

NONSELFINTERSECTING CLOSED EXTREMALS OF MULTIVALUED OR NOT EVERYWHERE POSITIVE FUNCTIONALS

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ABSTRACT. A proof is given for the theorem of Novikov and the author on the existence of a closed nonselfintersecting extremal for a single-valued functional corresponding to the motion of a charged particle in a strong magnetic field on a Riemannian manifold homeomorphic to the 2-sphere, and an analogue in the case of multivalued functionals is also proved.

§0. INTRODUCTION

Those periodic solutions of the Euler-Lagrange equations

$$(1) \quad \partial L / \partial x^i = (\partial L / \partial \dot{x}^i)'$$

with nonuniquely determined Lagrangian $L(x, \dot{x})$ (corresponding to the Kirchhoff equations) that lie on a given energy level are the critical points of the multivalued functionals on the space of closed curves on the 2-sphere that arise in the theory of the Kirchhoff equations for the free motion of a rigid body in a liquid. The classical Morse-Lyusternik-Schirelmann theory developed in [1]–[5] for length functionals turned out not to be applicable for proving the existence of critical points of such functionals and required an extension, which was dealt with in a cycle of papers by Novikov, Shmel'tser, and the author (see [6]–[12]), who considered not only multivalued but also not everywhere positive functionals. The topological aspect of the questions studied was treated thoroughly enough in these articles; however, as S. V. Bolotin has noted, convergence of the "gradient" deformations was not proved, with the exception of the paper [11] by Novikov and the author, where for proving convergence for single-valued functionals on the 2-sphere a method was presented that leads to the following result (see also the survey [12], where important questions related to this method are considered): if the single-valued functional

$$l(\gamma) = \int_{\gamma} \left(\sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} + A_i(x)\dot{x}^i \right) dt$$

on the space of closed contours on S^2 is not everywhere positive, then there exists a closed nonselfintersecting extremal.

The method of [11] turned out not to be applicable to multivalued functionals, and a completely rigorous proof of this theorem on the basis of the method was not given, since another approach also applicable to multivalued functionals was found. This paper is devoted to the presentation of a proof. Here we prove the existence of a closed nonselfintersecting extremal both for single-valued (Theorem 2)

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and for multivalued functionals (Theorem 3) of the form indicated above for "strong" magnetic fields $F = dA$ on the 2-sphere; we arrive at the necessity of extending the Morse theory by replacing the space of closed contours by the space of films* (§4).

§1. STATEMENT OF THE PROBLEM

We consider the collection of piecewise smooth closed curves on a manifold M^n , i.e., the set of mappings $\gamma: [0, 1] \rightarrow M^n$ such that $\gamma(0) = \gamma(1)$ and γ is piecewise smooth, and we introduce a topology on it by the standard method of Morse theory [3]. The infinite-dimensional space thus obtained is denoted by $\widehat{\Omega}(M^n)$. The group G of piecewise smooth orientation-preserving homeomorphisms $S^1 \rightarrow S^1$ acts on it: $g(\gamma)(t) = \gamma(g(t))$, $\gamma \in \widehat{\Omega}$, $g \in G$. The quotient space of $\widehat{\Omega}(M^n)$ by the action of G is denoted by $P(M^n)$.

In the study [6] of the Kirchhoff equations describing the motion of a rigid body in a liquid it was discovered that they form a Hamiltonian system on the Lie algebra $E(3)$ of the group of motions of three-dimensional Euclidean space and have two first integrals f_1 and f_2 that are trivial in the sense that they are integrals of any Hamiltonian system on $E(3)$, and any level surface $\{f_1 = a \neq 0, f_2 = b\}$ is diffeomorphic to the tangent bundle over the 2-sphere S^2 . Restricting the Hamiltonian system to it, we get a system equivalent to the Lagrangian system (i.e., of the form (1)) with Lagrange function

$$(2) \quad L^{(\alpha)}(x, \dot{x}) = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j + A_i^{(\alpha)}(x) \dot{x}^i - U(x),$$

where for $b \neq 0$ the functions $L^{(\alpha)}$ are defined on the tangent bundles over domains $U_\alpha \subset S^2$ and cannot be extended to the whole tangent bundle. This has to do with the fact that in (1) the 1-form $A_i^{(\alpha)} dx^i$ appears in terms of its exterior derivative

$$F_{ij} dx^i \wedge dx^j = (\partial_i A_j - \partial_j A_i) dx^i \wedge dx^j.$$

For $b \neq 0$ the form F appearing in the "Kirchhoff equations" on TS^2 is closed but not exact—it becomes exact when it is restricted to any domain $U_\alpha \subset S^2$ with zero second real cohomology group (however, all proper subdomains with smooth boundaries are such domains in the situation $M^n = S^2$), and then there is a 1-form $A_i^{(\alpha)} dx^i$ defined in U_α such that $dA^{(\alpha)} = F$. In this situation the closed extremals of the Euler-Lagrange equations will be critical points of the multivalued functionals on $\widehat{\Omega}(S^2)$: $\gamma \subset S^2 \setminus x_\alpha$, $x_\alpha \in S^2$, the 1-form $A^{(\alpha)}$ and thus the Lagrangian (2) are defined in $U_\alpha = S^2 \setminus x_\alpha$, and the branch $S^{(\alpha)}$ of the functional takes at γ the value

$$S^{(\alpha)}(\gamma) = \int_\gamma L^{(\alpha)}(x, \dot{x}) dt.$$

We remark that on the intersection of the domains of the branches $S^{(\alpha)}$ and $S^{(\beta)}$ their variational derivatives coincide: $\delta S^{(\alpha)} = \delta S^{(\beta)}$.

The "energy"

$$H = \dot{x}^i \partial L / \partial \dot{x}^i - L$$

is a first integral of the Lagrangian system. According to the Maupertuis principle, the trajectories of a point whose motion is described by equations (1) for the Lagrangian (2) on the energy level E are trajectorywise isomorphic to extremals of the Euler-Lagrange equations for

$$(3) \quad L_E^{(\alpha)}(x, \dot{x}) = \sqrt{(E - U(x))} g_{ij} \dot{x}^i \dot{x}^j + A_i^{(\alpha)} \dot{x}^i.$$

* Editor's note. The Russian term is "пленка", but, as the author points out in §4 below, he means an oriented two-dimensional surface with nonempty boundary.

