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Modified Novikov–Veselov Equation and Differential Geometry of Surfaces

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§1. Introduction

In the present paper we consider global soliton deformations of surfaces immersed in three-dimensional Euclidean space.

The local deformation of surfaces represented via the generalized Weierstrass formulas (3.3)–(3.4) were introduced by Konopelchenko ([**Kon**]) by using the modified Novikov–Veselov (mNV) equation (2.11), which in turn was introduced by Bogdanov ([**Bg1**]). This equation is a modification of the Novikov–Veselov equation in the same sense as the modified Korteweg–de Vries equation is a modification of the ordinary Korteweg–de Vries equation. Speaking of the geometric meaning of the Novikov–Veselov equation, we note that it has important applications to the theory of algebraic curves ([**T, Sh**]).

Here we shall discuss applications of the modified Novikov–Veselov equation to the differential geometry of surfaces. Investigation of the global properties of mNV-deformations of a surface in the case of tori of revolution was started in [**KT**], where it was shown in particular that tori of revolution are preserved under these deformations. But at that time the relation of these deformations to conformal geometry was not understood.

We consider global deformations of surfaces of general type and their relationship to the theory of the Willmore functional, which is defined as the integral of squared mean curvature (5.1).

Every regular surface is locally representable via the generalized Weierstrass formulas and, moreover, each analytic surface is globally representable in this manner (see Propositions 1 and 2). Thus, at least for analytic surfaces, the mNV-deformation is well defined.

We show that *the mNV-deformation transforms tori into tori and preserves their conformal structures and the values of the Willmore functional* (*Theorems* 1 *and* 2).

We also consider the following conjecture.

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CONJECTURE. *A nonstationary (with respect to an mNV-deformation) torus cannot be a local minimum of the Willmore functional.*

We discuss its relation to the famous Willmore conjecture (see §5). The author thanks M. V. Babich, B. G. Konopelchenko, and S. V. Manakov for helpful conversations.

§2. Modified Novikov–Veselov equation

2.1. **Novikov–Veselov equation.** The Novikov–Veselov equation (NV),

$$U_t = \partial^3 U + \bar\partial^3 U + \partial(VU) + \bar\partial(\overline{V}U), \qquad \bar\partial V = 3\partial U, \tag{2.1}$$

was introduced by Novikov and Veselov in [**VN1**] within the framework of the theory of two-dimensional potential Schrödinger operators that are finite-zone on one level of energy ([**DKN, VN2**]).

Here the functions U and V are defined on the complex plane and the partial derivatives ∂ and $\bar\partial$ are defined by the usual formulas

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \bar\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad z = x + iy \in \mathbb{C}.$$

This equation is a natural two-dimensional generalization of the famous Korteweg–de Vries equation (KdV),

$$U_t = \frac{1}{4}U_{xxx} + \frac{3}{2}UU_x, \tag{2.2}$$

to which the NV equation reduces by a suitable renormalization of the variable x in the case when the function $U(z,\bar z)$ does not depend on the variable y. Investigation of the KdV equation and the discovery of its remarkable properties were the starting point of the intensive development of soliton theory. Since the theory of this equation is presented in many monographs (see, for instance [**N**]), here we dwell only on some facts which are essential to our further exposition.

The KdV equation may be presented as the commutativity relation

$$\left[L^{\mathrm{KdV}}, \frac{\partial}{\partial t} - A\right] = 0, \tag{2.3}$$

for two scalar differential operators

$$L^{\mathrm{KdV}} = \frac{\partial^2}{\partial x^2} + U, \qquad A = \frac{\partial^3}{\partial x^3} + \frac{3}{2}U\frac{\partial}{\partial x} + \frac{3}{4}U_x\,. \tag{2.4}$$

In this case we say that the equation is represented by an (L, A)*-pair*.

