

## THE TOPOLOGY OF RIEMANNIAN MANIFOLDS WITH INTEGRABLE GEODESIC FLOWS

UDC 517.938.5

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### 1. INTRODUCTION

Existence of an integrable geodesic flow on a Riemannian manifold imposes strong restrictions on the manifold's topology. The problem of describing the topological properties of those manifolds on which the geodesic flow cannot be integrable—i.e., of determining the topological obstructions to integrability—has of late seen substantial progress. The purpose of this paper is to describe and discuss the results of this study; they delineate a large class of manifolds on which the geodesic flow cannot be integrable.

We denote by  $M^n$  a connected closed smooth Riemannian manifold of dimension  $n$ ; this will be given no further mention, except to specify the degree of smoothness. By  $T^*M^n$  we denote the cotangent bundle to  $M^n$ , and by  $g_{ij}(x) dx^i dx^j$  the tensor defining the Riemannian metric. On  $T^*M^n$  there exists a natural symplectic structure, defined by the closed 2-form  $\Omega$  given locally by  $\Omega = \sum dx^i \wedge dp_i$ , where  $\{x^i, \dots, x^n\}$  are coordinates on  $M^n$  and  $\{p_1, \dots, p_n\}$  are the corresponding conjugate coordinates on the fibers of the cotangent bundle. On the space of smooth functions on  $T^*M^n$  the form  $\Omega$  determines the Poisson bracket

$$\{f, g\} = h^{ij} \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial y^j},$$

where locally  $\Omega = h_{ij} dy^i \wedge dy^j$  in any coordinates  $\{y^i\}$  on the cotangent bundle. (Repetition of upper and lower indices, here and subsequently, means summation with respect to them.)

The geodesic flow is the Hamiltonian system with Hamiltonian  $H(x, p) = \frac{1}{2} g^{ij}(x) p_i p_j$ ; the change in a smooth function on the cotangent bundle under translation along a trajectory of the system is given by the equation  $\dot{f} = \{f, H\}$ . The Hamiltonian is a first integral:  $\dot{H} = 0$ . Consequently, any of its level surfaces is invariant with respect to the flow; furthermore, as is easily seen, in the case of a geodesic flow the restrictions of the flow to different nonzero level surfaces of the Hamiltonian are trajectoryally isomorphic: if  $(x(t), p(t))$  is a trajectory on the level surface  $\{H = c \neq 0\}$ , then  $(x(t), \frac{d}{c} p(t))$  is a trajectory on the level surface  $\{H = d\}$ . This warrants the following definition.

**Definition.** The geodesic flow on  $M^n$  is (*Liouville-*) *integrable* if there is a nonzero level surface  $L$  of the Hamiltonian in some neighborhood of which there exist on  $T^*M^n$ , besides the Hamiltonian  $H = I_1$ ,  $n - 1$  other first integrals  $I_2, \dots, I_n$  such that:

1991 *Mathematics Subject Classification.* Primary 58F05, 58F17; Secondary 58F11, 54H20, 28D20.

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(a) All the first integrals of this set are an involution with each other:  $\{I_i, I_j\} = 0$ ,  $1 \leq i, j \leq n$ .

(b) On  $L$ , the first integrals are functionally independent at every point of some invariant open everywhere dense subset  $U \subset L$ .

(c) If  $M^n$  is a real-analytic Riemannian manifold, then by analytic integrability is meant the existence of a set of first integrals as above that are real-analytic functions.

Before proceeding directly to an account of the results on topological obstructions to integrability, we make some remarks on the strategy of the proofs.

In the non-simply-connected case, the proofs of nonintegrability of geodesic flows split into two steps:

(a) Proof that some geometric disposition of a geodesic flow cannot exist.

(b) Description of the functional conditions on complete sets of first integrals that distinguish geodesic flows with the given geometric behavior.

To formalize the second step the notion of a geometrically simple geodesic flow was introduced in [1].

**Definition.** The geodesic flow on  $M^n$  is *geometrically simple* if there exists a nonzero level  $L = \{H = c \neq 0\}$  of the Hamiltonian such that:

(a) The surface  $L$  contains a closed subset  $\Gamma$  whose complement is invariant, is everywhere dense, and has only finitely many arcwise connected components:  $L \setminus \Gamma = \bigcup_{i=1}^m V_i$ .

(b) Each component  $V_i$  fibers into invariant  $n$ -dimensional tori over an  $(n-1)$ -dimensional disk.

(c) For any point  $q \in L$  and any neighborhood  $W \subset L$  of  $q$ , there exists a neighborhood  $W_1$  such that  $W_1 \subset W$  and  $W_1 \cap (L \setminus \Gamma)$  has only finitely many arcwise connected components.

The same paper [1] proved

**Theorem 1.** *If the geodesic flow on a real-analytic Riemannian manifold is analytically integrable, it is geometrically simple.*

The most general topological obstructions to integrability of geodesic flows on non-simply-connected manifolds that have been obtained so far are in fact obstructions to geometric simplicity.