From the physical point of view the appearance of the expression $A_i dx^i$ in the Lagrangian is equivalent to the switching on of a magnetic field

$$F_{ij} dx^i \wedge dx^j = (\partial_i A_j - \partial_j A_i) dx^i \wedge dx^j,$$

which is a uniquely determined closed (by the Maxwell equations) but not necessarily exact 2-form. This paper is devoted to the use of topological methods for proving the existence of closed extremals of the Euler-Lagrange equations for the Lagrangian function (2)—periodic extremals of the functional for the “motion of a particle in a magnetic field” (we set the last expression in quotation marks in order to underscore that in real problems (the Kirchhoff equations, for example) the form F can fail to have the physical meaning of a magnetic field).

§2. FEATURES IN THE USE OF MORSE THEORY. THE METHOD OF THROWING OUT CYCLES

We proceed to the rigorous definitions. Let M^n be a smooth oriented manifold on which there are defined:

- 1) a Riemannian metric $g_{ij}(x) dx^i dx^j$, and
- 2) a closed 2-form (the “magnetic field”) $F = F_{ij} dx^i \wedge dx^j$.

The metric and the form F are assumed to be smooth.

In the case when F is not an exact form, i.e., is not an exterior derivative of a 1-form, we consider a covering $\{U_\alpha\}$ of M^n with the following properties:

1) Each closed piecewise smooth curve γ on M^n lies entirely in at least one of the elements U_α of the covering (the subset of $P(M^n)$ consisting of the curves lying in U_α is denoted by W_α).

2) The restriction of F to each element U_α of the covering is exact (i.e., if $j_\alpha: U_\alpha \rightarrow M^n$ is the imbedding, then there is a 1-form $A_i^{(\alpha)} dx^i$ on U_α such that $dA^{(\alpha)} = j_\alpha^*(F)$).

According to property 1), the family $\{W_\alpha\}$ forms a covering of $P(M^n)$. On each set W_α we define the functional

$$(4) \quad l_\alpha(\gamma) = \int_\gamma \left(\sqrt{g_{ij} \dot{x}^i \dot{x}^j} + A_i^{(\alpha)} \dot{x}^i \right) dt.$$

If $\gamma \in W_\alpha \cap W_\beta$, then $\delta l_\alpha(\gamma) = \delta l_\beta(\gamma)$. If F is exact, then the functional is single-valued: $l(\gamma) = l_\alpha(\gamma)$ for all α .

A study of possible ways to apply Morse theory to such functionals was begun in [6]–[9], where the restriction of the functional to the subspace $P_0(M^n) \subset P(M^n)$ formed by the contours contractible to a point was considered. In this case the difference between the functional (4) and a length functional is especially marked:

- 1) In the case when it is multivalued it has the form

$$l_\alpha(\gamma) = \int_\gamma \sqrt{g_{ij} \dot{x}^i \dot{x}^j} dt + \int_{\Pi_\alpha} F_{ij} dx^i \wedge dx^j,$$

where Π_α is a film spanning the contour γ , and the multivaluedness has to do with the choice of various equivalence classes of films corresponding to elements of $H_2(M^n)/\text{Torsion}$.

2) Even if $[F] \neq 0$ in $H^2(M^n; \mathbf{R})$, it is possible (for example, when M^n is a torus) that $[p^*F] = 0$ in $H^2(\widehat{M}^n; \mathbf{R})$, where $p: \widehat{M}^n \rightarrow M^n$ is the universal covering, and the branches of the functional l_α over the domains W_α can be glued together into a single-valued functional on $P_0(M^n)$, which in the case of a torus, for example, will always be not everywhere positive.

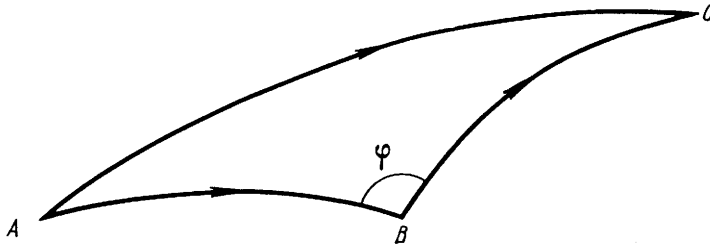


FIGURE 1

3) If F is an exact form, then the functional is single-valued but not necessarily everywhere positive.

In all three cases the classical Morse theory is not applicable.

A new device was presented in [7] for finding the critical points of l_α , namely, the throwing out of cycles. We explain it using the example of a single-valued functional l that is not everywhere positive (i.e., there exists a curve $\hat{\gamma} \in P_0 (= P_0(M^n))$ such that $l(\hat{\gamma}) < 0$). We join $\hat{\gamma}$ to a one-point curve by a segment, i.e., we consider a mapping (throwing out a zero-dimensional cycle in $H_0(M^n)$ into the domain $\{l < 0\}$) $\tau: [0, 1] \rightarrow P_0$ such that $\tau(0)$ is the one-point curve, $\tau(1) = \hat{\gamma}$, and τ is continuous. This mapping realizes a nontrivial cycle in $H_1(P_0, \{l \leq 0\})$. If there exists a deformation of this cycle reducing the value of l and such that under its action the lengths of the closed curves remain bounded by some constant, then the cycle "hangs" on a saddle extremal, since it cannot be deformed into the set $\{l \leq 0\}$, because the one-point curves form the manifold of local minima of the functional l .

The question of the existence of a deformation not increasing lengths unboundedly was not discussed in [6]–[11]. As S. V. Bolotin indicated to the authors of these papers, this question is nontrivial and requires additional discussion, and it is to such a discussion that we proceed, after making a remark: there was a discrepancy in [6] and [7] in applying the Lyusternik-Schnirelmann theorem on three closed geodesics on the 2-sphere to an irreversible functional l ; the singularities of l associated with its irreversibility (the dependence of its value on the orientation of the contour) were considered in this connection in [11] (see also [12], Russian p. 47, English pp. 47–48).

The existence of a deformation of the relative cycles in $H_i(P_0, \{l \leq 0\})$ that reduces the value of l and does not increase unboundedly the lengths of contours corresponding to points of the polyhedra imbedded in P_0 and realizing the relative cycles has not yet been proved for any closed manifold M^n . The difficulties arising can be clarified using the example of a polygonal deformation (see [1] and [2]): we consider triangles ABC formed by arcs of extremals of the Euler-Lagrange equation for the Lagrangian

$$L = \sqrt{\dot{x}^2 + \dot{y}^2} + Hx\dot{y},$$

where x, y are the coordinates on \mathbf{R}^2 and $H \neq 0$ is a constant (i.e., arcs of the Larmor orbits in a constant magnetic field on the plane [13]), such that the lengths of the arcs AB and BC are equal to some sufficiently small constant. If AB and BC belong to a closed curve $\gamma \in P$, then replacement of the part $AB \cup BC$ by the arc AC is a shortening deformation reducing the value of l by Δ_1 and increasing the value of the length functional by Δ_2 . With the help of uncomplicated computations it can be established that as $\varphi \rightarrow \pi$ the ratio Δ_2/Δ_1 increases unboundedly (see Figure 1). Therefore, it is not excluded that relative cycles can "hang on points at infinity".

