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PRYM VARIETIES OF BRANCHED COVERINGS AND NONLINEAR EQUATIONS

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ABSTRACT. An efficacious realization is presented of finite-gap solutions of the Veselov-Novikov equation expressed in terms of the theta function of Prym varieties of double coverings of algebraic curves with two branch points. For the given Prym mapping equations are obtained which locally solve a problem of Riemann-Schottky type, and a local Torelli theorem is proved.

The methods of finite-gap integration make it possible not only to effectively find solutions expressed in terms of the theta function of Abelian varieties of a number of important equations of mathematical physics (the Korteweg-de Vries and Kadomtsev-Petviashvili equations and others [1]–[3]) but also to study the Abelian varieties themselves, namely, those arising in the finite-gap theory of the Jacobi and Prym varieties. There exist two hierarchies of nonlinear equations whose finite-gap solutions can be expressed in terms of the theta function of Prym varieties (Prymians) of branched coverings: 1) the BKP equations obtained as a result of modification of the Kadomtsev-Petviashvili equations [4], and 2) the Veselov-Novikov equations [5] arising in the theory of two-dimensional Schrödinger operators which are finite-gap operators on one energy level [6], [7].

In this paper we describe the finite-gap solutions of the first nontrivial equation of the Veselov-Novikov hierarchy and consider their applications in the theory of Prym varieties of double coverings with two branch points; in particular, we obtain equations locally distinguishing the Prymians in the moduli space of principally polarized Abelian varieties and prove a local Torelli theorem for the given Prym mapping. Some of the results were announced in [8] and [9].

§1. Two-dimensional finite-gap potential Schrödinger operators and Prym varieties

A. Basic definitions.

DEFINITION. Let Γ be a nonsingular irreducible algebraic curve of genus g, let $Q_1, \dots, Q_l$ be a set of distinct points on Γ, let $k_1^{-1}, \dots, k_l^{-1}$ be local parameters in a neighborhood of them $(k_j^{-1}(Q_j) = 0)$, let $q_1(k), \dots, q_l(k)$ be a set of polynomials, and let D be a positive divisor of degree g on $\Gamma\backslash(Q_1 \cup \dots \cup Q_l)$. The *l-point Baker-Akhiezer function* corresponding to the indicated data is a meromorphic function

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$\psi(P)$ on $\Gamma\backslash(Q_1 \cup \cdots \cup Q_l)$ such that

a) the divisor of the zeros and poles $(\psi) \geq -D$, and

b) as $P \to Q_j$ the product $\psi(P)\exp(-q_j(k_j(P)))$ is analytic $(j = 1, \dots, l)$.

Such functions corresponding to a fixed collection of data form a linear space whose dimension is equal to one if the divisor D is nonspecial and the polynomials $q_1, \dots, q_l$ are in general position [2], [3].

With the help of the Baker-Akhiezer functions, the concept of a two-dimensional finite-gap (on one energy level) Schrödinger operator of the form

$$L = \partial\overline{\partial} + A(z, \overline{z})\overline{\partial} + u(z, \overline{z}), \tag{1}$$

where $\partial = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$, $\overline{\partial} = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$, and $z = x + iy$, was introduced in [6].

DEFINITION. An operator of the form (1) is said to be a *finite-gap operator* (of rank 1) on the energy level E if there exists a function $\psi(z, \overline{z}, P)$ meromorphic in P on $\Gamma\backslash(Q_1 \cup Q_2)$, where Γ is an algebraic curve of genus g and $Q_1, Q_2 \in \Gamma$, having poles $P_1, \dots, P_g$ on $\Gamma\backslash(Q_1 \cup Q_2)$ and the asymptotics

$$\begin{aligned} &\psi(z, \overline{z}, P) \sim e^{k_1 z}(1 + \xi_1^+ k_1^{-1} + O(k_1^{-2})), \qquad P \to Q_1; \\ &\psi(z, \overline{z}, P) \sim a(z, \overline{z})e^{k_2 \overline{z}}(1 + O(k_2^{-1})), \qquad P \to Q_2, \end{aligned} \tag{2}$$

where k_1^{-1} and k_2^{-1} are local parameters in neighborhoods of the points Q_1 and Q_2 $(k_j^{-1}(Q_j) = 0)$, such that $(L - E)\psi = 0$.

On the basis of the spectral data $\{\Gamma, Q_1, Q_2, k_1, k_2, D\}$ it is possible to recover an operator L of the form (1) such that the Baker-Akhiezer function (more precisely, a smooth z-family of Baker-Akhiezer functions) $\psi(z, \overline{z}, P)$ corresponding to the data $\{\Gamma, Q_1, Q_2, k_1, k_2, D, q_1(k) = kz, q_2(k) = k\overline{z}\}$ and normalized by the condition $\psi(P)\exp(-q_1(k_1(P))) \to 1$ as $P \to Q_1$ satisfies the equation $L\psi = 0$. The coefficients of L are found from the asymptotics of the form (2) of the eigenfunction $\psi(z, \overline{z}, P)$ [6]:

$$A = -\partial \ln a/\partial z, \qquad u = -\partial\xi_1^+/\partial\overline{z}. \tag{3}$$

In [7] a sufficient condition for L to be a potential operator was found: if a holomorphic involution σ such that

$$\sigma(Q_j) = Q_j, \quad \sigma(k_j) = -k_j, \quad j = 1, 2, \qquad D + \sigma(D) \sim C(\Gamma) + Q_1 + Q_2, \tag{4}$$

acts on Γ, where $C(\Gamma)$ is the divisor of zeros of holomorphic 1-forms on Γ and $\sim$ denotes equivalence of divisors, then the operator L constructed on the basis of $\{\Gamma, Q_1, Q_2, k_1, k_2, D\}$ (the divisor D is nonspecial, so the construction is possible) is a potential operator:

$$L = \partial\overline{\partial} + v(z, \overline{z}). \tag{5}$$

B. The theta-function formula for the potential v. On a nonsingular algebraic curve Γ of genus g we choose a basis of cycles $a_1, \dots, a_g$, $b_1, \dots, b_g$ in $H_1(\Gamma)$ with intersection form $a_k \circ a_j = b_k \circ b_j = 0$, $a_k \circ b_j = \delta_{kj}$ (i.e., a canonical basis). To it there corresponds a dual basis of holomorphic 1-forms $\omega_1, \dots, \omega_g$ normalized by the conditions

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk},$$

and the b-periods of these differentials define the Riemann matrix

$$B_{jk} = \int_{b_j} \omega_k, \tag{6}$$

which is symmetric and has a negative definite real part. On the basis of each symmetric $g \times g$ matrix B with $\operatorname{Re} B < 0$ it is possible to construct a series of theta functions

$$\theta(z|B) = \sum_{N \in \mathbf{Z}^g} \exp\{\tfrac{1}{2}\langle BN, N\rangle + \langle N, z\rangle\}, \tag{7}$$

$$\theta[\alpha, \beta](z|B) = \exp\{\tfrac{1}{2}\langle B\alpha, \alpha\rangle + \langle z + 2\pi i\beta, \alpha\rangle\}\theta(z + 2\pi i\beta + B\alpha|B), \tag{8}$$

where $z \in \mathbf{C}^g$, $\langle N, z\rangle = \sum_1^g N_j z_j$, and $\alpha, \beta \in \mathbf{R}^g$. The theta functions constructed according to (6) are called the theta functions of the algebraic curve Γ. The functions (8) are called theta functions with characteristics α and β.