The Novikov–Veselov equation, as opposed to the KdV equation, is represented by an (L, A, B)*-triple*

$$\frac{\partial L}{\partial t} + [L, A] - BL = 0, \tag{2.5}$$

where

$$L^{NV} = \partial\bar\partial + U, \quad A = (\partial^3 + V\partial) + (\bar\partial^3 + \overline{V}\bar\partial), \quad B = \partial V + \bar\partial\overline{V}. \tag{2.6}$$

Here B is a scalar operator of multiplication by a function.

The representation of nonlinear equations by (L, A, B)-triples was introduced by Manakov [**M**] as a two-dimensional generalization of the representation (2.3). Indeed, equation (2.3) preserves the spectrum of the operator L and deforms its eigenfunctions as follows:

$$\frac{\partial \psi}{\partial t} = A\psi, \qquad L\psi = \lambda\psi . \tag{2.7}$$

The operator (2.6) is multidimensional and its eigenspaces are infinite-dimensional in general. Equation (2.5) does not deform all the eigenspaces, but only the kernel of the operator L via the equation

$$\frac{\partial \phi}{\partial t} = A\psi, \qquad L\phi = 0 . \tag{2.8}$$

In some sense the Novikov–Veselov equation is a more natural two-dimensional generalization of the KdV equation than the famous Kadomtsev–Petviashvili equation

$$\left(U_t - \frac{1}{4}(U_{xxx} + 6UU_x)\right)_x = \frac{3}{4}U_{yy} .$$

This equation also reduces to the KdV equation in the case when the function U does not depend on the space variable y and it is represented by an (L, A)-pair, where the operator L has the form

$$\frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2} - U(x, y) .$$

This operator differs from the two-dimensional operator (2.6), which is the usual two-dimensional Schrödinger operator, i.e., it is the most natural two-dimensional generalization of the one-dimensional Schrödinger operator (2.4).

Notice that there exist two different deformations of the operator (2.6) that have the form (2.5):

$$\frac{\partial L^{\mathrm{NV}}}{\partial t^{\pm}} + [L^{\mathrm{NV}}, A^{\pm}] - B^{\pm}L = 0 .$$

These deformations are represented by (L, A, B)-triples for the operators

$$A^+ = \partial^3 + V\partial, \quad \bar\partial V = 3\partial U, \quad B^+ = \partial V,$$
$$A^- = \bar\partial^3 + \overline{V}\bar\partial, \quad \partial\overline{V} = 3\bar\partial U, \quad B^+ = \bar\partial\overline{V}.$$

But these deformations do not preserve real potentials. In its turn, equation (2.1), which in fact is their linear superposition, transforms the real potentials U into real ones.

2.2. **Modified Novikov–Veselov equation.** There exists another very well-known $(1+1)$-dimensional integrable equation called the *modified Korteweg–de Vries* equation (mKdV):

$$U_t = U_{xxx} + 24U^2U_x . \tag{2.9}$$

This equation is represented by an (L, A)-pair which we shall not present here. We only mention that the L-operator has the following form:

$$L^{\mathrm{mKdV}} = \frac{\partial}{\partial x} - \frac{1}{2}\begin{pmatrix} -1 & 4U \\ -4U & 1 \end{pmatrix} . \tag{2.10}$$

In [**Bg1**] Bogdanov introduced a two-dimensional generalization of the mKdV equation, the modified Novikov–Veselov (mNV) equation

$$U_t = \left(U_{zzz} + 3U_zV + \frac{3}{2}UV_z\right) + \left(U_{\bar z\bar z\bar z} + 3U_{\bar z}\overline{V} + \frac{3}{2}U\overline{V}_{\bar z}\right), \tag{2.11}$$

where

$$V_{\bar z} = (U^2)_z. \tag{2.12}$$

Just as the NV equation, this equation is also a linear superposition of two deformations of the form (2.5) represented by (L, A, B)-triples with the common operator L defined by

$$L^{\mathrm{mNV}} = \begin{pmatrix} \partial & -U \\ U & \bar\partial \end{pmatrix}, \tag{2.13}$$