Apparently, such a deformation nevertheless exists (at least in the two-dimensional

case.⁽¹⁾ But since its existence has not yet been established, we remark that the results in [6]–[11] concerning the existence of periodic extremals of the functionals we are considering (with the corrections inserted in [11] and [12] taken into account) are true only for functionals for which such deformations exist.

Below in §4 we prove the existence of periodic extremals in the case of “strong magnetic fields” on the 2-sphere without the assumption about the existence of such a deformation.

In conclusion we make two remarks.

1) Recently methods of symplectic topology (and not Morse theory), more precisely, the “geometric Poincaré theorem” and its generalizations, were used to prove the existence of periodic extremals for certain nonzero magnetic fields on two-dimensional manifolds (see [14]–[16]).

2) It is easy to see that in contrast to the case of geodesic flows, the two-endpoint problem for flows generated by Lagrangians of the form (2) on an arbitrary energy level can fail to have a solution even in the case $U = 0$ [7]: a strong magnetic field localizes the motions of points with small energies, as is clear from the example of Larmor orbits [13].

§3. THE THEOREM ON THROWING OUT CYCLES

In this section we prove the theorem on throwing out cycles for single-valued functionals (the main result in [10]).

On the tangent bundle of the manifold M^n we consider a function $F(x, \dot{x})$ satisfying the condition

$$(5) \quad F(x, k\dot{x}) = kF(x, \dot{x}), \quad k > 0.$$

On $\widehat{\Omega}(M^n)$ and $P(M^n)$ we consider functionals of the form

$$(6) \quad l(\gamma) = \int_{\gamma} F(x, \dot{x}) dt.$$

If $\alpha, \beta \in \widehat{\Omega}(M^n)$ and $\alpha(0) = \alpha(1) = \beta(0) = \beta(1)$, then it is possible to define the product $\alpha\beta$ of the loops by the following formula:

$$(7) \quad \alpha\beta(t) = \begin{cases} \beta(2t), & 0 \leq t \leq 1/2, \\ \alpha(2t - 1), & 1/2 \leq t \leq 1. \end{cases}$$

According to (5), l is an additive functional on $\widehat{\Omega}(M^n)$: $l(\alpha\beta) = l(\alpha) + l(\beta)$. In defining l on $P(M^n)$, we actually use another consequence of (5): the functional l is G -invariant on $\widehat{\Omega}$ ($= \widehat{\Omega}(M^n)$), where G is the group of piecewise smooth homeomorphisms $S^1 \rightarrow S^1$.

Denote by $\widehat{\Omega}_0$ the component of $\widehat{\Omega}$ formed by the contours that are contractible to a point, and define the mapping $p: \widehat{\Omega} \rightarrow M^n$ assigning distinguished points to curves: $p(\gamma) = \gamma(0) \in M^n$.

Theorem 1. *If the manifold M^n is connected and closed, and the functional l is not positive everywhere on $\widehat{\Omega}_0$ (i.e., there exists a contour $\gamma \in \widehat{\Omega}_0$ such that $l(\gamma) < 0$), then there exists a continuous mapping $g: M^n \times [0, 1] \rightarrow \widehat{\Omega}_0$ for which the following assertions are true:*

⁽¹⁾ In Novikov’s opinion, even in the two-dimensional case there may be no convergent deformations on surfaces of genus ≥ 2 (nor, possibly, on a torus), and the question of convergence is connected in an essential way with the geometry of the manifold: an elliptic geometry (the 2-sphere) evidently ensures convergence, and a parabolic geometry (the torus) possibly does not preclude it.

1) $g(M^n \times 1)$ is the image of M^n under the imbedding in $\widehat{\Omega}_0$ as the manifold of one-point contours.

2) On $g(M^n \times 0)$ the functional l takes negative values: $g(M^n \times 0) \subset \{l < 0\}$.

3) $p(g(x, t)) = x$ for all $x \in M^n$ and $t \in [0, 1]$.

The proof given below is essentially a reworked variant of the original proof [10]. The changes were made to simplify the construction of the throwing out mappings, and were suggested by D. V. Anosov.

Proof. We triangulate M^n and imbed it in the Euclidean space \mathbf{R}^k ($\varphi: M^n \rightarrow \mathbf{R}^k$) in such a way that the restriction of φ to the interior of each simplex is smooth. Let r be the number of zero-dimensional simplexes. Then we number all the zero-dimensional simplexes by the integers from 1 to r . We imbed $M^n \times [0, 1]$ in $\mathbf{R}^{k+1} = \mathbf{R}^k \times \mathbf{R}$ in such a way that $\tilde{\varphi}(x, \tau) = (\varphi(x), \tau)$, where $\tilde{\varphi}$ is the imbedding $M^n \times [0, 1] \rightarrow \mathbf{R}^{k+1}$, $x \in M^n$, and $\tau \in [0, 1]$. We construct a simplicial decomposition of $M^n \times [0, 1]$ as follows: let σ be the l -dimensional simplex in M^n spanned by the vertices $\alpha_{i_0}, \dots, \alpha_{i_l}$, where $i_0 < \dots < i_l$; the prism $\sigma \times [0, 1]$ is broken up into simplexes $\sigma'_0, \dots, \sigma'_l$ such that σ'_m ($0 \leq m \leq l$) spans the vertices $\alpha_{i_0} \times 0, \dots, \alpha_{i_m} \times 0$ and $\alpha_{i_m} \times 1, \dots, \alpha_{i_l} \times 1$.

We imbed \mathbf{R}^{k+1} in \mathbf{R}^{k+2} as the plane $x_{k+2} = -1$. In each simplex in M^n we introduce barycentric coordinates: Let σ be an l -dimensional simplex in M^n with vertices β_0, \dots, β_l , each of which is associated with its radius vector in \mathbf{R}^{k+2} ; then the radius vector of any point x in σ can be uniquely represented as a sum $\sum \lambda_i(x) \beta_i$, where $\sum \lambda_i = 1$, $\lambda_i \geq 0$, and β_i are the radius vectors of the vertices; the coefficients $\lambda_0(x), \dots, \lambda_l(x)$ are called the *barycentric coordinates* of x .