By means of the Riemann matrix it is possible to construct the Jacobi variety of the curve Γ: $J(\Gamma) = \mathbf{C}^g/\Lambda$, where the lattice Λ has the form

$$\Lambda = \{2\pi i N + BN' \colon N, N' \in \mathbf{Z}^g\}.$$

Suppose $P_0 \in \Gamma$ is some fixed point; then the Abel mapping $A\colon \Gamma \to J(\Gamma)$ with initial point P_0 is defined by

$$A_k(P) = \int_{P_0}^{P} \omega_k, \qquad 1 \le k \le g.$$

If a holomorphic involution $\sigma\colon \Gamma \to \Gamma$, $\sigma^2 = 1$, with two fixed points Q_1 and Q_2 acts on the curve Γ, then the genus of Γ is even $(= 2h)$, and we choose the canonical basis of cycles on Γ so that $\sigma_*(a_j) + a_{j+h} = \sigma_*(b_j) + b_{j+h} = 0$, $1 \le j \le h$, and the images $\pi(a_1), \dots, \pi(a_h)$, $\pi(b_1), \dots, \pi(b_h)$ of some of the basis cycles under the projection $\pi\colon \Gamma \to \Gamma/\sigma$ form a canonical basis of cycles on Γ/σ [10]. We denote by B and B_0 the Riemann matrices of the curves Γ and Γ/σ constructed on the basis of these bases.

Since $A(S^{2h}\Gamma) \supset J(\Gamma)$, where $S^{2h}\Gamma$ is the $2h$th symmetric power of Γ and A is extended to it by linearity, it follows that σ induces an involution $\hat{\sigma}\colon J(\Gamma) \to J(\Gamma)$: if $\xi \in J(\Gamma)$ and $(P_1, \dots, P_{2h}) \in A^{-1}(\xi)$, then $\hat{\sigma}(\xi) = A(\sigma(P_1)) + \dots + A(\sigma(P_{2h}))$. If Q_1 $(\sigma(Q_1) = Q_1)$ is taken as the initial point of the Abel mapping, then $\hat{\sigma}$ takes the form

$$\hat{\sigma}((z_1, \dots, z_{2h})) = (-z_{h+1}, \dots, -z_{2h}, -z_1, \dots, -z_h).$$

DEFINITION. The *Prym variety* $P(\Gamma, \sigma)$ is the subvariety of $J(\Gamma)$ distinguished by the equation $\hat{\sigma}(\xi) = -\xi$ for $\xi \in P(\Gamma, \sigma)$.

The Prym variety is itself an Abelian variety obtained by factoring $\mathbf{C}^h$ by the period lattice $\{2\pi i N + \Pi N' \colon N, N' \in \mathbf{Z}^h\}$ where Π is the Riemann matrix of the Prym variety:

$$\Pi_{ij} = \int_{b_i} (\omega_j + \omega_{j+h}), \qquad 1 \le i, j \le h.$$

The matrix Π is symmetric, $\operatorname{Re}\Pi < 0$, and it is connected with B and B_0 in a rather simple manner [10]:

$$B = \begin{bmatrix} \frac{1}{2}(B_0 + \Pi) & \frac{1}{2}(\Pi - B_0) \\ \frac{1}{2}(\Pi - B_0) & \frac{1}{2}(B_0 + \Pi) \end{bmatrix}. \tag{9}$$

We define imbeddings $\varphi\colon P(\Gamma, \sigma) \to J(\Gamma)$, $\varphi((z_1, \dots, z_h)) = (z_1, \dots, z_h, z_1, \dots, z_h)$, and $\pi^*\colon J(\Gamma/\sigma) \to J(\Gamma)$, $\pi^*((z_1, \dots, z_h)) = (z_1, \dots, z_h, -z_1, \dots, -z_h)$ and a mapping $\eta\colon \Gamma \to P(\Gamma, \sigma)$ of the form

$$(\eta(P))_k = \int_{Q_1}^{P} (\omega_k + \omega_{k+h}), \qquad 1 \le k \le h.$$

ASSERTION 1 [10]. *If $\theta(e|\Pi) \neq 0$, then the divisor of zeros D of the function $\theta(\eta(P) - e|\Pi)$ satisfies*

$$A(D) = [\varphi(e) + A(Q_1) + A(Q_2) - \pi^*(K_0)] \pmod{\Lambda}, \tag{10}$$

where K_0 is the vector of Riemann constants of the curve Γ/σ computed on the basis of the Abel mapping with initial point $\pi(Q_1)$.

We shall construct an eigenfunction of an operator L of the form (5) on the basis of data satisfying (4). Let Ω_j $(j = 1, 2)$ be a differential of the second kind on Γ having a single pole at Q_j with principal part $dk_j + \cdots$ and uniquely determined by the condition that its integrals over a-cycles be equal to zero. It is obvious that $\sigma^*\Omega_j = -\Omega_j$, and hence

$$\int_{b_k} \Omega_1 = \int_{b_{k+h}} \Omega_1 = U_k, \qquad \int_{b_k} \Omega_2 = \int_{b_{k+h}} \Omega_2 = V_k,$$

$1 \le k \le h$. We define ψ by

$$\psi(z, \bar{z}, P) = \exp\left[z\left(\int_{Q_0}^{P} \Omega_1 - \alpha\right) + \bar{z}\int_{Q_1}^{P} \Omega_2\right] \frac{\theta(\eta(P) + Uz + V\bar{z} - e|\Pi)\theta(e|\Pi)}{\theta(Uz + V\bar{z} - e|\Pi)\theta(\eta(P) - e|\Pi)}, \tag{11}$$

where $Q_0 \in \Gamma\backslash(Q_1 \cup Q_2)$, $\int_{Q_0}^{P} \Omega_1 - \alpha \sim k_1 + O(k_1^{-1})$ as $P \to Q_1$; the integral in the definition of $\eta(P)$ and in the argument of the exponential is chosen in a consistent manner that is achieved by fixing the path from Q_0 to Q_1 in the definition of α and making it possible to assign a meaning to the expression $\int_{Q_1}^{P} \Omega_1$.

For $e \in P(\Gamma, \sigma)$ of general position the function ψ has a divisor of poles D (not depending on z) which is nonspecial and satisfies (10) and (4). The asymptotics of ψ as $P \to Q_1$, Q_2 have the form (2) with $a^2(z, \bar{z}) \equiv 1$ [7]. We therefore have

ASSERTION 2 [7]. *The function ψ of (11) is an eigenfunction of the operator $L = \partial\bar{\partial} + v$ where*

$$v(z, \bar{z}) = 2\partial\bar{\partial} \ln \theta(Uz + V\bar{z} - e|\Pi) + c, \qquad c = \text{const}. \tag{12}$$

C. One-dimensional finite-zone potentials as two-dimensional potentials. Let $H = -\partial_{xx}^2 + v(x)$ be a finite-gap operator with periodic potential $v(x + T) = v(x)$. The Bloch eigenfunction $\varphi(x, w, E)$ $(H\varphi = E\varphi,\ \varphi(0, w, E) = 1$, and $\varphi(x + T, w, E) = \mu(w, E)\varphi(x, w, E))$ is meromorphic on the Riemann surface $\Gamma = \{(w, E)\colon w^2 + \prod_{j=0}^{2g}(E - E_j) = 0\}$ with an excised infinitely distant point. By adding a constant to v we arrange that all $E_j \neq 0$. The function $\psi(x, y, w, \lambda) = e^{\lambda y}\varphi(x, w, \lambda^2)$ is meromorphic on the surface $\hat{\Gamma} = \{(w, \lambda)\colon w^2 + \prod_{j=0}^{2g}(\lambda^2 - E_j) = 0\}$

with the pair of excised infinitely distant points Q_1 and Q_2 in neighborhoods of which we can take $k^{-1} = (i\lambda)^{-1}$ as local parameters. ψ has the asymptotics

$$\psi \sim e^{k(x+iy)}(1 + O(k^{-1})), \qquad P \to Q_1;$$
$$\psi \sim e^{k(x-iy)}(1 + O(k^{-1})), \qquad P \to Q_2.$$

The involution $\sigma((w, \lambda)) = (-w, -\lambda)$, $\sigma(Q_j) = Q_j$, $\sigma(k_j) = -k_j$, $j = 1, 2$, acts on $\widehat{\Gamma}$. We denote by D the divisor of poles of ψ. The divisor $D+\sigma(D)-Q_1-Q_2$ is the divisor of zeros and poles of the form $d\lambda/\chi(\lambda, w)$, where $\chi(\lambda, w) = w/\prod_1^g(\lambda^2-E_j)$ and E_j are the E-coordinates of the poles of φ.