**Theorem 2** (see [1] and [2]). *If the geodesic flow on  $M^n$  is geometrically simple (in particular, if it is analytically integrable), then:*

(a) *The fundamental group  $\pi_1(M^n)$  is almost commutative, i.e., contains a commutative subgroup of finite index.*

(b) *If  $\dim H_1(M^n; \mathbb{Q}) = d$ , then  $H^*(M^n; \mathbb{Q})$  contains a subring isomorphic to the rational cohomology ring of the  $d$ -dimensional torus.*

(c) *If  $\dim H_1(M^n; \mathbb{Q}) = n$ , then the rational cohomology rings of the manifold  $M^n$  and of the  $n$ -dimensional torus are isomorphic.*

Topological obstructions to integrability of Hamiltonian flows on simply-connected manifolds (and so on manifolds with finite fundamental group) can be obtained by another approach, based on a study of the topological obstructions to existence of a geodesic flow with zero topological entropy. Such an approach was presented by Paternain [3], [4], who generalized a theorem of Kozlov [5]—weakening the analyticity requirement and obtaining a new approach to the proof—and used algebraic means to demonstrate the vanishing of the topological entropy of geodesic flows integrable by the Thimm method [6]. The main inconvenience of this approach is that the topological entropy of an integrable flow is equal to that of the restriction

of the flow to the “singular set” (formed by the points at which the first integrals are functionally dependent), whereas the Liouville theorem gives a good description of the dynamics only on the complements to this set. If a flow is integrable by the Thimm method, then all the trajectories of the Hamiltonian flows generated by the first integrals  $I_1, \dots, I_n$  as Hamiltonians are closed.

Paternain has recently demonstrated the vanishing of the topological entropy of an integrable flow all of whose  $I$ -orbits are nondegenerate in the sense of Ito (preprint, Max Planck Institute, 1992). To explain this result, we give a definition of  $I$ -orbit:

**Definition.** By the  $I$ -orbit of a point  $x \in T^*M^n$  is meant the totality of points obtained from  $x$  by translations along trajectories of Hamiltonian systems with Hamiltonians  $a_1 I_1 + \dots + a_n I_n$ , where the  $(a_1, \dots, a_n)$  are all possible sets of real numbers and  $(I_1, \dots, I_n)$  is a complete set of first integrals of an integrable geodesic flow on  $M^n$ .

The definition has been formulated here for the case of geodesic flows, but it can be formulated for any integrable Hamiltonian system on an arbitrary symplectic manifold. The Ito nondegeneracy condition for  $I$ -orbits is a condition on the dynamics on the “singular set” and the noncompact  $n$ -dimensional  $I$ -orbits adjoining it.

We formulate below another condition on the dynamics on the “singular set”, and demonstrate the vanishing of the topological entropy of integrable flows that satisfy this condition. Let  $\Phi: L \rightarrow R^{n-1}$  be a moment mapping, where  $L = \{H (= I_n) = \text{const}\}$ ,  $\Phi(x) = (I_1(x), \dots, I_{n-1}(x))$ , and the flow is integrable on the level surface  $L$ .

**Condition I.** 1. For each value of the moment mapping, its inverse image contains at most countably many connected components. 2. If  $W$  is a component of  $\Phi^{-1}(c)$ ,  $x \in W^{(k)}$  (the subset of  $W$  consisting of those points at which the rank of the extended moment mapping  $T^*M^n \rightarrow R^n$  given by  $x \rightarrow (I_1(x), \dots, I_n(x))$  is equal to  $k$ ), and the  $I$ -orbit of  $x$  is noncompact and homeomorphic to a manifold  $F$ , then there exists a  $W^{(k)}$  neighborhood of the  $I$ -orbit of  $x$  that is diffeomorphic to the direct product of  $F$  by some domain  $V$ , and every fiber diffeomorphic to  $F$  is the  $I$ -orbit of some point.

Part 2 of Condition I can be replaced as follows.

**Condition II.**  $2^*$ . Suppose  $x \in W^{(k)}$  and  $v_1 (= (v_{11}, \dots, v_{1n}))$ ,  $\dots$ ,  $v_{n-k} (= (v_{(n-k)1}, \dots, v_{(n-k)n}))$  is a set of linearly independent vectors such that

$$(*) \quad \text{grad}(v_{11}I_1 + \dots + v_{1n}I_n) = \dots = \text{grad}(v_{(n-k)1}I_1 + \dots + v_{(n-k)n}I_n) = 0.$$

Then in the neighborhood of the point  $x$  in  $W^{(k)}$  equations  $(*)$  determine a  $k$ -dimensional submanifold.

These conditions are entirely natural. For example, in the “general position” case, a drop in rank by unity means a decrease by at least unity in the dimension of a submanifold of the form  $\{\text{rank} = \text{const}\}$ , and therefore in this case the neighborhood of the  $I$ -orbit in  $W^{(k)}$  coincides with the  $I$ -orbit itself. Condition I is satisfied, for example, in the case of an analytically integrable flow.

