theta_{KMMM}[n]) = 0,$$
$$\sum(a_n^{KM}\hat\theta_{KMMM}[n] - a_n^{MM}\hat\theta_{MMMM}[n] + 3a_n^{KK}\hat\theta_{KKMM}[n]) = 0,$$
$$1 \le K \ne M \le 2;$$
$$\sum(a_n^{11}\hat\theta_{1112}[n] - a_n^{22}\hat\theta_{1222}[n]) = 0$$

(*the summation is everywhere understood to be over* $n \in \frac{1}{2}(\mathbf{Z}_2)^2$).

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