On $\widehat{\Gamma}$ and $\widehat{\Gamma}/\sigma$ we can introduce a coordinated pair of canonical bases of 1-cycles (as in subsection B) so that the projection of half of the basis onto the quotient space under the action of ν $(\nu((w, \lambda)) = (w, -\lambda))$ forms a canonical basis $p(a_1), \dots, p(a_g)$, $p(b_1), \dots, p(b_g)$ of the curve $\widehat{\Gamma}/\nu = \Gamma$ $(p: \widehat{\Gamma} \to \Gamma$ is the projection). Such bases can be chosen simply: when all the E_j are in $\mathbf{R}$ it suffices to take for a-cycles on Γ g real ovals and extend this collection to a canonical basis; a basis on $\widehat{\Gamma}$ is obtained from this by lifting to the covering space $p: \widehat{\Gamma} \to \Gamma$. To the basis on $\widehat{\Gamma}$ there correspond a basis of holomorphic differentials $\omega_1, \dots, \omega_{2g}$ and a basis of holomorphic Prym differentials (i.e., such that $\sigma^*\omega = -\omega$) $u_1 = \omega_1 + \omega_{g+1}, \dots, u_g = \omega_g + \omega_{2g}$ uniquely determined by the normalization

$$\int_{a_j} u_k = \int_{a_{j+g}} u_k = 2\pi i\delta_{jk}, \qquad 1 \le j \le g.$$

We note that this condition is satisfied by the differentials $p^*(\omega'_1), \dots, p^*(\omega'_g)$, where $\omega'_1, \dots, \omega'_g$ is a basis of holomorphic 1-forms on Γ dual to the basis $p(a_1), \dots, p(b_g)$ in $H_1(\Gamma)$, $\int_{p(a_j)} \omega'_k = 2\pi i\delta_{jk}$. Hence, $\int_{b_k} u_j = \int_{p(b_k)} \omega'_j$, and the Prym variety $P(\widehat{\Gamma}, \sigma)$ is isomorphic to the Jacobian variety of the curve Γ.

If Ω is a differential of the second kind on Γ with a single pole at $p(Q_1) = p(Q_2)$ having principal part $dw+\cdots$ and normalized by equating the integrals over a-cycles to zero, then Ω_1 and Ω_2 defined in subsection B are such that

$$p^*(\Omega) = \Omega_1 + \Omega_2, \qquad U_j = V_j = \tfrac{1}{2}\int_{p(b_j)} \Omega.$$

In summary, we have

Assertion 3. *The potential operator constructed on the basis of the "spectral data" $\{\widehat{\Gamma}, \sigma, Q_1, Q_2, k_1^{-1}, k_2^{-1}, D\}$ satisfying* (4) *(D is in general position) has the form*

$$L = \partial\overline{\partial} + \tfrac{1}{2}\partial^2_{xx} \ln \theta((U + V)x - e) + \text{const}$$

and its theta-function expression coincides exactly with the analogous expression obtained in solving the one-dimensional inverse problem [1].

§2. Finite-gap solutions of the Veselov-Novikov equations

Let $\{\Gamma, \sigma, Q_1, Q_2, k_1^{-1}, k_2^{-1}, D\}$ be the "spectral data" of the two-dimensional potential operator $L = \partial\overline{\partial} + v$. On the basis of them and the polynomials $q_1(k) = kz + k^3t_1$ and $q_2(k) = k\overline{z} + k^3t_2$ we construct the Baker-Akhiezer function

$\psi(z, \bar{z}, t_1, t_2, P)$. It has the asymptotics

$$\begin{aligned} \psi &\sim e^{k_1 z + k_1^3 t_1}(1 + \xi_1^+ k_1^{-1} + \xi_2^+ k_1^{-2} + O(k_1^{-3})), \qquad P \to Q_1; \\ \psi &\sim e^{k_2 \bar{z} + k_2^3 t_2} a(1 + \xi_1^- k_2^{-1} + O(k_2^{-2})), \qquad P \to Q_2. \end{aligned} \tag{13}$$

According to [7], $a^2 = 1$. The explicit theta formula is

$$\begin{aligned} \psi = &\left[z\left(\int_{Q_0}^P \Omega_1 - \alpha\right) + \bar{z}\int_{Q_1}^P \Omega_2 + t_1\left(\int_{Q_0}^P \Omega_3 - \beta\right) + t_2 \int_{Q_1}^P \Omega_4 \right] \\ &\times \frac{\theta(\eta(P) + Uz + V\bar{z} + W^+ t_1 + W^- t_2 - e|\Pi)\theta(e|\Pi)}{\theta(Uz + V\bar{z} + W^+ t_1 + W^- t_2 - e|\Pi)\theta(\eta(P) - e|\Pi)}, \end{aligned} \tag{14}$$

where Ω_1, Ω_2, U, V, and α are the same as in (11); Ω_j $(j = 3, 4)$ is a differential of second kind on Γ such that all its integrals over a-cycles are equal to zero, and it has a single pole at Q_{j-2} with principal part $dk_{j-2}^3 + \cdots$. The vectors W^+, $W^- \in \mathbf{C}^h$ (the genus of Γ is equal to $2h$) have the form

$$W_j^+ = \int_{b_j} \Omega_3, \qquad W_j^- = \int_{b_j} \Omega_4, \qquad 1 \le j \le h.$$

The integrals of Ω_1 and Ω_3 in the argument of the exponential are taken over the same paths, and $\int_{Q_0}^P \Omega_3 - \beta = dk_1^3 + \cdots$ as $P \to Q_1$.

LEMMA 1. *The function* ψ *of* (14) *satisfies*

$$(\partial/\partial t_1 - A_1)\psi = 0,$$

where $A_1 = \partial^3 + u\partial$, $u = -3\partial\xi_1^+$.

PROOF. We consider the operator

$$A_1 - \partial^3 - (3\partial\xi_1^+)\partial - 3(\partial\xi_2^+ + \partial^2\xi_1^+) + 3\xi_1^+\partial\xi_1^+.$$

The function $(\partial/\partial t_1 - A_1)\psi$ is the Baker-Akhiezer function with the same "spectral data" as ψ, and is hence a multiple of it. Now $e^{-k_1 z - k_1^3 t_1}(\partial/\partial t_1 - A_1)\psi \to 0$ as $P \to Q_1$, and hence $(\partial/\partial t_1 - A_1)\psi = 0$. As $P \to Q_2$,

$$(\partial/\partial t_1 - A_1)\psi \sim 3a(-\partial\xi_2^+ - \partial^2\xi_1^+ + \xi_1^+\partial\xi_1^+ \exp(k_2\bar{z} + k_2^3 t_2).$$

Since $a \ne 0$ it follows that $\partial\xi_2^+ + \partial^2\xi_1^+ - \xi_1^+\partial\xi_1^+ = 0$. The lemma is proved.
In a similar way we can prove

LEMMA 2. *The function* ψ *of* (14) *satisfies*

$$(\partial/\partial t_2 - A_2)\psi = 0,$$

where $A_2 = \bar{\partial}^3 + w\bar{\partial}$, $w = -3\bar{\partial}\xi_1^-$.

From (14) we find that

$$\begin{aligned} u(z, \bar{z}, t_1, t_2) &= 6\partial^2 \ln\theta(Uz + V\bar{z} + W^+ t_1 + W^- t_2 - e) + d_1, \\ w(z, \bar{z}, t_1, t_2) &= 6\bar{\partial}^2 \ln\theta(Uz + V\bar{z} + W^+ t_1 + W^- t_2 - e) + d_2, \end{aligned} \tag{15}$$

where d_1 and d_2 are constants depending on Q_1, Q_2, k_1^{-1}, and k_2^{-1}, and θ is the theta function of the Prym variety.