It is easily seen that both Condition I and Condition II are satisfied by integrable 2-dimensional systems with Bott first integrals [18].

We show below that if an integrable flow satisfies at least one of Conditions I or II, then its topological entropy vanishes (Theorem 9).

Combining this result with those of Yomdin [8] and Gromov [9], we obtain

**Theorem 3.** *If the geodesic flow of an infinitely differentiable Riemannian metric on a closed manifold  $M^n$  is integrable and satisfies at least one of Conditions I or II, then  $M^n$  is rationally elliptic. In particular,*

- (a)  $\dim(\pi_*(M^n) \otimes \mathbb{Q}) \leq n$ ;
- (b)  $\dim H_*(M^n; \mathbb{Q}) \leq 2^n$ ;
- (c) *the Euler-Poincaré characteristic  $\chi(M^n)$  is nonnegative; and*
- (d)  $\chi(M^n) > 0$  *if and only if the odd-dimensional homology groups of  $M^n$  are finite.*

A number of other topological properties of rationally elliptic manifolds are indicated in [19] and [20].

Theorem 2 is dealt with in §2, Theorem 3 in §§3 and 4.

We should mention separately Kozlov's theorem—the first result on topological obstructions to integrability of geodesic flows.

**Theorem 4** [5]. *If the geodesic flow on a 2-dimensional closed orientable real-analytic Riemannian manifold is analytically integrable, then the manifold is homeomorphic to either a sphere or a torus.*

Several different proofs are presently known for this theorem; furthermore, the analyticity requirement can be weakened (see, for example, Theorem 2). Kozlov has recently obtained an interesting generalization of his result, proving that an analytic field of symmetries of the geodesic flow on a real-analytic surface of genus  $> 1$  is proportional to the field determined by the flow [10]. The proof, however, relies on analytic rather than geometric or topological methods.

§5, at the end of the paper, has to do with various conjectures concerning topological obstructions to integrability.

The author expresses his thanks to D. V. Anosov, Victor Bangert, V. V. Kozlov, and G. P. Paternain for helpful discussions.

## 2. OBSTRUCTIONS TO INTEGRABILITY OF GEODESIC FLOWS ON NON-SIMPLY-CONNECTED MANIFOLDS

The proof of Theorem 2 is based on the following result, which was actually proved by the author in [1] but not separated out in its own right.

**Theorem 5.** *If the geodesic flow on  $M^n$  is geometrically simple, then the cotangent bundle  $T^*M^n$  contains an invariant  $n$ -dimensional torus  $T^n$  such that the image of the fundamental-group homomorphism  $p_*: \pi_1(T^n) \rightarrow \pi_1(M^n)$  induced by the projection  $p: T^n \rightarrow M^n$  along the fibers of  $T^*M^n$  has finite index in  $\pi_1(M^n)$ .*

We note that the proof of the theorem, as given in [1], goes through when condition (c) in the definition of geometric simplicity is replaced by a weaker one:

(c') There exist a point  $x_0 \in M^n$  and a sufficiently small neighborhood  $U \subset M^n$  of  $x_0$  such that condition (c) is satisfied only for those points of  $L$  that lie over  $U$  in the fibers of the cotangent bundle.

Furthermore, it suffices to require merely that  $L \setminus \Gamma$  be dense on the set of points lying over  $U$ , i.e., that the flow be geometrically simple only locally.

Of the properties of a geodesic flow, what is essential in the proof is only the realizability of the Hopf-Rinow theorem; it is necessary that the surface  $L$  contain extremal loops beginning and ending at  $x_0$  and representing the elements of a subgroup of  $\pi_1(M^n, x_0)$  of finite index. Thus, the theorem is in particular automatically true for geodesic flows in Finsler metrics.

*Proof of Theorem 2* (modulo Theorem 5). Suppose there exists on  $M^n$  a geometrically simple geodesic flow. That the fundamental group of  $M^n$  is almost commutative follows at once from Theorem 5, since  $p_*(\pi_1(T^n))$  is commutative and has finite index in  $\pi_1(M^n)$ .

From the finiteness of the index it follows also that  $p_*: H_1(T^n; \mathbb{Q}) \rightarrow H_1(M^n; \mathbb{Q})$  is a monomorphism, and therefore that  $p^*: H^1(M^n; \mathbb{Q}) \rightarrow H^1(T^n; \mathbb{Q})$  is an epimorphism. Let  $a_1, \dots, a_k$  be generators of  $H^1(M^n; \mathbb{Q})$ , and  $b_1, \dots, b_k$  their images in  $H^1(T^n; \mathbb{Q})$ . For  $j_1, \dots, j_m$  distinct,

$$b_{j_1} \cup \dots \cup b_{j_m} = p^*(a_{j_1} \cup \dots \cup a_{j_m}) \neq 0.$$

It is now easily seen that the linear span of all the products of the form  $a_{j_1} \cup \dots \cup a_{j_m}$  constitutes a subring of  $H^*(M^n; \mathbb{Q})$  isomorphic to the rational cohomology ring of the  $k$ -dimensional torus.

