

TOPOLOGICAL OBSTRUCTIONS
TO INTEGRABILITY OF GEODESIC FLOWS
ON NON-SIMPLY-CONNECTED MANIFOLDS

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ABSTRACT. In this paper, (Liouville) integrability of geodesic flows on non-simply-connected manifolds is studied. In particular, the following result is obtained: A geodesic flow on a real-analytic Riemannian manifold cannot be integrable in terms of analytic functions if either 1) the fundamental group of the manifold contains no commutative subgroup of finite index, or 2) the first Betti number of the manifold over the field of rational numbers is greater than the dimension (the manifold is assumed to be closed).

Bibliography: 11 titles.

Let M^n be a closed manifold and let TM^n and T^*M^n be the tangent and cotangent bundles of M^n , respectively. Let $\Omega = \sum_{\alpha=1}^n dx^\alpha \wedge dp_\alpha$ be the standard nondegenerate closed skew-symmetric two-form on T^*M^n , where (x^α) are coordinates on M^n and (p_α) the corresponding coordinates on the fibers of the cotangent bundle. The form Ω gives T^*M^n the structure of a symplectic manifold and it provides the space of smooth functions on T^*M^n with a Poisson bracket

$$\{f, g\} = h^{ij} \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial y^j},$$

where $\Omega = h_{ij} dy^i \wedge dy^j$ locally (see [1]).

Let $g_{ij}(x)$ be a Riemannian metric on M^n . The corresponding geodesic flow on TM^n is given by the Lagrangian $L(x, \dot{x}) = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j$. The resulting Hamiltonian system on T^*M^n has Hamiltonian $H(x, p) = \frac{1}{2} g^{ij}(x) p_i p_j$. Such systems are very important geometric examples of Hamiltonian systems on symplectic manifolds. Many important problems in mechanics reduce to the study of geodesic flows on Riemannian and Finsler metrics (see [2] and [3]).

It is natural to consider the problem of whether geodesic flows are integrable.

DEFINITION. A geodesic flow on a closed Riemannian manifold M^n is (Liouville) *integrable* if, in addition to the Hamiltonian (the energy integral) $H = I_1$, there exist $n - 1$ more first integrals I_2, \dots, I_n on T^*M^n (where $n = \dim M^n$) with the property that $\{I_i, I_j\} = 0$ for all i and j , and the integrals are functionally independent at each point of some invariant open dense subset $U \subset L$ of some nonzero level of the energy integral $L = \{H = \text{const} \neq 0\}$. If M^n is a real analytic manifold and the metric $g_{ij}(x)$ is

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real-analytic, then the flow is said to be *real-analytic integrable* if the indicated integrals I_1, \dots, I_n exist and can be chosen to be real analytic.

This definition is based on the well-known Liouville theorem [1], [4]. In [5] and [6], it was proved that the geodesic flow of any real analytic metric on an orientable two-dimensional real analytic closed surface of genus $g > 1$ is not analytically integrable. The proofs there make essential use of properties particular to two dimensions. In this paper, we will find topological obstructions to analytic integrability of geodesic flows which do not depend on the dimension of the manifold. We actually will prove a more general fact (Theorem 1).

In the general case, the nonintegrability proof breaks into two parts: 1) proving that a certain geometric disposition of a geodesic flow cannot exist, and 2) establishing functional conditions on a complete set of first integrals of an integrable geodesic flow which distinguish integrable flows with a given geometrical disposition.

We introduce the main definition.

DEFINITION. A geodesic flow on M^n is *geometrically simple* if there exists a nonzero level of energy integral $L = \{H = \text{const} \neq 0\}$ such that the following conditions are satisfied:

1) L contains a closed set Γ with invariant, everywhere dense complement $L \setminus \Gamma$ consisting of a finite number of arcwise connected components U_α , $\alpha = 1, \dots, d$:

$$L \setminus \Gamma = \bigcup_{\alpha=1}^d U_\alpha.$$

2) Each component U_α fibers over an $(n-1)$ -dimensional closed disk with fiber an n -dimensional torus:

$$\varphi_\alpha: U_\alpha \rightarrow D^{n-1}, \quad n = \dim M^n.$$

3) For any given point $q \in L$ and any neighborhood W of q there exists a neighborhood W_1 such that $q \in W_1 \subset W$ and $W_1 \cap (L \setminus \Gamma)$ has a finite number of arcwise connected components.