According to [5] and [7], for each fixed pair t_1, t_2 on the basis of $\psi(z, \bar{z}, t_1, t_2, P)$ it is possible to recover an operator $L = \partial\bar{\partial} + v$ such that $L\psi = 0$, i.e., we obtain a family of potentials, depending on t_1 and t_2,

$$v(z, \bar{z}, t_1, t_2) = 2\partial\bar{\partial} \ln \theta(Uz + V\bar{z} + W^+ t_1 + W^- t_2 - e). \tag{16}$$

We remark that $\bar{\partial} u = 3\partial v$ and $\partial w = 3\bar{\partial} v$.

If for L, A_1, A_2, $B_1 = A_1 + \partial u$, and $B_2 = A_2 + \bar{\partial} w$ we consider equations of the type of "L, A, B triples" [12]

$$\frac{\partial L}{\partial t_1} + LA_1 - B_1 L = 0, \qquad \frac{\partial L}{\partial t_2} + LA_2 - B_2 L = 0,$$

then we obtain equations for the evolution of v with respect to t_1 and t_2:

$$\frac{\partial v}{\partial t_1} = \partial^3 v + \partial(uv), \qquad 3\partial v = \bar{\partial} u; \tag{17}$$

$$\frac{\partial v}{\partial t_2} = \bar{\partial}^3 v + \bar{\partial}(wv), \qquad 3\bar{\partial} v = \partial w. \tag{18}$$

These are called the *Veselov-Novikov equations*. Coordinated deformation of v with respect to $t = t_1 = t_2$ is described by the equation

$$\frac{\partial v}{\partial t} = \partial^3 v + \bar{\partial}^3 v + \partial(uv) + \bar{\partial}(wv),$$

which was actually indicated in [5] and in the case $u = \bar{w}$ describes deformations preserving real potentials. These equations are the simplest in the entire hierarchy of equations found in [5], and were studied in [21].

As a result, we have proved

LEMMA 3. *The functions u, w, and v of* (15) *and* (16), *where e is of general position* (*in particular, $\theta(e|\Pi) \neq 0$*), *satisfy equations* (17), (18).

We consider the following problem.

If the symmetric $g \times g$ matrix B with negative definite real part, on the basis of which the theta function $\theta(z) = \theta(z|B)$ is constructed, and the vectors $U, V \in \mathbf{C}^g \backslash \{0\}$ and $W^+, W^- \in \mathbf{C}^g$ and the constants c, d_1, d_2 are such that for e of general position the theta formulas (15) and (16) give solutions of (17) and (18), then how are B, U, V, W^+, W^- and the constants c, d_1, d_2 related?

By general position we mean membership in an open dense set. Fulfillment of (17) and (18) for general position implies the relations

$$\partial_U[2\partial_{W^+}\partial_V \ln\theta - \partial_U^2(2\partial_U\partial_V \ln\theta) - (2\partial_U\partial_V \ln\theta + c)(6\partial_U^2 \ln\theta + d_1)] = 0, \tag{19}$$

$$\partial_V[2\partial_{W^-}\partial_U \ln\theta - \partial_V^2(2\partial_U\partial_V \ln\theta) - (2\partial_U\partial_V \ln\theta + c)(6\partial_V^2 \ln\theta + d_2)] = 0, \tag{20}$$

where σ_V, for example, is understood as the derivative in the direction V in $\mathbf{C}^g$. We note that (20) can be obtained from (19) by the changes $W^+ \to W^-$, $U \to V$, $V \to U$, and $d_1 \to d_2$.

We suppose that the Abelian variety $M = \mathbf{C}^g/\Lambda$, where $\Lambda = \{2\pi i N + BN' \colon N, N' \in \mathbf{Z}^g\}$ is irreducible, i.e., it is not the direct product of Abelian varieties of smaller dimension. In the space of moduli of principally polarized Abelian varieties this is a case of general position. It is known that if on an irreducible Abelian variety we have $\partial_U f = 0$, where $U \neq 0$ and f is meromorphic, then $f = \text{const}$ [3]. It therefore follows from (19) that the expression in square brackets is constant $(= a_1')$, which after transformation implies that

$$\begin{aligned}(\partial_V \partial_{W^+}\theta)\theta - \partial_V\theta\partial_{W^+}\theta - (\partial_U^3\partial_V\theta)\theta + \partial_U^3\theta\cdot\partial_V\theta \\ + 3\partial_U\theta\cdot\partial_U^2\partial_V\theta - 3\partial_U^2\theta\cdot\partial_U\partial_V\theta - 3c(\partial_U^2\theta)\theta \\ + 3c(\partial_U\theta)^2 - d_1(\partial_U\partial_V\theta)\theta + d_1\partial_U\theta\partial_V\theta = 2a_1\theta^2,\end{aligned} \tag{21}$$

where $2a_1 = a_1' + cd_1/2$.

A method of analyzing such relations was developed in [3]. The following two facts form the basis for it:

1) The addition theorem for theta functions:

$$\theta(z^1)\theta(z^2) = \sum \hat{\theta}[n](w^1)\hat{\theta}[n](w^2), \tag{22}$$

where the summation goes over $n \in \frac{1}{2}(\mathbf{Z}_2)^g$, $z^1 + z^2 = w^1$, $z^1 - z^2 = w^2$, and $\hat{\theta}[n](z) = \theta[n, 0](z|2B)$, where $\theta(z) = \theta(z|B)$.

2) The 2^g functions $\hat{\theta}[n](2z)$, $n \in \frac{1}{2}(\mathbf{Z}_2)^g$, are linearly independent.

We introduce the operators

$$\widetilde{X}_i = \sum U_j\partial/\partial z_j^i, \qquad \widetilde{Y}_i = \sum V_j\partial/\partial z_j^i, \qquad \widetilde{T}_i = \sum W_j^+\partial/\partial z_j^i,$$
$$X_i = \sum U_j\partial/\partial w_j^i, \qquad Y_i = \sum V_j\partial/\partial w_j^i, \qquad T_i = \sum W_j^+\partial/\partial w_j^i$$

($i = 1, 2$; the summation goes over $j = 1, \dots, g$). They are connected by relations of the form

$$\widetilde{A}_1 = A_1 + A_2, \qquad \widetilde{A}_2 = A_1 - A_2, \qquad A \in \{X, Y, T\}.$$

We rewrite (21) in the form

$$\begin{aligned}\{[\widetilde{T}_1\widetilde{Y}_1 - \widetilde{T}_2\widetilde{Y}_2 - \widetilde{X}_1^3\widetilde{Y}_1 + 3\widetilde{X}_1^2\widetilde{Y}_1\widetilde{X}_2 + \widetilde{X}_1^3\widetilde{Y}_2 - 3\widetilde{X}_1^2\widetilde{X}_2\widetilde{Y}_2 - 3c\widetilde{X}_1^2 \\ + 3c\widetilde{X}_1\widetilde{X}_2 - d_1\widetilde{X}_1\widetilde{Y}_1 + d_1\widetilde{X}_1\widetilde{Y}_2 - 2a_1]\theta(z^1)\theta(z^2)\}|_{z^1=z^2} = 0.\end{aligned} \tag{23}$$

We express $\widetilde{T}_i$, $\widetilde{X}_i$, and $\widetilde{Y}_i$ in terms of T_i, X_i, and Y_i and apply the operator in square brackets to the right side of the addition formula (22) for $w^1 = 2z$ and $w^2 = 0$. By the parity of $\hat{\theta}[n](w)$ it suffices to leave only terms with even powers of X_2, Y_2, and T_2. We obtain

$$\left[(Y_2T_2 - 4X_2^3Y_2 - d_1X_2Y_2 - 3cX_2^2 - a_1)\sum \hat{\theta}[n](w^1)\hat{\theta}[n](w^2)\right]\Big|_{w^1=2z,\, w^2=0} = 0$$

(summation over n), which by the linear independence of $\hat{\theta}[n](2z)$ implies that for all n

$$(\partial_V\partial_{W^+} - 4\partial_U^3\partial_V - d_1\partial_U\partial_V - 3c\partial_U^2 - a_1)\hat{\theta}[n](0) = 0. \tag{24}$$