Let $\gamma: [0, 1] \rightarrow M^n$ be a closed path in $\widehat{\Omega}_0$ such that $l(\gamma) < 0$. By assumption, this path is connected by a homotopy γ_s with a one-point contour γ_1 ($0 \leq s \leq 1$; it can be assumed that $\gamma_s(0) = \gamma(0)$ for all s). Denote by $\gamma_{N,s}$ the paths of the form

$$\gamma_{N,s}(t) = \begin{cases} \gamma(Nt), & 0 \leq t \leq i/N, \\ \gamma_{Ns-i}(Nt - i), & i/N \leq t \leq (i+1)/N, \\ \gamma(0), & t \geq (i+1)/N, \end{cases}$$

where $s \in [i/N, (i+1)/N]$.

With each vertex α_m of the complex M^n we associate a smooth path in M^n from α_m to $\gamma(0)$, denoted by $b_m: [0, 1] \rightarrow M^n$. With each point x lying in the interior of σ , which has α_m as a vertex, we associate a path from x to α_m , realizable in \mathbf{R}^k by the segment from x to α_m . Denote this path by $c_{mx}: [0, 1] \rightarrow M^n$. The product of the paths b_m and c_{mx} is denoted by $d_{mx} = b_m c_{mx}$.

For each i we also define an s -parameter family of contours ($0 \leq s \leq 1$, t the parameter on the contour) depending also on the points in M^n :

$$f_i(x, s, t) = \begin{cases} d_{ix}(4t), & 0 \leq t \leq s/2, \\ d_{ix}(2s), & s/2 \leq t \leq 1 - s/2, \\ d_{ix}(4 - 4t), & 1 - s/2 \leq t \leq 1, \end{cases}$$

for $s \in [0, 1/2]$, and

$$f_i(x, s, t) = \begin{cases} d_{ix}(4t), & 0 \leq t \leq 1/4, \\ \gamma_{N,2s-1}(2t - 1/2), & 1/4 \leq t \leq 3/4, \\ d_{ix}(4 - 4t), & 3/4 \leq t \leq 1, \end{cases}$$

for $s \in [1/2, 1]$.

We also define a function $q: \mathbf{R} \rightarrow \mathbf{R}$:

$$\begin{cases} q(t) = 0 & \text{for } t \leq 0, \\ q(t) = 1 & \text{for } t \geq 1/(2(n+1)), \\ q(t) = 2(n+1)t & \text{for } 0 \leq t \leq 1/(2(n+1)). \end{cases}$$

Let us now proceed to a direct construction of a throwing out.

Suppose that x lies in the interior of an l -dimensional simplex $\sigma \subset M^n$ with vertices $\alpha_{i_0}, \dots, \alpha_{i_l}$ ($i_0 < \dots < i_l$), which we renumber for simplicity of subsequent expressions: $\delta_0 = \alpha_{i_0}, \dots, \delta_l = \alpha_{i_l}$. Denote by $\lambda_0, \dots, \lambda_l$ the barycentric coordinates corresponding to them, and by Δ_j ($0 \leq j \leq l$) the simplexes in $M^n \times [0, 1]$ spanned by the vertices $\delta_0 \times 0, \dots, \delta_j \times 0, \delta_j \times 1, \dots, \delta_l \times 1$ in \mathbf{R}^{k+1} . Obviously, $(x, \tau) \in \Delta_j$ for $\tau \in [\lambda_{j+1}(x) + \dots + \lambda_l(x), \lambda_j(x) + \lambda_{j+1}(x) + \dots + \lambda_l(x)]$.

With each point $x \in \Delta_j$ we now associate the family of paths $g(x, s)$, $s \in [0, 1]$, defined as follows:

$$(8) \quad g(x, s, t) = \begin{cases} x & \text{for } t = \lambda_{j+1}(x) + \dots + \lambda_l(x), \lambda_j(x) = 0; \\ f_i \left(x, (1-s)q(\lambda_j(x)), \frac{t - \lambda_{j+1}(x) - \dots - \lambda_l(x)}{\lambda_j(x)} \right) & \\ \text{otherwise} & \end{cases}$$

(here $t \in [0, 1]$ is a parameter on the path).

We define the following constants:

$$c_1 = \max_i |b_i|, \quad c_2 = \max_{i,x} |c_{ix}|, \quad c_3 = \max_i l(\gamma_i)$$

(where $||$ is the length of the curve in some chosen Riemannian metric).

It now remains to see that for

$$N > \frac{(n+1)(2c_1 + 2c_2 + c_3)}{|l(\gamma)|}$$

the family of paths (8) gives a throwing out of cycles, because $g(x, 0) \in \{l < 0\}$, and $g(x, 1)$ is the one-point contour corresponding to the point x . The theorem is proved.

Corollary 1. *If under the conditions of Theorem 1 the manifold $M^n \subset P_0(M^n)$ of one-point curves is the manifold of local minima of the functional l , then*

$$(9) \quad \dim H_i(P_0, \{l \leq 0\}; \mathbf{R}) \geq \dim H_{i-1}(M^n; \mathbf{R}).$$

Proof. Since M^n is the manifold of local minima,

$$H_i(\{l \leq 0\}; \mathbf{R}) = H_i(M^n; \mathbf{R}) \oplus H_i(\{l \leq 0\} \setminus M^n; \mathbf{R}).$$

Let $u \in H_i(M^n; \mathbf{R})$. By Theorem 1, it is homologous to some cycle $w \in H_i(\{l \leq 0\} \setminus M^n; \mathbf{R})$ that is the image of u under a throwing out. We consider the exact homology sequence of the pair $(P_0, \{l \leq 0\})$:

$$\begin{aligned} \dots \rightarrow H_{i+1}(P_0, \{l \leq 0\}; \mathbf{R}) &\xrightarrow{\partial} H_i(M^n; \mathbf{R}) \oplus H_i(\{l \leq 0\} \setminus M^n; \mathbf{R}) \\ &\xrightarrow{j} H_i(P_0; \mathbf{R}) \rightarrow \dots \end{aligned}$$

Since $j(u - w) = 0$, it follows that $u - w \in \partial H_{i+1}(P_0, \{l \leq 0\}; \mathbf{R})$, and hence

$$\dim H_{i+1}(P_0, \{l \leq 0\}; \mathbf{R}) \geq \dim H_i(M^n; \mathbf{R}).$$

The corollary is proved.