and the following A- and B-operators

$$\begin{aligned} A^+ &= \partial^3 + 3\begin{pmatrix} 0 & -U_z \\ 0 & V \end{pmatrix}\partial + \frac{3}{2}\begin{pmatrix} 0 & 2UV \\ 0 & V_z \end{pmatrix}, \\ B^+ &= 3\begin{pmatrix} 0 & U_z \\ -U_z & 0 \end{pmatrix}\partial + 3\begin{pmatrix} 0 & -UV \\ -U_{zz} - UV & 0 \end{pmatrix} \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} A^- &= \bar\partial^3 + 3\begin{pmatrix} \overline{V} & 0 \\ U_{\bar z} & 0 \end{pmatrix}\bar\partial + \frac{3}{2}\begin{pmatrix} \overline{V}_{\bar z} & 0 \\ -2U\overline{V} & 0 \end{pmatrix}, \\ B^- &= 3\begin{pmatrix} 0 & U_{\bar z} \\ -U_{\bar z} & 0 \end{pmatrix}\bar\partial + 3\begin{pmatrix} 0 & U_{\bar z\bar z} + U\overline{V} \\ U\overline{V} & 0 \end{pmatrix}. \end{aligned} \tag{2.15}$$

These triples represent the equations

$$U_{t^+} = U_{zzz} + 3U_zV + \frac{3}{2}UV_z \quad \text{and} \quad U_{t^-} = U_{\bar z\bar z\bar z} + 3U_{\bar z}\overline{V} + \frac{3}{2}U\overline{V}_{\bar z},$$

where the function V is defined by formula (2.12).

As in the case of the NV equation, we can derive that

1) if the function U depends only on one spatial variable x, then the mNV equations reduce to the mKdV equation;

2) equation (2.11) transforms real potentials into real ones, as opposed to the equations represented by $(L^{\mathrm{mNV}}, A^\pm, B^\pm)$-triples;

3) the modified Novikov–Veselov equations deform the kernel of the operator L via the equations

$$\frac{\partial\psi}{\partial t^\pm} = A^\pm\psi, \qquad L^{\mathrm{mNV}}\psi = 0, \tag{2.16}$$

and the deformation of eigenfunctions of L^{mNV} via (2.11) is defined by

$$\begin{aligned} \frac{\partial}{\partial t}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \Bigg(\partial^3 + \bar\partial^3 &+ 3\begin{pmatrix} 0 & -U_z \\ 0 & V \end{pmatrix}\partial + 3\begin{pmatrix} \overline{V} & 0 \\ U_{\bar z} & 0 \end{pmatrix}\bar\partial \\ &+ \frac{3}{2}\begin{pmatrix} 0 & 2UV \\ 0 & V_z \end{pmatrix} + \frac{3}{2}\begin{pmatrix} \overline{V}_{\bar z} & 0 \\ -2U\overline{V} & 0 \end{pmatrix}\Bigg)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \end{aligned} \tag{2.17}$$

2.3. **Hierarchies of equations.** One of the remarkable properties of equations integrable by the inverse scattering method is that they are included into hierarchies of such equations, which are recursively defined.

For instance, the KdV equation and its modification (mKdV) are only the first members (for $k=1$) of hierarchies of equations of the form

$$U_{t_{2k+1}} = N_{2k+1}(U),$$

where $N_{2k+1}(U)$ are nonlinear operators. These equations are represented by (L,A)-pairs with operators L^{KdV} and L^{mKdV}, respectively. For the KdV hierarchy, the operators A have the following form

$$A_k^{\mathrm{KdV}} = \frac{\partial^{2k+1}}{\partial x^{2k+1}} + \dots,$$

where by dots we denote terms of lower orders. These terms are defined by the condition that the commutators of the operators L and A are operators of multiplication by scalars.

Thus we can say that the KdV hierarchy is attached to the operator L^{KdV}. The mKdV hierarchy is defined similarly.