Similarly, for all n we obtain

$$(\partial_U\partial_{W^-} - 4\partial_V^3\partial_U - d_2\partial_U\partial_V - 3c\partial_V^2 - a_2)\hat{\theta}[n](0) = 0. \tag{25}$$

We have thus proved

THEOREM 1. *If the Abelian variety* $M = \mathbf{C}^g/\{2\pi i N + BN' : N, N' \in \mathbf{Z}^g\}$ *is irreducible and for* $e \in \mathbf{C}^g$ *of general position the functions* v, u, *and* w *of the form* (15), (16), *where* $\theta(z) = \theta(z|B)$, *are solutions of* (17) *and* (18), *then relations* (24) *and* (25) *are satisfied (for all* $n \in \frac{1}{2}(\mathbf{Z}_2^g)$), *where* a_1 *and* a_2 *are constants.*

Relations (24) and (25) are invariant under the transformations

I) $U \to \lambda U$, $V \to \mu V$, $W^+ \to \lambda^3 W^+$, $W^- \to \mu^3 W^-$,

$$c \to \lambda\mu c,\ d_2 \to \lambda^2 d_1,\ d_2 \to \mu^2 d_2,\ a_1 \to \lambda^3\mu a_1,\ a_2 \to \lambda\mu^3 a_2, \qquad \lambda, \mu \in \mathbf{C}\backslash\{0\}; \tag{26}$$

II) $W^+ \to \alpha U$, $W^- \to W^- + \beta V$, $d_1 \to d_1 + \alpha$, $d_2 \to d_2 + \beta$,

$$\{U, V, c, a_1, a_2\} \to \{U, V, c, a_1, a_2\}, \qquad \alpha, \beta \in \mathbf{C}.$$

Using (26) to reduce (24) and (25) to the form $d_1 = d_2 = 0$, we obtain

THEOREM 2. *If the Prym variety* $P(\Gamma, \sigma)$ *is irreducible, then there exist vectors* $U, V \in \mathbf{C}^h\backslash\{0\}$ *and* $W^+, W^- \in \mathbf{C}^h$ *and constants* c, a_1, a_2 *such that for them and the theta function of the Prymian* ($\dim P(\Gamma, \sigma) = h$) *the following relations are satisfied*:

$$(\partial_V\partial_{W^+} - 4\partial_U^3\partial_V - 3c\partial_U^2 - a_1)\hat{\theta}[n](0) = 0,$$
$$(\partial_U\partial_{W^-} - 4\partial_V^3\partial_U - 3c\partial_V^2 - a_2)\hat{\theta}[n](0) = 0, \qquad n \in \tfrac{1}{2}(\mathbf{Z}_2^h). \tag{27}$$

The following criterion of irreducibility is important for effective computations.

CRITERION FOR IRREDUCIBILITY [12]. *An Abelian variety*

$$M = \mathbf{C}^g/\{2\pi i N + BN' : N, N' \in \mathbf{Z}^g\}$$

is irreducible if and only if the rank of the matrix

$$(\hat{\theta}[n]\hat{\theta}_{11}[n] \cdots \hat{\theta}_{ij}[n] \cdots \hat{\theta}_{gg}[n])$$

$$\left(\hat{\theta}[n](z) = \theta[n, 0](z, 2B),\ \hat{\theta}_{ij}[n] = \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\hat{\theta}[n](0),\ \hat{\theta}[n] = \hat{\theta}[n](0)\right)$$

is maximal, i.e., is equal to $g(g+1)/2 + 1$.

We remark that the Jacobi varieties of algebraic curves are irreducible (Mertens' theorem).

THEOREM 3. *If the conditions of Theorem* 1 *are satisfied,* $g \geq 2$, *and the vectors* U *and* V *are linearly independent, then* W^+, W^-, c, d_1, *and* d_2 *are uniquely determined on the basis of* B, U, *and* V *up to the transformations* (26.II) *relative to which the constant* c *is fixed.*

PROOF. It suffices to demonstrate uniqueness of the recovery of W^+, W^-, and c on the basis of B, U, and V for $d_1 = d_2 = 0$. Since an Abelian variety is irreducible, there exists a set of characteristics $n_1, \ldots, n_r$, $r = 1 + g(g+1)/2$, such that the matrix

$$\begin{pmatrix} \hat{\theta}[n_1]\hat{\theta}_{11}[n_1] & \cdots & \hat{\theta}_{ij}[n_1] & \cdots & \hat{\theta}_{gg}[n_1] \\ \vdots & & \vdots & & \vdots \\ \hat{\theta}[n_r]\hat{\theta}_{11}[n_r] & \cdots & \hat{\theta}_{ij}[n_r] & \cdots & \hat{\theta}_{gg}[n_r] \end{pmatrix}$$

is invertible, and hence the system

$$\sum_{i\le j} Q_{ij}\hat\theta_{ij}[n_k] + Q\hat\theta[n_k] = 4\partial_U^3\partial_V\hat\theta[n_k],$$

$1 \le k \le r$, is solvable, and Q_{ij} and Q are uniquely determined as polynomials in $U_1, \dots, U_g, V_1, \dots, V_g$, which we denote by $Q_{ij}(U, V)$ and $Q(U, V)$. The system (24) implies that

$$\begin{cases} V_1W_1^+ - 3cU_1^2 = Q_{11}(U, V), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ V_gW_g^+ - 3cU_g^2 = Q_{gg}(U, V), \\ W_1^+V_2 + W_2^+V_1 - 6cU_1U_2 = Q_{12}(U, V), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ W_i^+V_j + W_j^+V_i - 6cU_iU_j = Q_{ij}(U, V), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ W_g^+V_{g-1} + W_{g-1}^+V_g - 6cU_{g-1}U_g = Q_{(g-1)g}(U, V), \end{cases}$$

$1 \le i < j \le g$. According to Theorem 1, this system is consistent. We can easily see that it is uniquely solvable for W^+ and c if U and V are linearly independent. Unique recovery of W^- and c for $d_2 = 0$ follows in a similar way from (25). The theorem is proved.

REMARKS. 1) For $g = 2$ all irreducible Abelian varieties are Jacobi varieties of algebraic curves, while U and V can be any nonzero vectors. The Jacobi varieties of hyperelliptic curves are the Prymians of the type we consider (§1.B). Theorems 2 and 3 imply the existence of a direct procedure for constructing finite-gap solutions of (17) and (18) on the basis of the set $\{B, U, V\}$. Explicit formulas and the identities for the theta constant following from them are presented in [9].

2) If $U = V$, then the potential (16) for fixed t_1 and t_2 depends only on $x = \operatorname{Re} z$, i.e., it is one-dimensional, and from §1.B it follows that c is not uniquely determined, i.e., in this case the assertion of Theorem 3 is not true.

3) Under the conditions of Theorem 3 it is possible to find sufficient conditions on B, U, and V for the potentials to be smooth and real. For example, if B is real and either $U = \overline{V}$ and $e \in \mathbf{R}^g$, or $U = -\overline{V}$ and $e \in i\mathbf{R}^g$, the potential is smooth and real for $t_1 = t_2$. Sufficient conditions for smoothness and realness in the language of "spectral data" were found in [22] with use of the sufficient conditions for realness found in [7].

4) The problem of whether the conditions (4) are necessary for (1) to be a potential operator in the class of periodic operators constructed on the basis of nonsingular curves was solved independently by the author [23] and I. M. Krichever [24].

3. The Prym mapping

On each Abelian variety $\widetilde{M}$, i.e., on an algebraic variety diffeomorphic to a real torus of even dimension, there exists at least one Hodge form ω, and each Hodge form on it represents the first Chern class of a positive linear fibering L which is uniquely determined by the form up to translation. The homology class of the Hodge form is called the *polarization*. A polarization is called *principal* if $\dim H^0(\widetilde{M}, O(L)) = 1$ [13].