We remark that condition 3) holds, for example, if L has a simplicial decomposition in which the set Γ is a subcomplex. This is the situation in the well-known integrable cases. It is clear from the proofs below that all the results remain true if condition 2) is weakened by demanding only that U_α be fibered into k -dimensional tori over $(2n-k-1)$ -dimensional disks for some $k \leq n$. This situation arises, for example, in "noncommutative" integration [7].

The main result is as follows.

THEOREM 1. *If a closed manifold M^n satisfies at least one of the following conditions:*

a) $\pi_1(M^n)$ is not almost commutative; that is, it does not contain a commutative subgroup of finite index; or

b) the dimension of the first rational homology group of the manifold M^n exceeds the dimension of M :

$$\dim H_1(M^n; \mathbf{Q}) > \dim M^n,$$

then no geodesic flow on M^n is geometrically simple.

We remark that the considerations connected with condition b) first arose in the proof of the main theorem of [5] and led to the previously mentioned restriction on the genus of the surface. The existence of obstructions to integrability connected with noncommutativity of the fundamental group of the configuration space M^n (condition a)) has not, apparently, been noticed to date.

Application of Theorem 1 is related to the study of the geometry of integrable systems. Here, it is most effective to consider functional properties of first integrals under which “the singular set” Γ is a subcomplex of a simplicial decomposition of L (note that L is compact since we have assumed that the metric is positive definite). This situation occurs, for example, if the first integrals are smooth with sufficiently simple critical values and levels of the “moment map” $F: L \rightarrow R^{n-1}$, where $F(q) = (I_2(q), \dots, I_n(q))$. Citing examples of conditions on the critical values and the levels of F does not present difficulties. The following theorem is very interesting.

THEOREM 2. *If a geodesic flow on a real-analytic manifold M^n with a real analytic metric is analytically integrable, then it is geometrically simple.*

Theorems 1 and 2 imply the following.

THEOREM 3. *If a real-analytic closed manifold M^n with a real analytic metric satisfies at least one of the conditions:*

- a) $\pi_1(M^n)$ is not almost commutative; or
- b) $\dim H_1(M^n; \mathbf{Q}) > \dim M^n$,

then the geodesic flow on M^n is not analytically integrable.

REMARKS. 1) From a topological point of view, almost commutativity of the fundamental group is equivalent to the existence of a finite-sheeted cover $\hat{M}^n \rightarrow M^n$ with abelian fundamental group $\pi_1(\hat{M}^n)$. An example of a manifold with an analytically integrable geodesic flow and a noncommutative, almost commutative fundamental group is the Klein bottle; in this case, an example of a metric with an integrable geodesic flow is the metric “taken off” the usual Euclidean plane from which the Klein bottle is obtained as a quotient space by the action of a discrete group of isometries.

2) It is well known that if a closed manifold M^n admits a metric with strictly negative curvature, then $\pi_1(M^n)$ is not abelian and the maximal abelian subgroup of $\pi_1(M^n)$ is infinite cyclic [8]. It is clear that such fundamental groups are not almost commutative. We obtain the following corollary.

COROLLARY. *If a closed manifold M^n admits a metric with strictly negative curvature, then for any real-analytic structure on M^n and for any real-analytic metric on M^n the geodesic flow is not analytically integrable.*

§1. The main construction

Let (A, G, μ) be a triple, where $A = \{a_1, \dots, a_r\}$ is a finite set, G is a group, and $\mu: G \setminus \{e\} \rightarrow A \times A$ is a map of $G \setminus \{e\}$ (e is the identity element of the group G) to the direct product of two copies of A . In this section we construct a graph $\Pi(A, G, \mu)$ for an arbitrary triple (A, G, μ) . The triples (A, G, μ) arise below in describing simple geodesic flows, and the corresponding graphs $\Pi(A, G, \mu)$ turn out to carry the main geometric information concerning the flows.

Suppose that we are given a triple (A, G, μ) , where A consists of r elements. We let Π_0 be a graph whose vertices are r points which we identify in a one-to-one manner with the elements of A . If $g \in G \setminus \{e\}$ and $\mu(g) = (a_i, a_j) \in A \times A$, then we join the vertices by an arrow from a_i to a_j and we label the arrow by the element g of G . We thus obtain a graph Π_0 in which the set of vertices is identified with the set A , and the set of arrows (oriented edges) with the set $G \setminus \{e\}$. We augment Π_0 to obtain a graph Π as follows: if Π_0 contains an arrow from a_i to a_j with label g , then we add to Π_0 the arrow from a_j to a_i with label g^{-1} . We obtain a graph $\Pi = \Pi(A, G, \mu)$ together with a function σ on the set of oriented edges which associates to each edge its label.