If $F(x, \dot{x})$ has the form (3), then the functional (6) satisfies the conditions of Theorem 1 if the magnetic field dA is sufficiently strong (there exist curves $\gamma \in \Omega_0$ such that $l(\gamma) < 0$; of course, we assume that $E > \max U$), and the conditions of Corollary 1 (in a neighborhood of the one-point curves the A -addition to l behaves like the area of a small area element bounded by the curve, and it behaves like $O(l^2)$ upon contraction of the curve to a point). If there exists a deformation of curves in Ω_0 that decreases the value of l and does not increase the lengths of curves unboundedly under a finite change in the value of l , and all the closed extremals of l are nondegenerate in the Morse sense, then Theorem 1 gives us the Morse-Novikov inequalities:

$$(10) \quad R_d(l) \geq \dim H_{d-1}(M^n; \mathbf{R}),$$

where $R_d(l)$ is the number of closed extremals (critical points of $l: P_0 \rightarrow \mathbf{R}$, including multiple ones), which means that the Morse index is equal to d .

Throwing out of cycles for multivalued functionals is discussed in detail in [7] and [9].

§4. CLOSED EXTREMALS IN STRONG MAGNETIC FIELDS ON THE 2-SPHERE

Let M^2 be a closed smooth oriented two-dimensional surface on which we are given:

1) a Finsler metric, i.e., a function $f(x, \dot{x})$ such that $f(x, \dot{x}) \geq 0$ and $f(x, \dot{x}) = 0$ only for $\dot{x} = 0$, and $f(x, k\dot{x}) = kf(x, \dot{x})$ for $k > 0$ (a special case is a Riemannian metric $f(x, \dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}$); and

2) a magnetic field—a 2-form $F_{ij}dx^i \wedge dx^j$.

The metric and the field are assumed to be smooth.

Extremals are defined to be curves on M^2 that are solutions of the Euler-Lagrange equations (1) for the Lagrangian

$$(11) \quad L^{(\alpha)}(x, \dot{x}) = f(x, \dot{x}) + A_i^{(\alpha)}(x)\dot{x}^i,$$

where $dA^{(\alpha)} = F$, $x \in U_\alpha \subset M^2$, and $[F] = 0$ in $H^2(U_\alpha; \mathbf{R})$ (the extremals are well defined in $U_\alpha \cap U_\beta$, since the form $A^{(\alpha)}$ appears in (1) through the relation $dA^{(\alpha)} = F$).

Denote by $L(M^2)$ the space of films Π in M^2 (i.e., oriented two-dimensional surfaces with nonempty boundaries imbedded in M^2) such that their boundaries are unions of finitely many closed curves γ_α satisfying the following conditions:

1) The γ_α are oriented according to the orientation of Π .

2) The γ_α are nonselfintersecting polygons formed by finitely many extremal segments with lengths not exceeding δ (the definition of this constant, which depends on the choice of metric on M^2 , will be given below).

3) The boundary contours are disjoint; that is, $\gamma_\alpha \cap \gamma_\beta = \emptyset$ for $\alpha \neq \beta$.

Note that films do not have to be connected.

Each continuous deformation of the boundary of a film gives rise to a deformation of the film that the boundary “drags” after itself (we do not dwell on a detailed description of this natural process). For each film $\Pi \in L(M^2)$ we consider the family $L(\Pi) \subset L(M^2)$ of films formed by the films obtained from Π by piecewise smooth deformations of the form indicated above. The sets of the form $L(\Pi)$ make up a disjoint partition $L = \bigcup L_\alpha$.

In each such set we introduce a metric: let $\Pi_1, \Pi_2 \in L_\alpha$, and let $\Phi (= \Phi(\Pi_1, \Pi_2))$ be the set of deformations of the film Π_1 into Π_2 in $L(M^2)$. To each deformation

there corresponds a mapping $h: \Gamma \times [0, 1] \rightarrow M^2$, where $h = h(\tau)$ ($\tau \in \Phi$), Γ is a union of circles, $h(\Gamma \times 0) = \partial\Pi_1$, $h(\Gamma \times 1) = \partial\Pi_2$, $h(\Gamma \times t)$ consists of a union of polygons for all t , and h is piecewise smooth (i.e., smooth everywhere except at the points of the one-dimensional skeleton of some simplicial decomposition of $\Gamma \times [0, 1]$). A form $h^*(d\mu)$ (where $d\mu$ is the volume form on M^2) of the form $\lambda(x)d\hat{\mu}$ is thereby defined on an open dense subset of $\Gamma \times [0, 1]$, where $d\hat{\mu}$ is the volume form on $\Gamma \times [0, 1]$, and the integral of $d\hat{\mu}$ over any component of $\Gamma \times [0, 1]$ is positive. We associate with a deformation τ the number $c(\tau) = \int |\lambda| d\hat{\mu}$. The distance in L_α is defined by

$$\rho(\Pi_1, \Pi_2) = \inf_{\tau \in \Phi} c(\tau).$$

Let $L(M^2, N)$ be the subspace of $L(M^2)$ formed by the films whose boundaries consist of contours such that the total number of extremal segments forming them does not exceed N . Denote by $L_\alpha(N)$ the component of $L(M^2, N)$ lying in L_α . We complete $L_\alpha(N)$ in the metric ρ to form the space $D_\alpha(M^2, N)$. The spaces $D_\alpha(M^2, N)$ and $D_\beta(M^2, N)$ can have common points for different α and β : films whose boundary contours are tangent to each other or to themselves, coincide with each other or with themselves along extremal segments, or are one-point contours. Gluing all the D_α together along common points, we get a compact connected space $D(M^2, N)$.

If the form F is not exact, then we assume without loss of generality that

$$\int_{M^2} F_{ij} dx^i \wedge dx^j > 0$$

(for this it suffices to change the orientation of M^2 if necessary). Consider the subspace $L^+(M^2) \subset L(M^2)$ of films whose orientations are induced by the orientation of M^2 when they are imbedded in M^2 . Similarly, we define $L^-(M^2)$ as the subspace of films whose orientations are induced by the negative (with respect to the chosen orientation) orientation of M^2 . Denote by $L^*(M^2)$ the union $L^*(M^2) = L^+(M^2) \cup L^-(M^2)$ formed by the films whose components have orientations with the same behavior with respect to the orientation of M^2 . The constructions of $D^+(M^2, N)$ and $D^*(M^2, N)$ are analogous to that of $D(M^2, N)$.

On $L(M^2)$ we define the functional

$$l(\Pi) = \int_{\partial\Pi} f(x, i) dt + \int_{\Pi} F_{ij} dx^i \wedge dx^j,$$

which extends to a continuous functional on $D(M^2, N)$ for any N . The following result is obvious.