The set of all symmetric $g\times g$ matrices with negative definite real part is called the Siegel half plane H_g. On the basis of any matrix $B \in H_g$ it is possible to construct an

Abelian variety $\mathbf{C}^g/\{2\pi i N + BN' : N, N' \in \mathbf{Z}^g\}$ with principal polarization defined by the form

$$\omega = \frac{1}{4\pi i}\sum_{k,l}(\operatorname{Re} B)^{-1}_{kl}\,dz_k \wedge d\bar{z}_l.$$

All principally polarized Abelian varieties can be obtained in this manner.

The modular Siegel group G_g is the group $\mathrm{Sp}(2g, \mathbf{Z})$, where $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{Sp}(2g, \mathbf{Z})$ if α, β, γ, and δ are integral $g \times g$ matrices and

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\alpha^t & \gamma^t\\ \beta^t & \delta^t\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

(t denotes the transpose). G_g acts on H_g:

$$B \to B' = 2\pi i(\alpha B + 2\pi i\beta)(\gamma B + 2\pi i\delta)^{-1}. \tag{28}$$

The theta functions θ and θ' constructed on the basis of B and B' are connected by the relation

$$\theta'(z') = \kappa \exp\left\{\frac{1}{2}\sum_{i\le j} z_i z_j \frac{\partial \ln \det M}{\partial B_{ij}} + \sum_i l_i z_i\right\}\theta(z+\lambda), \tag{29}$$

where $M = (\gamma B + 2\pi i\delta)$, $z' = 2\pi i z M^{-1}$, and κ, l_i, and λ do not depend on z [10]. The Abelian varieties $\mathbf{C}^g/\{2\pi i N + BN'\}$ and $\mathbf{C}^g/\{2\pi i N + B'N'\}$ constructed above are isomorphic if and only if B and B' are connected by the transformation (28). The quotient space H_g/G_g is the moduli space of principally polarized Abelian varieties of dimension g. After suitable compactification it becomes an irreducible algebraic variety A_g of dimension $g(g+1)/2$.

Suppose $\widehat{\Gamma} \to \Gamma$ is a two-sheeted covering, and $\widehat{\Gamma}$ and Γ are nonsingular. On $\widehat{\Gamma}$ there acts an involution σ permuting the sheets, and to the covering there corresponds a Prym variety Pr imbedded in $J(\widehat{\Gamma})$ by $i\colon \mathrm{Pr} \to J(\widehat{\Gamma})$. If $[\omega]$ is a principal polarization of $J(\widehat{\Gamma})$, then $i^*([\omega])$ is a polarization of Pr. The Prym variety can be principally polarized only for three classes of coverings:

1) $\Gamma = \mathbf{C}P^1$—the covering is hyperelliptic and $\mathrm{Pr} = J(\widehat{\Gamma})$;
2) the covering is unbranched; and
3) the covering has two branch points.

In cases 2) and 3) the class $\frac{1}{2}i^*([\omega])$ determines the principal polarization on the Prym variety.

Below we shall consider only coverings with two branch points. We denote by L_g the moduli space of double coverings of curves of genus g with two branch points. On it there is defined the Prym mapping $P\colon L_g \to A_g$ assigning to the covering a Prym manifold with polarization $\frac{1}{2}i^*([\omega])$.

The covering can be reconstructed from Γ and the branch points $Q_1, Q_2 \in \Gamma$ up to 2^{2g} possibilities, where g is the genus of Γ. Namely, the induced covering over $\Gamma\backslash(Q_1 \cup Q_2)$ is unbranched and is uniquely determined by the representation $\rho\colon \pi_1(\Gamma\backslash(Q_1 \cup Q_2)) \to \mathbf{Z}_2$ such that $\rho(a) = 0$ if the loop realizing a becomes disconnected on lifting to the covering space, and $\rho(a) = 1$ otherwise. There exists a collection $a_1, \dots, a_g, b_1, \dots, b_g, c_1, c_2$ of generators of $\pi_1(\Gamma\backslash(Q_1 \cup Q_2))$ with the single relation $[a_1, b_1]\cdots[a_g, b_g]c_1c_2 = 1$ such that c_1 and c_2 are realized by contours contracting on Γ to the points Q_1 and Q_2 respectively. For all coverings we have $\rho(c_1) = \rho(c_2) = 1$. The collection $\{\rho(a_1), \dots, \rho(b_g)\}$, which can be chosen arbitrarily, uniquely determines the covering.

From the general theory (see [15] and [16]) it follows that, for $g \geq 2$, L_g can be compactified up to an irreducible scheme of dimension $3g-1$ (the dimension of the moduli space of algebraic curves of genus g is equal to $3g-3$ plus two branch points). By the way, irreducibility follows from the existence of the following deformations. Suppose ρ_1 and ρ_2 are two representations defining the coverings l_1 and l_2 over Γ with branches at Q_1 and Q_2, and suppose the vectors $S_1 = (\rho_1(a_1), \dots, \rho_1(b_g))$ and $S_2 = (\rho_2(a_1), \dots, \rho_2(b_g))$. We consider the path $w\colon [0, 1] \to \Gamma \backslash Q_2$, $w(0) = w(1) = Q_1$, such that the vector of the intersection indices of the contour $w([0, 1])$ with the a-cycles and b-cycles modulo 2 $(w \circ a_1, \dots, w \circ b_g)$ is equal to $S_1 - S_2$. By deforming l_1 by means of a shift of the branch point along the path w, we reduce it to l_2.

CONJECTURE (S. P. Novikov). The equations obtained in solving (27) for B distinguish the closure of the range of the Prym mapping in A_g.

This conjecture is an analogue of Novikov's conjecture regarding the description of Jacobi varieties, in which the role of (17) is played by the Kadomtsev-Petviashvili equation. It was proved in [17] that the equations obtained distinguish in A_g a subvariety containing the closure of the family of Jacobians as an irreducible component. Irreducibility of this subvariety was proved in [18], and the problem of describing Jacobi varieties in terms of Abelian varieties was thus solved (the Riemann-Schottky problem).

We shall prove an analogue of Dubrovin's theorem [17] for the Novikov conjecture regarding Prymians. We consider the system

$$(\partial_V \partial_W - 4\partial_U^3 \partial_V - 3c\partial_U^2 - a)\hat{\theta}[n](0) = 0, \tag{30}$$

where $n \in \frac{1}{2}(\mathbf{Z}_2)^g$.

We introduce the variety X_g whose points are the collections (U, V, W, c, a, B), $U, V \in \mathbf{C}^g \backslash \{0\}$, $W \in \mathbf{C}^g$, $c, a \in \mathbf{C}$, $B \in H_g$, factored with respect to the action of the groups:

I. $U \to \lambda U$, $V \to \mu V$,

$$W \to \lambda^3 W,\ c \to \lambda\mu c,\ a \to \lambda^3 \mu a,\ B \to B; \qquad \lambda, \mu \in \mathbf{C}\backslash\{0\}; \tag{31}$$

II. The group $\mathrm{Sp}(2g, \mathbf{Z})$; B is transformed according to (29),

$$\begin{gathered} U \to 2\pi i U M^{-1}, \qquad V \to 2\pi i V M^{-1}, \qquad W \to 2\pi i (W + 6[z, z]) U M^{-1}, \\ a \to a + \tfrac{1}{2}[\bar{z}, t] - \tfrac{3}{2}[z, z](c - 2[z, \bar{z}]), \qquad c \to c - 2[z, \bar{z}], \end{gathered} \tag{32}$$

where M is the same as in (29), while

$$[x, y] = \frac{1}{2} \frac{\partial^2}{\partial x \partial y} \sum_{i \leq j} \xi_i \xi_j \frac{\partial \ln \det M}{\partial B_{ij}} \quad \text{for } \xi = Uz + V\bar{z} + Wt.$$

From (20) and (29) we immediately obtain

LEMMA 4. *Solutions of system* (30) *are invariant under* (31) *and* (32), *and therefore form a subvariety* Y_g *in* X_g.

The variety X_g after suitable compactification becomes algebraic, and Y_g is an algebraic subvariety [17].