We consider paths on the graph Π of the form $\xi = \gamma_k \gamma_{k-1} \cdots \gamma_1$, where γ_i is an arrow and the terminal point of γ_j coincides with the initial point of γ_{j+1} . The initial point of γ_1 is deemed to be the initial point of the path ξ , and the terminal point of the arrow γ_k is deemed to be the terminal point of the path ξ . For each vertex a_i , we consider the set of loops Ω_i consisting of paths with initial point and terminal point at a_i . On all the $\Omega_i, i = 1, \dots, r$, we naturally define a function σ_1 , the "path label", as follows: if $\xi \in \Omega_i$, where $\xi = \gamma_k \cdots \gamma_1$ and the γ_j are arrows, then $\sigma_1(\xi) = \sigma(\gamma_k)\sigma(\gamma_{k-1}) \cdots \sigma(\gamma_1)$.

LEMMA 1. $\sigma_1(\Omega_i)$ is a subgroup of G .

The proof is obvious. We let $H_i = \sigma_1(\Omega_i)$.

LEMMA 2. The group G is a set-theoretic union of a finite number of left cosets of the subgroups $H_i, i = 1, \dots, r$, defined above.

PROOF. To each pair of vertices a_k, a_l which are joined in Π by at least one arrow from a_k to a_l , associate one (fixed) such arrow and its label, and denote it by g_{kl} . Then

$$G = \left(\bigcup_i H_i \right) \cup \left(\bigcup_{i,j} g_{ij} H_i \right).$$

We prove this. Suppose there is another arrow from a_k to a_l with label h . Then $g_{kl}^{-1}h \in H_k$ and, hence, $h \in g_{kl}H_k$. It remains to add that all elements in $G \setminus \{e\}$ occur as labels of oriented edges of Π .

LEMMA 3. Suppose that an element $g \in G$ has infinite order. Then there exist $k \neq 0$ and s such that $g^k \in H_s$.

PROOF. We consider all possible arrows with labels g^l , where l runs over all nonzero integers. There are infinitely many such arrows, and the number of vertices of the graph Π is finite. Hence, there exist vertices a_s, a_t and p, q with $p \neq q$ such that there exist arrows with labels g^p and g^q pointing from a_s to a_t . It follows that $g^{p-q} \in H_s$.

Lemma 4 is of a general nature; however, we retain the preceding notation to facilitate seeing how it manifests itself in our construction.

LEMMA 4. If a group G is a set-theoretic union of a finite number of left cosets of a finite number of subgroups $H_i, i = 1, \dots, r$, then at least one of these subgroups H_j has finite index in $G: |G:H_j| < \infty$.

PROOF. We proceed by a simple induction on the number r of subgroups. For $r = 1$, the assertion is obvious. Suppose that it is satisfied for $r \leq k - 1$. Then

$$G = \bigcup_{i,j} g_{ij} H_i. \tag{1}$$

Suppose that H_k has infinite index in G . Then some coset hH_k does not appear in the formal representation (1). Then $hH_k \subset \bigcup_{i \neq k,j} g_{ij} H_i$ and, hence, $H_k \subset \bigcup_{i \neq k,j} h^{-1}g_{ij} H_i$. We obtain

$$G = \left(\bigcup_{i \neq k,j} g_{ij} H_i \right) \cup \left(\bigcup_{i \neq k,j,l} g_{kj} h^{-1} g_{il} H_i \right).$$

Hence G is a set-theoretic union of a finite number of left cosets of the $k - 1$ subgroups H_1, \dots, H_{k-1} . By the induction hypothesis, there is a $j, 1 \leq j \leq k - 1$, such that H_j has finite index in G . Lemma 4 is proved.