Lemma 1. *Let $\Pi \in L(M^2)$. Then $\delta l(\Pi) = 0$ if and only if the boundary of Π consists of a union of smooth closed extremals.*

The spaces we have constructed are basic in the treatment of Morse theory to be presented below, in which we consider the behavior of the singular points of l on $D(M^2, N)$ and not on the spaces of closed curves, as in [1]–[4] and [6]–[11]. Another feature of this extension is the fact that the films can change their topological type under gradient deformations (decreasing the value of l); for example, under a shortening deformation two components Π_1 and Π_2 of the film Π (Figure 2a; see p. 368) touch each other at a point A (Figure 2b; see p. 368), the degenerate film obtained will not be an extremal of l , and under the action of a gradient deformation it passes into a connected nondegenerate film obtained as a result of cutting off

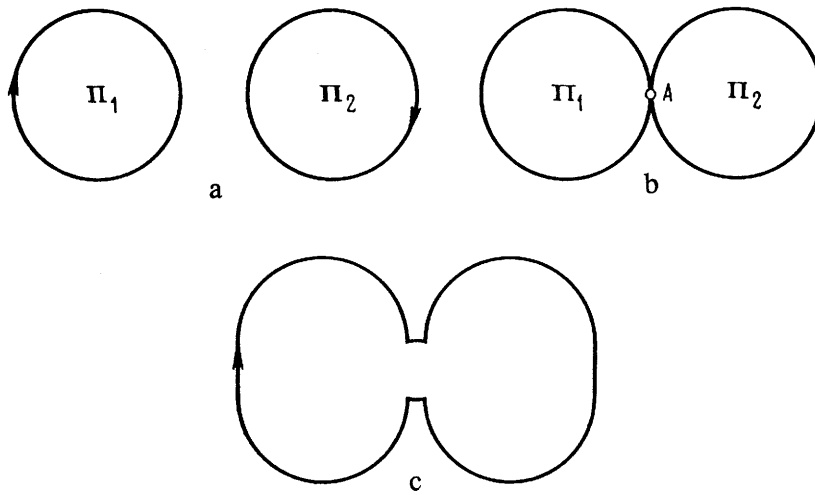


FIGURE 2

the angles (Figure 2c). We do not present detailed descriptions of the shortening deformations, but we prove, by using them in the form we need, that there are closed extremals in strong magnetic fields on the 2-sphere.

Theorem 2. *If the form F is exact on S^2 and on $L^*(S^2)$ the functional l takes also negative values, then there exists a closed nonselfintersecting extremal γ_0 on S^2 such that*

$$S(\gamma) = \int_{\gamma} L(x, \dot{x}) dt \geq S(\gamma_0)$$

for curves $\gamma \in P(S^2)$ sufficiently close to γ_0 .

Theorem 3. *If the form F is not exact on S^2 and on $L^+(S^2)$ the functional l takes also negative values, then there exists a closed nonselfintersecting extremal γ_0 on S^2 such that*

$$S^{(\alpha)}(\gamma) = \int_{\gamma} L^{(\alpha)}(x, \dot{x}) dt \geq S^{(\alpha)}(\gamma_0)$$

for curves $\gamma_0 \in P(S^2)$ ($\gamma_0 \subset U_{\alpha}$) sufficiently close to γ_0 , and γ_0 is not a single point.

Remark. 1) In the formulation of Theorem 2, $L(x, \dot{x})$ ($= L^{(\alpha)}(x, \dot{x})$ for all α) is of the form (11).

2) The conditions of Theorems 2 and 3 hold in the case of an arbitrary nonzero exact form (or any inexact form taking both positive and negative values) F for fields λF with $\lambda \geq \lambda_0(F)$ ($\lambda = \text{const}$). Therefore, we refer to such fields as “strong” fields.

We mention two important facts.

1) Suppose that a Riemannian metric is defined on M^2 , giving rise to a metric space structure on M^2 . Then there exist constants $\varepsilon > \delta > 0$ such that any point $x \in M^2$ can be joined to any point $y \in M^2$ whose distance $\rho(x, y)$ from x is at most δ by an extremal segment directed from x to y and lying completely in the disk of radius ε about x , and such a segment is unique and nonselfintersecting, and its union with any path γ from y to x disjoint from it and lying in such a disk bounds a film Π in the disk with

$$l(\Pi) \leq \int_{\gamma} (f(x, \dot{x}) + f(x, -\dot{x})) dt$$

(moreover, equality is attained only if the interior of the film is empty).

2) For any fixed metric f and any field F there exists a number $\omega(f, F) > 0$ such that any piecewise smooth nonselfintersecting transversal contour of length $\leq \omega(f, F)$ is contractible to a point in any disk of radius $2\omega(f, F)$ about a point of the contour and bounds a film (possibly degenerate) Π in such a disk for which $l(\Pi) \geq 0$ (moreover, equality is attained only if the contour is a one-point contour), and this is also true for any piecewise smooth nonselfintersecting transversal contour lying interior to such a film.

The first fact is a consequence of the compactness of M^2 and of a theorem on the existence and uniqueness of a solution of the Euler-Lagrange equation. The second fact, which lies at the basis of the method of throwing out cycles, was already discussed in §3.

Theorems 2 and 3 follow from the next result.

Theorem 4. *Under the conditions of Theorem 2 (3) the functional l attains its minimum in $L^*(S^2)$ ($L^+(S^2)$) on some film Π with nonempty boundary whose boundary contours are thus all closed extremals satisfying the requirements of Theorem 2 (3).*

Proof. We carry out the proof under the conditions of Theorem 2, and then show how to modify it under the conditions of Theorem 3.

We introduce on S^2 a Riemannian metric g_{ij} and denote by ε ($= \varepsilon(g)$), δ ($= \delta(g)$), and $\omega = \min(\omega(f, F), \omega(f, -F))$ the constants indicated above.

Lemma 2. *The functional l is bounded below, and the sums of the lengths (in the chosen metric) of the boundary contours of the films on which l takes negative values are bounded above by a constant $\alpha < \infty$.*

Proof. The form F reduces to the form $\varphi d\mu$, where $d\mu$ is the volume form on S^2 . Obviously,

$$l \geq - \int_{S^2} |\varphi| d\mu.$$

Since S^2 is compact, there exist constants c_1 and c_2 such that

$$0 \leq c_1 \leq \frac{f(x, \dot{x})}{g(x, \dot{x})} \leq c_2 < \infty,$$

where $g(x, \dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}$. If $l(\Pi) < 0$, then

$$\int_{\partial\Pi} f(x, \dot{x}) dt < \int_{S^2} |\varphi| d\mu,$$

and the sum of the lengths of the contours in $\partial\Pi$ is less than $c_1^{-1} \int_{S^2} |\varphi| d\mu = \alpha$. The lemma is proved.