When  $\dim H^1(M^n; \mathbb{Q}) = n$ , the preceding argument shows that  $p_*: H_n(T^n; \mathbb{Q}) \rightarrow H_n(M^n; \mathbb{Q})$  is an isomorphism, i.e.,  $M^n$  is orientable and the mapping  $p: T^n \rightarrow M^n$  has nonzero degree. This implies that  $\dim H_j(M^n; \mathbb{Q}) \leq \dim H_j(T^n; \mathbb{Q})$  for all  $j$ . But then, again by the preceding argument, the opposite inequalities also hold; i.e., the subring determined by elements of the form  $a_{j_1} \cup \dots \cup a_{j_m}$  coincides with the whole ring  $H^*(M^n; \mathbb{Q})$ . This completes the proof of Theorem 2.

### 3. TOPOLOGICAL OBSTRUCTIONS TO EXISTENCE OF GEODESIC FLOWS WITH ZERO TOPOLOGICAL ENTROPY

Let  $L$  be a nonzero level surface of the Hamiltonian of a geodesic flow. On it consider any metric, defining in particular a metric-space structure, and denote distance between points by  $\rho$ . Denote by  $f_t(x)$  the point in  $L$  obtained from a point  $x \in L$  by translation along the trajectory of the flow at time  $t$ , and by  $\rho_t$  the metric on  $L$  given by

$$\rho_t(x, y) = \max_{0 \leq s \leq t} \rho(f_s(x), f_s(y)).$$

For  $\varepsilon$  and  $t$  any positive numbers, define the  $\varepsilon$ -entropy of the metric space  $(L, \rho_t)$  as  $\ln N_t(\varepsilon)$ , where  $N_t(\varepsilon)$  is the minimal cardinality of the coverings of the space  $(L, \rho_t)$  by disjoint subsets of diameter at most  $2\varepsilon$ .

**Definition.** The limit

$$\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow 0} \frac{\ln N_t(\varepsilon)}{t} = h(g)$$

is called the *topological entropy* of the geodesic flow of the metric  $g_{ij}$  ( $= g$ ).

The value of the topological entropy is independent of the choice of the metric  $\rho$  on  $L$ . The original definition of a topological entropy, independent of the choice of a metric, was given in [11] and is different from the above; equivalence of the two definitions was proved in [12].

As pointed out by Gromov in his Bourbaki seminar report [9], if the metric is infinitely smooth and there exists a positive constant  $C$  such that a pair of points in general position on  $M^n$  are joined, for any sufficiently large  $\lambda$ , by at least  $C^\lambda$  geodesics of length  $\leq \lambda$ , then the topological entropy of the geodesic flow of this metric is positive. The derivation of this result is based on results obtained in the mid 80's by Yomdin [8]. Earlier, Gromov had pointed out conditions on  $M^n$  guaranteeing the exponential growth, as  $\lambda \rightarrow \infty$ , of the number of geodesics of length  $\leq \lambda$  that join a pair of points in general position on a Riemannian manifold homeomorphic to  $M^n$  [13]. Theorem 6 below follows from these results of Gromov and Yomdin.

In our presentation of the proof we follow the paper [4].

**Theorem 6.** *If the fundamental group of the manifold  $M^n$  is finite and the geodesic flow of an infinitely differentiable metric on  $M^n$  has zero topological entropy, then the Betti numbers of the loop space on  $M^n$  grow subexponentially:*

$$\lim_{m \rightarrow \infty} \frac{1}{m} \ln \left( \sum_{i=1}^m \dim H_i(\Omega M^n; \mathbb{Q}) \right) = 0.$$

*Proof.* We can suppose that  $L = \{H = 1\}$  (since we can always arrange for this by a homothety  $g_{ij} \rightarrow \mu g_{ij}$ ). For every pair of points  $x, y \in M^n$  and for every  $\lambda > 0$  we denote by  $n(x, y, \lambda)$  the number of geodesics of length  $< \lambda$  joining  $x$  and  $y$ . By  $I(x, \lambda)$  we denote the function

$$I(x, \lambda) = \int_{M^n} n(x, y, \lambda) dy.$$

We define the geodesic entropy  $\Phi(x)$  at a point  $x \in M^n$  as

$$\Phi(x) = \lim_{\lambda \rightarrow \infty} \frac{\ln I(x, \lambda)}{\lambda}.$$

**Lemma 1.** *If the metric, and with it the geodesic flow, is infinitely differentiable, then for every point  $x \in M^n$  we have  $\Phi(x) \leq h(g)$ .*

*Proof.* Let  $S_x$  be the set of covectors of unit length in the cotangent-bundle fiber over the point  $x \in M^n$ . Denote by Vol the  $(n - 1)$ -dimensional volume in  $L$ . By Yomdin's result [8],

$$h(g) \geq \overline{\lim}_{t \rightarrow \infty} \frac{\ln \text{Vol}(f_t S_x)}{t} = \Delta$$