The NV equation and its modification are also included in hierarchies for which the A-operators take the form

$$A_k^{\mathrm{NV}} = \partial^{2k+1} + \cdots \quad \text{and} \quad A_k^{\mathrm{mNV}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial^{2k+1} + \cdots,$$

respectively.

We can also say that these hierarchies are attached to the operators L^{NV} (i.e., to the two-dimensional Schrödinger operator) and L^{mNV} (i.e., to the Dirac operator), respectively.

In soliton theory, the method of defining hierarchies by using the so-called recursion operators is well known. For instance, the kth equation of the KdV hierarchy takes the form

$$U_{t_{2k+1}} = R^k(U_x),$$

where the recursion operator is given by

$$R = \frac{\partial^2}{\partial x^2} + 3U + 3U_x \left(\frac{\partial}{\partial x} \right)^{-1}.$$

Regretably, (2+1)-equations, and the NV equations among them, do not admit such simple representation in terms of a local operator R. There exists a method based on bilocal operators (see [**FS**]) but its realization is more difficult than in the case of the KdV equations. This is seen from the form of higher equations. We shall describe, for instance, only the second equations (NV2) of these hierarchies.

The NV2 equations

$$U_{t_3^+} = \Phi_{\mathrm{NV2}}(U), \qquad U_{t_3^-} = \overline{\Phi_{\mathrm{NV2}}(U)},$$

where

$$\begin{aligned} \Phi(U)_{\mathrm{NV2}} &= \partial^5 U + V\partial^3 U + 2V_z \partial^2 U + (W + V_{zz})\partial U + W_z U \\ &= \partial^5 U + \partial(V\partial^2 U + V_z \partial U + WU), \end{aligned}$$

are represented by (L, A, B)-triples with the following operators

$$A^+ = \partial^5 + V\partial^3 + V_z\partial^2 + W\partial, \qquad B^+ = V_z\partial^2 + V_{zz}\partial + W_z,$$
$$A^- = \bar{A}^+, \qquad B^- = \bar{B}^+,$$

and

$$\bar\partial V = 5\partial U, \qquad \bar\partial W = 5\partial^3 U + 3V\partial U + V_z U.$$

The second equations of the mNV hierarchy are more complicated:

$$U_{t_3^+} = \Phi_{\mathrm{mNV2}}(U), \qquad U_{t_3^-} = \overline{\Phi_{\mathrm{mNV2}}(U)},$$

where

$$\Phi(U)_{\mathrm{mNV2}} = U_{zzzzz} + 5VU_{zzz} + \frac{15}{2}V_zU_{zz} + 5\Big(V^2 - \frac{3}{2}V_{zz} + W\Big)U_z + 5\Big(VV_z - V_{zzz} + \frac{1}{2}W_z\Big)U$$

and

$$V_{\bar z} = (U^2)_z, \qquad W_{\bar z} = (U^2V - U_z^2)_z.$$

The operators A and B are given by

$$A^+ = \partial^5 + 5\begin{pmatrix} 0 & -U_z \\ 0 & V \end{pmatrix}\partial^3 + 5\begin{pmatrix} 0 & UV - U_{zz} \\ 0 & \frac{3}{2}V_z \end{pmatrix}\partial^2 + 5\begin{pmatrix} 0 & \frac{1}{2}UV_z - U_zV - U_{zzz} \\ 0 & V^2 - \frac{3}{2}V_{zz} + W \end{pmatrix}\partial + 5\begin{pmatrix} 0 & UV^2 - 2UV_{zz} + U_{zz}V + \frac{1}{2}U_zV_z + UW \\ 0 & VV_z - V_{zzz} + W_z \end{pmatrix},$$