Let $E_g \subset L_g$ be the subset of coverings with irreducible Prymians; it is open and dense, since reducible Prymians are distinguished by algebraic equations [12] and there exist irreducible Prymians (Jacobians of hyperelliptic curves). According to Lemma 3 and Theorem 1 there exists a canonical mapping $\Phi: E_g \to X_g$ which are actually constructed in §1.B, and $\Phi(E_g) \subset Y_g$.

LEMMA 5. *$\dim Y_g^0 \le 3g-1$, where Y_g^0 is the irreducible component of Y_g containing $\Phi(E_g)$.*

PROOF. Suppose $\Gamma \to \Gamma/\sigma$ is a covering, canonical bases of cycles are selected on Γ and Γ/σ as in §1, the Riemann matrices are $J(\Gamma)$ and $J(\Gamma/\sigma)$, and the Prymians are connected by relation (9). We deform Γ/σ by contracting the a-cycles to points. This deformation lifts to a deformation of Γ. The diagonal elements of the Riemann matrices $J(\Gamma)$ and $J(\Gamma/\sigma)$ (and hence the Prymians, according to (9)) tend to $-\infty$, while the remaining elements remain bounded [10].

Let $\varepsilon_i = \exp B_{ii}$ (B is the Riemann matrix of the Prymian). To estimate $\dim Y_g^0$ we solve (30) according to "perturbation theory" in a neighborhood of the variety $\{\varepsilon_1 = \cdots = \varepsilon_g = 0\}$. From (30) we chose equations corresponding to the characteristics: $n=0$; $n = n_{(p)} = e_p/2$, where $e_p = (0, \dots, 1, \dots, 0)$ (one at the pth place, $1 \le p \le g$); and $n = n_{(p,q)} = (e_p + e_q)/2$, $1 \le p \ne q \le g$. After renormalization the functions $\hat\theta[n](z)$ become analytic in a neighborhood of $\{\varepsilon = 0\}$, and as $\varepsilon \to 0$ they have the asymptotics [17]

$$\hat\theta[0](z) = 1 + O(\varepsilon); \qquad \frac{1}{2\sqrt[4]{\varepsilon_p}}\hat\theta[n_{(p)}](z) = \cosh\frac{z_p}{2} + O(\varepsilon), \tag{33}$$

$$\frac{\sqrt{\zeta_{pq}}}{2\sqrt[4]{\varepsilon_p\varepsilon_q}}\hat\theta[n_{(p,q)}](z) = \zeta_{pq}\cosh\frac{z_p+z_q}{2} + \cosh\frac{z_p-z_q}{2} + O(\varepsilon),$$

where $\zeta_{pq} = \exp B_{pq}$.

The left sides of (30) with the corresponding renormalization we denote by $f[n]$, and we seek a, W_i, and ζ_{ij} in the form of series in powers of $\varepsilon_1, \dots, \varepsilon_g$.

From $f[0] = 0$ it follows that $a = 0 + O(\varepsilon)$.

We require that $V_1 \cdots V_g \ne 0$. From $f[n_{(p)}] = 0$ we obtain

$$W_p = (U_p^3 + 3cU_p^2/V_p) + O(\varepsilon),$$

and from $f[n_{(p,q)}] = 0$ it follows that $\zeta_{pq} = d_{pq}e_{pq}^{-1} + O(\varepsilon)$, where

$$\begin{aligned} d_{pq} &= V_pV_q(V_p - V_q)(U_p - U_q)U_pU_q - c(U_pV_q - U_qV_p)^2, \\ e_{pq} &= V_pV_q(V_p + V_q)(U_p + U_q)U_pU_q - c(U_pV_q - U_qV_p)^2. \end{aligned} \tag{34}$$

We compute the Jacobian matrix

$$\left(\frac{\partial f[n]}{\partial a}\,\frac{\partial f[n]}{\partial W_i}\,\frac{\partial f[n]}{\partial \zeta_{ij}}\right) \tag{35}$$

for $\varepsilon = 0$ (the arguments belong to Y_g^0):

$$\begin{matrix} n=0 \\ n = n_{(p)} \\ n = n_{(p,q)} \end{matrix} \qquad \begin{bmatrix} -1 & 0 & 0 \\ * & \delta_{ip}V_p & 0 \\ * & * & \delta_{ip}\delta_{jq}e_{pq} \end{bmatrix}.$$

It is lower triangular, and at a general point the diagonal elements are nonzero. From the complex-analytic implicit function theorem it follows that in a neighborhood of $\{\varepsilon = 0\}$ of the point Y_g^0 the values of $\varepsilon_1, \dots, \varepsilon_g$, $U_1, \dots, U_g$, $V_1, \dots, V_g$, and c are uniquely determined up to the transformations (31). The lemma is proved.

In [17] this method was used to obtain an exact value of $\dim Y_g^0$, since the lower bound for the dimension followed from Torelli's theorem for the Jacobi mapping. For the Prym mapping we shall prove a local Torelli theorem below with the use of Theorem 4, in whose proof we must use Osgood's theorem, which in turn is prepared for present use by Lemma 6.

LEMMA 6. *There exists a point $\omega \in \Phi(E_g)$ such that $\Phi^{-1}(\omega)$ consists of one point, and the matrix* (35) *is invertible at ω.*

PROOF. According to [17], §3.2, the subvariety $K_g \subset Y_g^0$ given by the equations $U = V$ and $c = 0$ contains a component K_g^0 abutting $\{\varepsilon = 0\}$ and parametrizing the families of one-dimensional finite-zone potentials; the projection of K_g^0 onto A_g has dimension $2g - 1$ and is formed by the Jacobians of hyperelliptic curves. According to Assertion 3, $K_g^0 \subset \overline{\Phi(E_g)}$, and the points of K_g^0 of general position belong to $\Phi(E_g)$. We find rational $q_1, \dots, q_g$ and sufficiently small $\varepsilon_1, \dots, \varepsilon_g$ such that at the point $\omega \in K_g^0$ given by the coordinates $U_k = 2\pi i q_k$ $(1 \le k \le g)$, $\varepsilon_1, \dots, \varepsilon_g$ the matrix (35) is invertible, and $\omega \in \Phi(E_g)$ ($\Phi(E_g) \cap K_g^0$ contains an open dense subset of K_g^0).

To the point ω there corresponds an e-family of potentials

$$u(x) = 2\partial_{xx}^2 \ln \theta(Ux + e).$$

Since $U \in 2\pi i \mathbf{Q}^g$, the U-winding of the Abelian variety forms a circle imbedded in it, and hence it is possible to choose e so that the circle $\{Ux + e\}$ does not intersect the theta divisor. Such a potential will be smooth and periodic. If $\tilde{\omega} \in \Phi^{-1}(\omega)$, then on the basis of $\tilde{\omega}$ it is possible to construct the Bloch function (11) which decomposes into the product

$$\psi(x, P)\exp(\lambda(P)y), \quad \text{where } (\partial_{xx}^2 + u(x) + \lambda^2(P))\psi(x, P) = 0,$$

i.e., we are in the situation studied in §1.B, from which it now follows that $\tilde{\omega} = \Phi^{-1}(\omega)$. The lemma is proved.

THEOREM 4. *In a neighborhood of a point in general position the mapping $\Phi\colon E_g \to Y_g^0$, $g \ge 2$, is a local diffeomorphism.*

PROOF. We recall Osgood's theorem [19]: suppose f is a holomorphic mapping of a neighborhood of a point $a \in \mathbf{C}^n$ into $\mathbf{C}^n$ and a is an isolated point of the set $f^{-1}(b)$, where $f(a) = b$; then in some neighborhood of a the mapping f is a covering (possibly branched) over the image.

The moduli space R_g of curves of genus g is the quotient space of the Teichmüller space T_g modulo the action of a discontinuous group. Let ω be the point of $\Phi(E_g)$ found in Lemma 6, and let $\tilde{\omega} = \Phi^{-1}(\omega)$ be the covering $\Gamma \to \Gamma/\sigma$.