(this is where infinite differentiability of the metric is essential). According to Berger and Bott [14],

$$I(x, \lambda) = \int_0^\lambda dq \int_{S_x} |\det A_u(\lambda)| d\theta,$$

where  $\theta$  is the canonical measure on  $S_x$  and  $A_u(t) : (f_0 u)^\perp \rightarrow (f_t u)^\perp$  is the family of linear mappings, defined by the Jacobi equation, along the geodesic with initial point  $x \in M^n$  and initial cotangent vector  $u \in S_x$  such that  $A_u(0) = 0$  and  $A'_u(0) = \text{Id}$ . We obtain

$$I(x, \lambda) \leq \int_0^\lambda dt \int_{S_x} |\det(df_t|_{T_u S_x})| d\theta = \int_0^\lambda V(t) dt.$$

It is easily seen that for any  $\varepsilon > 0$  there exists a  $T(\varepsilon)$  such that if  $\lambda \geq T(\varepsilon)$ , then  $V(\lambda) \leq \exp(v(\Delta + \varepsilon)\lambda)$ . This implies that

$$\begin{aligned} I(x, \lambda) &\leq \int_0^{T(\varepsilon)} V(t) dt + \int_{T(\varepsilon)}^\lambda V(t) dt \\ &\leq \int_0^{T(\varepsilon)} V(t) dt + \exp((\Delta + \varepsilon)\lambda)/(\Delta + \varepsilon). \end{aligned}$$

Since from this inequality it follows immediately that

$$\overline{\lim}_{\lambda \rightarrow \infty} (\ln I(x, \lambda))/\lambda \leq \Delta + \varepsilon$$

for all  $\varepsilon > 0$ , we find that  $\Phi(x) \leq \Delta$ , and so, by Yomdin's inequality,  $\Phi(x) \leq h(g)$ . This proves Lemma 1.

By Gromov's theorem [13], if the fundamental group of  $M^n$  is finite and the manifold is closed (we emphasize this here expressly), then for points  $x, y \in M^n$  that are not conjugate along any geodesic (the general-position case) we have

$$n(x, y, \lambda) \geq \sum_{j=1}^{c(\lambda-1)} \dim H_j(\Omega M^n; \mathbb{Q}),$$

where the constant  $c > 0$  is determined by the metric in  $M^n$ . Integrating with respect to  $y$ , we find

$$\|M^n\| \left( \sum_{j=1}^{c(\lambda-1)} \dim H_j(\Omega M^n; \mathbb{Q}) \right) \leq I(x, \lambda),$$

where  $\|M^n\| = \int_{M^n} dy$  is the volume of  $M^n$ . By Lemma 1,

$$\overline{\lim}_{m \rightarrow \infty} \frac{1}{m} \ln \left( \sum_{j=1}^m \dim H_j(\Omega M^n; \mathbb{Q}) \right) \leq \Phi(x)/c \leq h(g)/c.$$

Theorem 6 now follows immediately from this inequality.

In the non-simply-connected case, the topological obstructions to existence of a geodesic flow with zero topological entropy are given by

**Theorem 7** (a) [12]. *If the fundamental group of  $M^n$  has exponential growth, then  $h(g) > 0$  for any smooth metric on  $M^n$ .*

(b) [15]. *Furthermore, if  $V(x, r)$  is the volume of the ball of radius  $r$  and center  $x \in \widehat{M}^n$  (the universal covering space of  $M^n$ , with metric lifted from  $M^n$ ), then  $h(g) \geq \lim_{r \rightarrow \infty} (\ln V(x, r)/r)$ ; for a manifold with nonpositive sectional curvature this inequality becomes an equality.*

We note that although in [12] part (a) of the theorem is formulated for infinitely smooth metrics, the proof given goes through for  $C^1$ -metrics, i.e., for metrics for which geodesic flow is defined.

Theorems 6 and 7 exhaust the presently known results on topological obstructions to existence of geodesic flows with zero topological entropy. This problem is of interest separately from its connection with the problem of topological obstructions to integrability, which is dealt with in the next section.

Finally, we remark on rationally elliptic manifolds. A manifold  $M^n$  with finite fundamental group is called *rationally elliptic* if  $\text{rank}(\pi_*(M^n) \otimes \mathbb{Q}) < \infty$ .

**Theorem 8** [16]. *If the fundamental group of a manifold  $M^n$  is finite, then:*

(a) *The condition of subexponential growth of the Betti numbers of the loop space on  $M^n$  is equivalent to rational ellipticity.*

(b) *If  $M^n$  is rationally elliptic, then*

- 1)  $\dim(\pi_*(M^n) \otimes \mathbb{Q}) \leq n$ ;
- 2)  $\dim H_*(M^n, \mathbb{Q}) \leq 2^n$ ;
- 3) *the Euler-Poincaré characteristic  $\chi(M^n)$  is nonnegative, and*
- 4)  $\chi(M^n) > 0$  *if and only if all the odd-dimensional homology groups of  $M^n$  are finite.*

#### 4. TOPOLOGICAL ENTROPY OF INTEGRABLE FLOWS

**Theorem 9.** *If a geodesic flow on a closed Riemannian manifold  $M^n$  is integrable and satisfies at least one of Conditions I or II, then its topological entropy (i.e., the topological entropy of its restriction to the relevant level surface  $L$ ) is zero.*

*Proof.* By Bowen's theorem, the topological entropy of the flow is equal to  $h(f_t) = \sup_{\mu} h(f_t, \mu)$ , where the supremum is taken over all ergodic invariant Borel measures  $\mu$  [7] and  $h(f_t, \mu)$  is the entropy of the flow  $f_t$  with respect to  $\mu$ .