Definition. A contour γ is said to be *small* if either its length is $\leq \omega$, or it lies inside a contour of length $\leq \omega$ (i.e., interior to a film bounded by it lying in a disk of radius 2ω).

We introduce the operation of removing a small contour.

Definition. If a small contour lies on the boundary of a film Π ($\gamma \subset \partial\Pi$) and there are no other boundary contours inside it, then *removal* of the small contour γ is defined to mean the following operation $\Pi \rightarrow \Pi_1$:

(a) If the small (i.e., lying in a disk of radius 2ω) film Π_0 bounded by it is a component of Π , then $\Pi_1 = \overline{\Pi \setminus \Pi_0}$ (the bar denotes closure).

(b) If $\Pi_0 \cap \Pi = \gamma$, then $\Pi_1 = \Pi \cup \Pi_0$.

Obviously, if γ is not a one-point contour, then $l(\Pi_1) < l(\Pi)$. The boundary of Π_1 does not contain γ .

Lemma 3. *There exists a sufficiently large number N_0 ($= N_0(f, F)$) such that if the functional l attains a minimal value on $D^*(S^2, N)$ at the film $\Pi \in D^*(S^2, N)$, $N \geq N_0$, then $\Pi \in D^*(S^2, N - 3)$.*

Proof. For any sufficiently large N we consider a film Π such that:

(a) it belongs to $D^*(S^2, N)$, i.e., its boundary can be represented as a union of contours (possibly selfintersecting and tangent to one another) that are not one-point contours and are formed by k segments of extremals, $k \leq N$; and

(b) at Π the functional l takes its minimal value on $D^*(S^2, N)$ (and this value is negative).

The number of boundary contours of the film Π is at most $[\alpha/\omega]$ (the square brackets denote the integer part of a number), as otherwise at least one of them is small and can be removed, and then the film obtained after the removal belongs to $D^*(S^2, N)$, and l takes a value on it less than the value on Π , which contradicts the choice of Π .

Let $\partial\Pi = \gamma_1 \cup \dots \cup \gamma_k$ ($k \leq [\alpha/\omega]$). Denote by b_{ij} the number of segments of γ_i tangent to γ_j , and by c_i the number of segments of γ_i tangent to other segments of γ_i (the contact with the preceding and succeeding segments is not tangency). We remark that segments cannot be tangent to themselves, because of the restriction on their lengths: $\leq \delta$.

Assertion 1. $b_{ij} < 12[\alpha/\omega]$.

Proof. Each pair of points $Q_1, Q_2 \in \gamma_i$ that are points of tangency with γ_j gives rise to a partition of $\gamma_i \cup \gamma_j$ into a union of two other contours: the first obtained by taking the union of the part of γ_i from Q_1 to Q_2 and the part of γ_j from Q_2 to Q_1 , and the second obtained by taking the union of the part of γ_i from Q_2 to Q_1 and the part of γ_j from Q_1 to Q_2 (here is where it is important that $\Pi \in D^*(S^2, N)$). If there are more points of tangency, then $\gamma_i \cup \gamma_j$ is partitioned into a larger number of contours (here we make essential use of the fact that $M^2 = S^2$, since such a partition is not always possible on spheres with handles; in Figure 3 we represent such a situation, realized in \mathbf{R}^3 when the surface M^2 is imbedded in it), some of which are possibly small and can be removed, after which there are at most $[\alpha/\omega]$ contours in which the parts previously belonging to γ_i are not tangent to parts previously belonging to γ_j . At least $[b_{ij}/2] - 2[\alpha/\omega]$ segments of γ_i will be removed as part of the small contours ($[b_{ij}/2]$ is a lower bound for the number of points of tangency which, being endpoints, can be common for two segments); furthermore, the number of segments γ_i and γ_j that appeared in the remaining contours can increase (because of the partition of segments in γ_i and γ_j at points of tangency) by at most $4[\alpha/\omega]$. If $b_{ij} \geq 12[\alpha/\omega]$, then after such a reconstruction we get a film in $D^*(S^2, N)$ on which l takes a value less than on Π , which contradicts the choice of Π . The assertion is proved.

Assertion 2. $c_i < 80[\alpha/\omega]^3 + 32[\alpha/\omega]^2$.

On each segment in γ_i tangent to other segments we mark one point of tangency and form from distinct marked points a set \hat{P} (consisting of at least $[c_i/2]$ points). We transform the contour γ_i by using the following operation successively: if the contour contains a loop that has a vertex in \hat{P} , is not small, and does not have subloops with vertices in \hat{P} , then we cut it off, and otherwise we single out a maximal (with respect to inclusion) loop that has a vertex in \hat{P} and can be removed by successive removal of small loops, and cut it off. The loops of the first type are denoted by μ_1, \dots, μ_n , and those of the second type by ν_1, \dots, ν_m . Obviously,

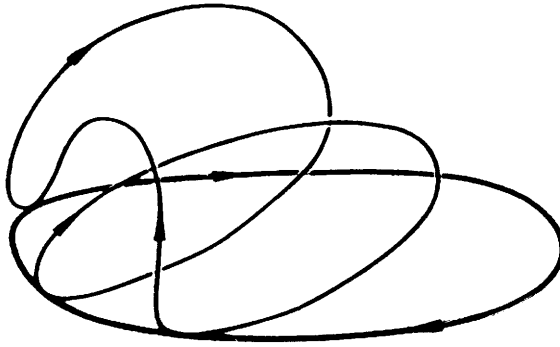


FIGURE 3

$n \leq [\alpha/\omega]$. Since the loops of the second type can arise only due to cutting off loops of the first type, and there are none of them initially on γ_i (otherwise, by cutting them off we would get a film in $D^*(S^2, N)$ with a smaller value of l), it follows that $m \leq [\alpha/\omega]$.

The points in \hat{P} break up into points of tangency of $\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_m$ [for each ordered pair there are at most $10[\alpha/\omega]$ of them, since otherwise we could pass from γ_i to $\mu_1 \cup \dots \cup \nu_m$ and remove the small contours arising when a pair is tangent (see the proof of Assertion 1), after which we could get a film in $D^*(S^2, N)$ with a smaller value of l (and the points of self-tangency of ν_1, \dots, ν_m); and there are no more than $8[\alpha/\omega]$ of them, as otherwise we could cut out ν_j (such that it contains more than $8[\alpha/\omega]$ points of self-tangency in \hat{P}) from γ_j and remove it, obtaining a film in $D^*(S^2, N)$ with a value of l smaller than on Π]. Collecting the estimates, we get a proof of Assertion 2.⁽²⁾

We can break up the boundary of Π into segments in such a way that the number of segments is minimal, and the smooth parts of length $> \delta$ will be broken up into the initial segments of length δ and a remaining segment of length $< \delta$. Under such a partition the smooth segments of length $< \delta$ that are not tangent to others can lie only on smooth contours, as otherwise if QP is the segment from Q to P of length $< \delta$ without points of tangency, then it is possible to choose a point P' lying after P in a circuit of the contour such that $\rho(Q, P') < \delta$ and upon replacing the part of the contour from Q to P' by an extremal segment from Q to P' (it is assumed that the contour is not smooth) lying in the disk of radius ε about Q , we reconstruct Π to form a film in $D^*(S^2, N)$ with value of l smaller than on Π .