We fix a pair of consistent (§1) canonical bases of 1-cycles on Γ and Γ/σ, i.e., points $S \in T_g$ and $\widetilde{S} \in T_{2g}$ actually corresponding to Γ and Γ/σ. In a neighborhood of $\tilde{\omega}$ the mapping $\widetilde{\Phi}$ is defined, which assigns to coverings equivalence classes (the pair of canonical bases can be extended continuously to this neighborhood) (B, U, V, W, c, a) modulo the action (31). If Φ is defined on $E_g \subset L_g$ and L_g is a 2^{2g}-sheeted covering of R_g, then $\widetilde{\Phi}$ is defined on a subset of 2^{2g}-covering of T_g, which in contrast to R_g is nonsingular, and we can therefore apply Osgood's theorem to $\widetilde{\Phi}$ for $a = (\widetilde{S}, \sigma)$ and $b = \widetilde{\Phi}(a)$, and find that the mapping $\widetilde{\Phi}$ factored through E_g without loss of dimension is a local diffeomorphism at a. Since E_g is the complement of an analytic subset of positive codimension a, E_g is connected, and hence Φ is a local diffeomorphism in a neighborhood of a general point. The theorem is proved.

From Theorem 4 we obtain

THEOREM 5. *The projection of Y_g^0 onto A_g coincides with the closure of the range of the Prym mapping $\overline{P(L_g)}$.*

Only here have we used the compactification of L_g to a scheme; otherwise, we could only assert that the projection of Y_g^0 onto A_g contains the image of the Prym mapping as an open subset. Theorem 5 is an analogue of Dubrovin's theorem [17] and provides a local solution of the problem of Riemann-Schottky type for the Prym mapping.

We shall compute the rank of the Jacobian matrix of the Prym mapping at a general point. According to Theorem 4, for this it suffices to compute the rank of the projection $\pi: Y_g^0 \to A_g$ at a general point; we compute it in a neighborhood of $\{\varepsilon = 0\}$.

LEMMA 7. *The rank of the Jacobian matrix of the projection $\pi: Y_g^0 \to A_g$ at a point in general position is maximal.*

PROOF. From the proof of Lemma 5 and from Theorem 4 it follows that in a neighborhood of $\{\varepsilon = 0\}$ the collections $(\varepsilon_1, \dots, \varepsilon_g, U_1, \dots, U_g, V_1, \dots, V_g, c)$ taken modulo (31) parametrize neighborhoods of points of Y_g^0 in general position. We consider the region $U_1 V_1 \neq 0$ and in them we normalize U and V by the conditions $U_1 = V_1 = 1$. We restrict π to this region. As local coordinates in A_g we take $\varepsilon_i = \exp B_{ii}$, $\zeta_{ij} = \exp B_{ij}$, $1 \leq i < j \leq g$. We shall compute the rank of the Jacobian matrix modulo $O(\varepsilon)$. It is equal to $g + \operatorname{rank}(A \bmod O(\varepsilon))$, where the matrix A is

$$\begin{pmatrix} \dfrac{\partial \zeta_{12}}{\partial U_2} & \cdots & \dfrac{\partial \zeta_{12}}{\partial V_g} & \dfrac{\partial \zeta_{12}}{\partial c} \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial \zeta_{(g-1)g}}{\partial U_2} & \cdots & \dfrac{\partial \zeta_{(g-1)g}}{\partial V_g} & \dfrac{\partial \zeta_{(g-1)g}}{\partial c} \end{pmatrix}.$$

According to (34) for $c = 0$ we have the formulas $(\bmod O(\varepsilon))$

$$\frac{\partial \zeta_{qp}}{\partial U_p} = 2\frac{(V_p^2 - V_q^2)U_q}{[(V_p + V_q)(U_p + U_q)]^2}; \qquad \frac{\partial \zeta_{qp}}{\partial V_p} = 2\frac{(U_p^2 - U_q^2)V_q}{[(V_p + V_q)(U_p + U_q)]^2}; \tag{36}$$
$$\frac{\partial \zeta_{qp}}{\partial U_r} = \frac{\partial \zeta_{qp}}{\partial V_r} = 0 \quad \text{for } r \notin \{p, q\};$$
$$\frac{\partial \zeta_{qp}}{\partial c} = -2\frac{(U_p V_q - V_p U_q)^2 (V_p U_q + V_q U_p)}{[(V_p + V_q)(U_p + U_q)]^2 V_p V_q U_p U_q}.$$

At a generic point, as we show, $\operatorname{rank}(A \bmod O(\varepsilon))$ is maximal and for $g \geq 5$ is equal to $2g-1$, while for $2 \leq g \leq 4$ it is equal to $g(g-1)/2$. In each case we simply indicate such a point and the maximal square minor of the matrix $A \bmod O(\varepsilon)$.

a) $g = 2$. For $c = 0$, $U = (1, -2)$, and $V = (1, 2)$ we have

$$\partial \zeta_{12}/\partial U_2 \bmod O(\varepsilon) = 1.$$

b) $g = 3$. For $c = 0$, $U = (1, -2, 3)$, and $V = (1, 2, 3)$ the minor formed by the derivatives ζ_{12}, ζ_{13}, and ζ_{23} with respect to V_2, V_3, and U_3 has rank 3 for sufficiently small ε.

c) $g = 4$. For $c = 0$, $U = (1, -2, 3, 1)$, and $V = (1, -2, 3, 3)$ the minor formed by the derivatives of ζ_{ij} with respect to U_2, U_3, U_4, V_2, V_3, and V_4 has rank 6 for sufficiently small ε.

d) $g = 5$. For $c = 0$, $U = (1, -2, 3, 1, 1)$, and $V = (1, 2, 3, 3, -4)$ the minor formed by the derivatives ζ_{ij}, $1 \leq i < 4$, $i < j$, has rank 9 for sufficiently small ε.

e) $g \geq 6$. Suppose $c = 0$, $U = (1, -2, 3, 1, 1, U_6, \ldots, U_g)$, $V = (1, 2, 3, 3, -4, V_6, \ldots, V_g)$, and $U_i(U_i + U_j)V_i(V_i + V_j) \neq 0$, $1 \leq i, j \leq g$. From (36) it follows that

$$\det\left[\begin{pmatrix} \frac{\partial \zeta_{1k}}{\partial U_k} & \frac{\partial \zeta_{2k}}{\partial U_k} \\ \frac{\partial \zeta_{1k}}{\partial V_k} & \frac{\partial \zeta_{2k}}{\partial V_k} \end{pmatrix} \bmod O(\varepsilon)\right] = \frac{R(U_k, V_k)}{S(U_k, V_k)},$$

where the constant term of the polynomial $R(U_k, V_k)$ is equal to 16 and

$$S(U_k, V_k) = (V_k + 1)^2 (U_k + 1)^2 (V_k + 2)^2 (U_k + 2)^2.$$

If $U_6, \ldots, U_g$ and $V_6, \ldots, V_g$ are sufficiently small, then for $k \geq 6$ we have $R(U_k, V_k) S(U_k, V_k) \neq 0$. Then

$$\operatorname{rank}(A \bmod O(\varepsilon)) = \operatorname{rank}(M \bmod O(\varepsilon)) = 2g - 1,$$

where M is the minor formed by the derivatives ζ_{ij}, $1 \leq i < 4$, $i < j \leq 5$, and ζ_{1k} and ζ_{2k}, $6 \leq k \leq g$.

Since by hypothesis Y_g^0 is connected and irreducible, from this we obtain the assertion of the lemma. By $f \bmod O(\varepsilon)$ we mean the constant term of the expansion of f in powers of $\varepsilon_1, \ldots, \varepsilon_g$. The lemma is proved.

From Lemma 7 and Theorem 4 we obtain

THEOREM 6. *The rank of the Jacobian matrix of the Prym mapping* $P\colon L_g \to A_g$ *is maximal at a point in general position, and for* $g \geq 5$ *in a sufficiently small neighborhood of a point of general position in* L_g *the covering can be reconstructed uniquely from the Prym variety.*

We have thus proved Torelli's theorem for the given Prym mapping at a generic point; the essential fact is that for all $g \geq 2$ there exist smooth families of coverings with isomorphic Prym varieties [20].

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