§2. The proof of Theorem 1

We consider a geometrically simple geodesic flow on a closed manifold M^n . Let $p: L \rightarrow M^n$ be the standard projection to the base. On the base, we choose a fixed point x_0 and neighborhoods U and V such that $x_0 \in U \subset \bar{U} \subset V$ (where \bar{U} is the closure of U) and both the universal cover $\hat{M}^n \rightarrow M^n$ and the cotangent bundle are trivial over V . Let $W = p^{-1}(\bar{U})$. We have $W \cap (L \setminus \Gamma) = \bigcup_{\alpha} U_{\alpha}$, where U_{α} is an arcwise connected component. There may be infinitely many of them. We introduce an equivalence relation on the set of components by agreeing that $U_{\alpha} \sim U_{\beta}$ if there are points $q_1 \in U_{\alpha}$ and $q_2 \in U_{\beta}$ which can be joined by a curve in $p^{-1}(V) \cap (L \setminus \Gamma)$. Property 3) of geometric simplicity implies that the number of equivalence classes is finite. In fact, if there were infinitely many, we could choose a representative U_{α} in each class and a point $q_{\alpha} \in U_{\alpha}$ in each U_{α} . Since the set $\{q_{\alpha}\}$ is infinite and W is compact, there is a limit point \bar{q} in W . The domain $p^{-1}(V)$ is a neighborhood of \bar{q} and, according to property 3), contains a subdomain $Z \subset p^{-1}(V)$ with $\bar{q} \in Z$ such that $Z \cap (L \setminus \Gamma)$ has a finite number of connected components. This contradicts the assertion that Z contains an infinite number of the points in $\{q_{\alpha}\}$. We choose a representative Φ_i , $i = 1, \dots, r$, in each class and a point $\varphi_i \in \Phi_i$ in each representative.

Since M^n is complete, we can apply the Hopf-Rinow theorem to the universal cover \hat{M}^n to deduce that each element $g \in \pi_1(M^n, x_0) \setminus \{e\}$ can be realized as a geodesic loop with vertex at the point x_0 . We lift this loop to L to obtain a loop $\gamma(g)$. If the loop lies in Γ , then a sufficiently small perturbation of the initial conditions gives another loop $\gamma'(g)$ (which is also a trajectory of the geodesic flow) with the property that $\gamma'(g)$ lies in $L \setminus \Gamma$; if the initial point and terminal point of its projection to M^n are joined in V by segments to x_0 , we obtain a loop realizing an element $g \in \pi_1(M^n, x_0)$. We join the initial points and terminal points of the indicated paths $\gamma(g)$ and $\gamma'(g)$ with points of the set $\{\varphi_i\}$ by segments in $p^{-1}(V) \cap (L \setminus \Gamma)$ (this procedure is obviously unique) and we let $\xi(g)$ denote the resulting paths. We now apply the construction of §1.

For this, we construct the triple (A, G, μ) as follows. We take A to be the set of points $\{\varphi_i\}$; $G = \pi_1(M^n, x_0)$. Let $g \in G \setminus \{e\}$, and let $\xi(g)$ be the path constructed above with initial point at φ_{1g} and terminal point at φ_{2g} . Then we consider $\mu(g) = (\varphi_{1g}, \varphi_{2g})$.

We now prove Theorem 1.

Suppose that $\pi_1(M^n)$ is not almost commutative. It is clear that paths on the graph $\Pi = \Pi(A, G, \mu)$ can be realized as a composition of paths of the form $\xi(g)$. We apply Lemma 4: some H_s has finite index in G . Consider the corresponding point $\varphi_s \in \Phi_s$. By the definition of geometrical simplicity, $\Phi_s \subset U_{\alpha(s)}$, where $U_{\alpha(s)}$ is homeomorphic to the direct product of an open $(n-1)$ -dimensional disk by the torus T^n . Let $f: U_{\alpha(s)} \rightarrow L$ be inclusion, ω a path from x_0 to $p(\varphi_s)$ lying in V , and $\omega_*: \pi_1(M^n, x_0) \rightarrow \pi_1(M^n, p(\varphi_s))$ the corresponding isomorphism. Then it is evident from the geometric construction that $\omega_*(H_s) \subset (pf)_*(\pi_1(U_{\alpha(s)}, f^{-1}(\varphi_s)))$, where $(pf)_*$ is the homomorphism of fundamental groups induced by composition of the inclusion f with the projection p . However, $\pi_1(U_{\alpha(s)}) = \pi_1(T^n)$ is a free abelian group and $\omega_*(H_s)$ is not abelian because it has finite index in $\pi_1(M^n, p(\varphi_s))$. At the same time $\omega_*(H_s)$ lies in the homomorphic image of an abelian group. This is a contradiction.