We prove that if an integrable geodesic flow satisfies at least one of Conditions I or II, then its entropy with respect to any ergodic invariant Borel measure is zero.

Let  $\mu$  be an ergodic invariant Borel measure.

Let  $\Phi: L \rightarrow R^{n-1}$  be a moment mapping. Cover its image by balls of radius 1. The inverse image of each ball is an invariant closed set, and its measure (since  $\mu$  is ergodic) is either zero or unity. Since there are only finitely many such balls and the measure of the inverse image of the moment mapping is unity, we can select one of the balls, call it  $S_1$ , whose inverse image has measure unity. Now cover  $S_1$  by balls of radius  $1/2$  and select from them, similarly, a ball  $S_2$  whose inverse image has measure unity. Continuing this process, we obtain a sequence of balls  $S_j$  of radius  $1/j$  such that the inverse image of each ball has measure unity and all the balls together have nonempty intersection—some point  $c \in R^{n-1}$ . Obviously (from the countable additivity of the measure) the measure of the inverse image of this point is unity, i.e., the whole measure  $\mu$  is concentrated on  $\Phi^{-1}(c)$ .

By part 1 of Conditions I and II, the inverse image of a point has at most countably many connected components (obviously they are invariant). The measure of each component is either zero or unity. From the countable additivity of the measure it follows that there exists a unique component  $W$  such that  $\mu(W) = 1$ .

It suffices now to prove that the topological entropy of the restriction of the flow to  $W$  is zero.

By another result of Bowen, it suffices in the definition of topological entropy to take the supremum of entropies with respect to those invariant Borel ergodic measures that are concentrated on the set of nonwandering points, since the topological entropy of a flow is the same as that of its restriction to this set [7], [12]. We recall that a point  $z$  is said to be nonwandering if for any neighborhood  $U$  of  $z$  there exists an arbitrarily large  $t$  such that the intersection  $U \cap f_t(U)$  is nonempty. The nonwandering points form a closed invariant set.

Our argument proceeds by induction. Suppose  $h(f_t|_W) > 0$ . Let  $W_k = \bigcup_{j \leq k} W^{(j)}$ . We show that the equality  $h(f_t|_W) = h(f_t|_{W_k})$  implies  $h(f_t|_W) = h(f_t|_{W_{k-1}})$ . Since  $W_0$  is empty, this brings us to a contradiction, and it follows that  $h(f_t|_W) = 0$ .

Suppose  $h(f_t|_W) = h(f_t|_{W_k})$ , and let  $\eta$  be an ergodic measure on  $W_k$  concentrated on the set of nonwandering points (nonwandering on  $W_k$ !). We show that if  $\eta(W^{(k)}) = 1$ , then  $h(f_t|_{W_k}, \eta) = 0$ . There are two cases to examine: that of Condition I being satisfied and that of Condition II.

*Case 1 (Condition I satisfied).* The  $I$ -orbit of a wandering point is noncompact. If the  $f_t$ -orbit of a point in  $W^{(k)}$  whose  $I$ -orbit is noncompact fails to lie on an imbedded compact invariant torus, it follows from part 2 of Condition I that the orbits of nearby points also fail to lie on compact invariant tori, and the point is wandering. We observe, therefore, an important fact: if the  $f_t$ -orbit of a point in  $W^{(k)}$  fails to lie on a compact invariant torus, then the point is wandering. Let  $Y$  be the set of points in  $W^{(k)}$  whose  $f_t$ -orbits lie on compact invariant tori (we can even be more specific: the closures of whose  $f_t$ -orbits are such tori). This set is closed in  $W^{(k)}$ , but not necessarily in  $W_k$ .

Consider the closure of  $Y$  in  $W_k$  (call it  $Y^*$ ) and a covering of  $Y^*$  by balls of radius 1. To each ball  $\Omega$ , associate the closure of the set of points in  $Y$  the closures of whose  $f_t$ -orbits intersect  $\Omega$ . Denote it by  $\Omega'$ . This set is closed and invariant. If  $\eta(W^{(k)}) = 1$ , there exist a ball  $\Omega_1$  such that  $\eta(\Omega'_1) = 1$ . Now consider a covering



of  $\Omega$  by balls of radius  $1/2$ ; etc. Repeating this process indefinitely, we obtain a sequence of balls  $\Omega_j$  of radius  $1/j$  with nonempty intersection and such that for every  $j$  the measure of  $\Omega'_j$  is unity:  $\eta(\Omega'_j) = 1$ . Since the closures of the  $f_t$ -orbits of all the points in  $Y$  are compact imbedded invariant tori, we conclude:

(a) If the point of intersection of all the balls  $\Omega_j$  lies in  $Y$ , then the measure of the closure of its  $f_t$ -orbit is unity (i.e., the measure  $\eta$  is concentrated on this torus); and since this torus  $T^*$  is a low-dimensional Liouville torus (i.e., the flow trajectories wind around it in a single direction (in some linear coordinate system on  $T^*$ )), we have  $h(f_t, \eta) = h(f_t|_{T^*}) = 0$ .