It is now easy to estimate N_0 : suppose that for $N \geq N_1$ the functional l takes also negative values on $D^*(S^2, N)$; then $N_0 \leq 3 + \max(N_1, [\alpha/\delta] + 125[\alpha/\omega]^4)$. Lemma 3 is proved.

Lemma 4. *If the functional l takes a smallest value (which, by Lemma 3, is negative) on $D^*(S^2, N_0)$ at the film $\Pi \in D^*(S^2, N_0)$, then the film is smooth.*

Proof. If the film is not smooth, then its boundary contains an angle with vertex at the point Q and extremal segments as sides, one going into Q and the other leaving (denote them by I_1 and I_2 , respectively), and in a sufficiently small neighborhood of Q the interior of the angle does not contain points in the boundary of the contour and contains an extremal segment joining points $Q_1 \in I_1$ and $Q_2 \in I_2$ (directed from Q_1 to Q_2) such that $\rho(Q_1, Q_2) < \delta$. Since $\Pi \in D^*(S^2, N_0 - 3)$ (by Lemma 3), it follows that upon cutting off the angle, i.e., replacing the part of I_1 from Q_1

⁽²⁾ Outwardly, the estimates seem doubly excessive; however, in their derivation we take into account that the segments in γ_i can be tangent at a pair of marked points (in \hat{P}).

to Q and the part of I_2 from Q to Q_2 by a small extremal segment from Q_1 to Q_2 with the addition of the triangle bounded by them to the interior of the film if its interior points did not previously belong to the film, or with removal of it otherwise, we get a film in $D^*(S^2, N_0)$ on which the functional l takes a value smaller than on Π , which contradicts the choice of Π . Lemma 4 is proved.

According to Lemma 3, the smallest value taken by l on $D^*(S^2, N)$ is the same for all $N \geq N_0$, and hence coincides with the limit inferior of the values of l on $L^*(S^2)$. According to Lemma 4, this limit value is attained on a smooth film. To prove Theorem 4 it remains to observe that the boundary of such a film cannot be empty: the film cannot be a one-point film, since l takes a negative value on it, and cannot coincide with the whole sphere, since l has the value zero on a spherical film (the form F is exact). We have proved Theorem 4 under the conditions of Theorem 2. The proof is the same under the conditions of Theorem 3—it suffices to replace $L^*(S^2)$ and $D^*(S^2, N)$ everywhere by $L^+(S^2)$ and $D^+(S^2, N)$, respectively, and to note that a minimal film cannot be spherical, since the integral of F over S^2 is positive (here is where it is important that l takes negative values on $L^+(S^2)$). Theorem 4 is proved, and this concludes the proof of Theorems 2 and 3.

We remark that nowhere did we actually use a shortening deformation acting on $D^*(S^2, N)$, having proved at once convergence for the deformations of a point in Lemma 3. The existence of deformations of integer cycles is a question that has not yet been cleared up. It turned out that the length-shortening deformation of nonselfintersecting curves in [2] is not applicable for continuous deformations of integer cycles of curves (this question is discussed in [17]; other approaches were presented in [17] and [18] for refining the proof of the Lyusternik-Schnirelmann theorem). It is intuitively clear that the assertions of Theorems 2 and 3 are true for all closed oriented two-dimensional manifolds, but the proof of an analogue of Lemma 3 (only in its proof is sphericity essential) for them is evidently quite laborious.⁽³⁾ Another approach to the proof of the existence of a nonselfintersecting closed extremal on S^2 in the case of a single-valued not everywhere positive functional was presented in [11] and [12], but a perfectly rigorous justification of it turned out to be very laborious and is not yet complete (this is partly the reason that other approaches were found that are applicable also to multivalued functionals).

In conclusion we remark that the problem of the existence of periodic solutions of equations (1) for a single-valued Lagrangian (2) with given period reduces to the following topological problem: prove the existence of critical points (that are not one-point curves) of the functional

$$S_T(\gamma) = \int_0^T [g_{ij}(\gamma(t))\dot{\gamma}^i\dot{\gamma}^j + A_i(\gamma(t))\dot{\gamma}^i + U(\gamma(t))] dt$$

on the space $\widehat{\Omega}(M^n, T)$ formed by the piecewise smooth mappings $\gamma: [0, T] \rightarrow M^n$ such that $\gamma(0) = \gamma(T)$. The periodic solutions with period T are precisely the critical points. We remark that the one-point curves are not critical points of S_T if the points themselves are not critical points of the potential $U(x)$. For different T the critical points of S_T as contours on M^n are distinct in the general case (in contrast to the geodesic flows). If M^n is closed, then the constants

$$U_0 = \max |U(x)|,$$

$$A_0 = \max \left(|A_i(x)\dot{x}^i| / \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} \right)$$

⁽³⁾ *Added in proof.* The author recently proved the assertions of Theorems 2 and 3 for all oriented closed two-dimensional manifolds. The proof will be published in *Sibirsk. Mat. Zh.*

are defined, and the (easily derived) inequality

$$S_T(\gamma) \geq \|\gamma\|^2 T^{-1} - A_0 \|\gamma\| - U_0 T$$

holds, where $\|\gamma\|$ is the length of a curve in the Riemannian metric g_{ij} (the 1-form $A_i dx^i$ is assumed to be single-valued); therefore, it follows from the inequality $S_T(\gamma) \leq h$ ($h \geq 0$) that

$$\|\gamma\| \leq \frac{1}{2} \left(A_0 T + \sqrt{A_0^2 T^2 + 4T(U_0 T + h)} \right).$$

The subspaces of the form $\{\gamma: S_T(\gamma) \leq h\}$ consist of contours of finite length, and in constructing a Morse theory for the functionals S_T on $\widehat{\Omega}(M^n, T)$ the difficulties described in §2, which are connected with trajectories of the “vector field” $-\text{grad } S_T$ “going to infinity”, do not arise. Apparently, such a theory can be completely constructed in the spirit of Morse theory for the energy functional in the theory of closed geodesics [3].

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