Suppose that $\dim H_1(M^n; \mathbf{Q}) > \dim M^n$. Let $h: \pi_1(M^n, x_0) \rightarrow H_1(M^n)$ be the Hurewicz homomorphism and $\sigma: H_1(M^n) \rightarrow H_1(M^n; \mathbf{Q})$ the homomorphism induced by inclusion of the coefficient group Z into \mathbf{Q} . Suppose $g \in \pi_1(M^n)$ has infinite order. According to Lemma 3, there exist $k \neq 0$ and s such that $g^k \in H_s$. Then

$\sigma \circ h(g) \in (pf)_*(H_1(U_{\alpha(s)}; \mathbf{Q}))$. From this we obtain

$$H_1(M^n; \mathbf{Q}) = \bigcup_{\alpha=1}^d (pf)_* H_1(U_\alpha; \mathbf{Q}), \quad (2)$$

where $f_\alpha: U_\alpha \rightarrow L$ is inclusion. But $H_1(U_\alpha; \mathbf{Q}) = \mathbf{Q}^n$. It follows that (2) is impossible if $\dim H_1(M^n; \mathbf{Q}) > \dim M^n = n$. The theorem is proved.

§3. The proof of Theorem 2

The proof of Theorem 2 is based on the theory of semianalytic sets and their projections [9]–[11]. A subset $A \subset X$ of a real-analytic manifold X is called *semianalytic* if it can be represented in a neighborhood of each point $x \in X$ as a finite union of sets of the form

$$\{f_i(x) = 0, i = 1, \dots, r; g_j(x) > 0, j = 1, \dots, s\},$$

where the f_i and g_j are real-analytic functions on a neighborhood of a point $x \in X$. A subset $B \subset Y$ is called *constructive* if it is the image of a relatively compact semianalytic set $A \subset X$ under some analytic map of manifolds $X \rightarrow Y$ [11]. This class of sets is closed under finite unions, intersections, and products, taking images and preimages under proper analytic maps, closure, and complementation. Each constructive set is a finite union of arcwise connected constructive sets (see [10] and [11]).

Consider an analytically integrable geodesic flow and let L be the nonzero level (mentioned in the definition of integrability) of the energy integral $I_1 = H$, and let I_2, \dots, I_n be an additional collection of involutory first integrals. Consider the “moment map” $F: L \rightarrow R^{n-1}$, where $F(q) = (I_2(q), \dots, I_n(q))$. Let $V \subset L$ be the set of critical points F and $C_1 = F(V)$ the set of critical values. Then V is semianalytic and C_1 is constructive. To prove Theorem 2, it suffices to find a constructive set C_2 in R^{n-1} such that $F(L) \setminus (C_1 \cup C_2)$ splits into a union of open disks. In this case, we take $\Gamma = F^{-1}(C_1 \cup C_2)$. Property 3) of geometric simplicity will then follow in an obvious way from the properties of constructible sets enumerated above.

We begin by noting the following completely obvious fact.

Let W be a domain in R^N with compact closure. If the boundary ∂W is constructive and has a constructive simplicial decomposition (that is, each simplex is a constructive set in R^N), then the decomposition extends to a constructive simplicial decomposition of W .

In [10], it is proved that a compact constructive set in R^N admits a decomposition into a finite union of constructive sets, each of which homeomorphically projects to a domain with constructive boundary under some projection to a linear subspace of R^N ([10], Theorem 2). Using this together with the above fact and a simple induction on the dimension of the constructive sets, we find that any compact constructive set in R^N admits a constructive simplicial decomposition.

In our situation, having constructed a constructive simplicial decomposition of the set of critical values C_1 , we extend it to a constructive simplicial decomposition of $F(L)$. We take the $(n-2)$ -dimensional skeleton of the complex $F(L)$. Denote it by C_3 . It is clear that $C_1 \subset C_3$. Therefore, we can set $\Gamma = F^{-1}(C_3)$. It remains to note that the fibrations mentioned in the definition of geometrical simplicity are obtained by the usual restriction of the “moment map” to the arcwise connected components of the preimages of $(n-1)$ -dimensional simplices of the constructive simplicial decomposition of $F(L)$. Theorem 2 is proved.

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