(b) If the point of intersection of all the balls  $\Omega_j$  lies in  $Y^* \setminus Y$ , then  $\eta(W^{(k)}) = 0$ , contradicting the choice of  $\eta$ .

We have shown that if  $\eta(W^{(k)}) = 1$ , then  $h(f_t|_{W_k}, \eta) = 0$ , and so  $h(f_t|_{W_k}) (= h(f_t|_W)) = h(f_t|_{W_{k-1}})$ .

Thus, we find that  $h(f_t|_W) = h(f_t|_{W_0}) = 0$ . Consequently, when Condition I is satisfied, the assertion of Theorem 9 has been proved.

*Case 2* (Condition II satisfied). In the same way as was shown above that any ergodic measure is concentrated on the inverse image (under the moment mapping) of a single point, we can prove that the measure  $\eta$  is concentrated on the set of solutions of equation (\*) for a fixed set of vectors  $v_1, \dots, v_{n-k}$ , regarded as points of the projecting space. Since equations (\*) determine in the neighborhood of every point of  $W^{(k)}$  a  $k$ -dimensional submanifold, the number of  $I$ -orbits of these points is at most countable. From the countable additivity of the measure it follows that there exists a single such  $I$ -orbit  $Z$  for which  $\eta(Z) = 1$  (it is easily seen that these orbits are Borel sets).

We can suppose also that the measure  $\eta$  is concentrated on the set of nonwandering points (as in the argument for Case 1).

The orbit  $Z$  is diffeomorphic to  $T^j \times R^{k-j}$ ; and in linear coordinates  $(x_1, \dots, x_j, x_{j+1}, \dots, x_k)$  (where  $x_1, \dots, x_j$  are defined mod  $2\pi$ ) the flow is given by equations

$$x_i = \omega_i, \quad 1 \leq i \leq k,$$

where  $\omega_i = \text{const}$ . If  $\omega_i$  is different from zero for at least one of the indices  $i$ ,  $j+1 \leq i \leq k$ , then all points of  $Z$  are wandering and we need not consider this case. If  $\omega_i = 0$  for  $i > j$ , we can show, by the same device used more than once above, that there exist constants  $c_{j+1}, \dots, c_k$ , such that  $\eta(Z^*) = 1$ , where  $Z^* = Z \cap \{x_{j+1} = c_{j+1}, \dots, x_k = c_k\}$ ; and since the entropy of the restriction of the flow to  $Z^*$  is zero, that  $h(f_t, \eta) = 0$ . Since  $\eta$  was taken to be an arbitrary ergodic measure concentrated on the set of nonwandering points of  $W^{(k)}$ , it follows that  $h(f_t|_{W_k}) = h(f_t|_{W_{k-1}})$ .

We conclude now by induction that  $h(f_t|_W) = h(f_t|_{W_0}) = 0$ , proving Theorem 9 when Condition II is satisfied.

This completes the proof of Theorem 9.

*Remark.* It is easily seen that the proof uses nowhere the fact that the flow is geodesic, so it applies to an arbitrary flow.

## 5. CONJECTURES AND CONCLUDING REMARKS

At the end of the 80's we advanced the following as a conjecture:

If the geodesic flow on an  $n$ -dimensional Riemannian manifold  $M^n$  is integrable, then  $\dim H_j(M^n; \mathbb{Q}) \leq \dim H_j(T^n; \mathbb{Q})$ .

The conjecture has as yet been neither proved nor disproved. It appears to us now that the inequality holds in a more general case: when the geodesic flow has zero topological entropy.

In [4] Paternain made the following conjecture:

If the geodesic flow on a closed Riemannian manifold is integrable, then

- 1) the fundamental group of the manifold has polynomial growth, and
- 2) the manifold is rationally elliptic.

By Theorem 2, the geodesic flow on a manifold whose fundamental group is not almost commutative cannot be geometrically simple; so by Theorem 1 the fundamental groups of manifolds with geometrically simple flows must have polynomial growth. Thus, the first part of Paternain's conjecture is proved (as a corollary to Theorems 1 and 2) in the case that the first integrals have sufficiently good analytic properties (in particular, if they are real-analytic).

The topicality of these conjectures is underlined by recent results of Paternain and Spatzier, who have constructed integrable geodesic flows on a number of manifolds not homeomorphic to homogeneous ones. All the integrable geodesic flows known until lately are listed in the monograph [18]; at present it suffices to supplement this list by the new ones to be found in [17].

#### APPENDIX

After completing this survey the author received from Paternain the text of a paper [21] prepared by him for publication, presenting new results based on his entropy approach (which is discussed in §3). These results have already been mentioned in the Introduction, in connection with integrable flows that are Ito-nondegenerate. We were familiar with the course of Paternain's work through correspondence, but can now present the new results in their entirety.

To start with, we give the definition of nondegeneracy.

Let  $O(x)$  be the  $I$ -orbit of a point  $x$  in a symplectic manifold  $M^{2n}$  with respect to an integrable Hamiltonian flow. Denote by  $F(O(x))$  the space of all infinitely differentiable functions defined in some neighborhood of  $O(x)$  and in involution with all the first integrals.

If  $\dim O(x) = k$ , then the neighborhood of  $x \in M^{2n}$  we can introduce coordinates  $(u, v) \in \mathbb{R}^{2k}$ ,  $z \in \mathbb{R}^{2(n-k)}$ , such that:  $I_j = v_j$ , where  $j = 1, \dots, k$  and the first integrals  $I_1, \dots, I_k$  are functionally independent at  $x$ ;  $x = (0, 0, 0)$ ; and the symplectic 2-form is given by

$$\Omega = \sum_{j=1}^k du_j \wedge dv_j + \sum_{j=1}^{(n-k)} d\zeta_j \wedge d\eta_j, \quad z = (\zeta, \eta).$$

**Definition.** The orbit  $O(x)$  is *nondegenerate* if there exists a function  $f \in F(O(x))$  such that  $f_z(0, 0) = 0$  and the matrix  $f_{zz}(0, 0)$  is invertible. An integrable flow is called *nondegenerate* if every  $I$ -orbit is nondegenerate.

**Theorem A.** *If an integrable flow is nondegenerate, then the topological entropy of its restriction to any compact invariant set is zero.*

Another result is also connected with the introduction of canonical coordinates in the neighborhoods of points of the "singular set".

**Definition.** An *action-angle coordinate system with singularities* (for the Hamiltonian  $H$ ) at a point  $x \in M^{2n}$  is given by a diffeomorphism  $\varphi: U \rightarrow T^k \times D_1 \times D_2$ , where  $U$  is a neighborhood of  $x$ ,  $T^k$  a  $k$ -dimensional torus with coordinates  $(\theta_1, \dots, \theta_k)$ ,  $D_1$  a domain in  $\mathbb{R}^k$  with coordinates  $(I_1, \dots, I_k)$ , and  $D_2$  a domain in  $\mathbb{R}^{2m}$  with coordinates  $(\zeta, \eta) = (\zeta_1, \dots, \zeta_m, \eta_1, \dots, \eta_m)$ , such that

- 1)  $\Omega = \varphi^* \Omega_0$ , where  $\Omega_0 = \sum_j d\theta_j \wedge dI_j + \sum_j d\zeta_j \wedge d\eta_j$ , and
- 2) the function  $H$  depends only on  $I_1, \dots, I_k$  and  $\tau_1, \dots, \tau_m$ , where  $\tau_j = (\zeta_j^2 + \eta_j^2)/2$ .

**Theorem B.** *If  $V^n$  is a compact smooth simply-connected Riemannian manifold and for every covector of unit length in the cotangent bundle there exist action-angle coordinates with singularities (for the Hamiltonian of the geodesic flow), then the dimensions of the homology groups of the loop space on  $V^n$  grow polynomially for any coefficient field (i.e., the manifold  $V^n$  is  $\mathbb{Z}$ -elliptic).*

The proofs of these theorems are based on the fact that under their hypotheses the flow has only trivial Poincaré first-return mappings and has therefore zero topological entropy. The argument can also be applied to the case that there exists on the manifold a sufficiently good Hamiltonian action by a compact Lie group. This approach was used successfully by Paternain in [4]; he recently sharpened the results in [22].

Let  $M^{2n}$  be a symplectic manifold with symplectic form  $\Omega$ , on which is given a smooth Hamiltonian action by a compact connected Lie group  $G$ . Denote by  $B_x$  the subspace of  $T_x M^{2n}$  formed by the vectors conjugate to  $\text{sgrad } f$ , where  $f$  is a  $G$ -invariant function. Let  $B_x^\perp$  be the orthogonal complement to  $B_x$  with respect to the form  $\Omega$ . If  $\dim(B_x/(B_x \cap B_x^\perp)) = k$  at a point in general position, we say that the action of  $G$  has *multiplicity  $k$* .

The main result of [22] is the following.

**Theorem C.** *If the Hamiltonian is  $G$ -invariant and the multiplicity of the action of  $G$  is  $\leq 2$ , then the restriction of the Hamiltonian flow to any flow-invariant and  $G$ -invariant compact subset has zero topological entropy.*

These results are obtained for arbitrary Hamiltonian flows. It is clear enough how to apply them to finding topological obstructions to integrability of geodesic flows.

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Translated by J. A. ZILBER

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\*Translator's note. This preprint has now appeared, in J. Geom. Phys. **13** (1994), 289–298.