$$B^+ = 5\begin{pmatrix} 0 & U_z \\ -U_z & 0 \end{pmatrix}\partial^3 + 5\begin{pmatrix} 0 & U_{zz} - UV \\ -UV - 2U_{zz} & 0 \end{pmatrix}\partial^2 + 5\begin{pmatrix} 0 & U_zV + U_{zzz} - \frac{1}{2}UV_z \\ -\frac{3}{2}UV_z - 2U_{zzz} - 3U_zV & 0 \end{pmatrix}\partial - 5\begin{pmatrix} 0 & U(V^2 + W - 2V_{zz}) + U_{zz}V + \frac{1}{2}U_zV_z \\ U(V^2 + W - \frac{3}{2}V_{zz}) + U_{zz}V + 3U_zV_z + U_{zzzz} & 0 \end{pmatrix},$$

and

$$A^- = \bar\partial^5 + 5\begin{pmatrix} \overline{V} & 0 \\ U_{\bar z} & 0 \end{pmatrix}\bar\partial^3 + 5\begin{pmatrix} \frac{3}{2}\overline{V}_{\bar z} & 0 \\ -U\overline{V} + U_{\bar z\bar z} & 0 \end{pmatrix}\bar\partial^2 + 5\begin{pmatrix} \overline{V}^2 - \frac{3}{2}\overline{V}_{\bar z\bar z} + \bar W & 0 \\ -\frac{1}{2}U\overline{V}_{\bar z} + U_{\bar z}V + U_{\bar z\bar z\bar z} & 0 \end{pmatrix}\bar\partial + 5\begin{pmatrix} \overline{V}\overline{V}_{\bar z} - \overline{V}_{\bar z\bar z\bar z} + \bar W_{\bar z} & 0 \\ -U\overline{V}^2 + 2U\overline{V}_{\bar z\bar z} - U_{\bar z\bar z}\overline{V} - \frac{1}{2}U_{\bar z}\overline{V}_{\bar z} - U\bar W & 0 \end{pmatrix},$$

$$B^- = 5\begin{pmatrix} 0 & U_{\bar z} \\ -U_{\bar z} & 0 \end{pmatrix}\bar\partial^3 + 5\begin{pmatrix} 0 & U\overline{V} + 2U_{\bar z\bar z} \\ -U_{\bar z\bar z} + U\overline{V} & 0 \end{pmatrix}\bar\partial^2$$
$$+ 5\begin{pmatrix} 0 & \frac{3}{2}U\overline{V}_{\bar z} + 2U_{\bar z\bar z\bar z} + 3U_{\bar z}\overline{V} \\ -U_{\bar z}\overline{V} - U_{\bar z\bar z\bar z} + \frac{1}{2}U\overline{V}_{\bar z} & 0 \end{pmatrix}\bar\partial$$
$$+ 5\begin{pmatrix} 0 & \begin{array}{l} U(\overline{V}^2 + \bar W - \frac{3}{2}\overline{V}_{\bar z\bar z}) \\ +U_{\bar z\bar z}\overline{V} + 3U_{\bar z}\overline{V}_{\bar z} + U_{\bar z\bar z\bar z\bar z} \end{array} \\ \begin{array}{l} U(\overline{V}^2 + \bar W - 2\overline{V}_{\bar z\bar z}) \\ +U_{\bar z\bar z}\overline{V} + \frac{1}{2}U_{\bar z}\overline{V}_{\bar z} \end{array} & 0 \end{pmatrix}.$$

Equations

$$U_t = \Phi_{\mathrm{NV2}}(U) + \overline{\Phi_{\mathrm{NV2}}(U)} \quad \text{and} \quad U_t = \Phi_{\mathrm{mNV2}}(U) + \overline{\Phi_{\mathrm{mNV2}}(U)}$$

take real potentials to real ones as in the case of the first equations.

§3. Weierstrass representation

3.1. **Construction of minimal surfaces.** The most general method for constructing minimal surfaces in three-dimensional Euclidean space was introduced by Weierstrass, and we begin explanation of how to represent surfaces by presenting it.

Let us take a pair of functions (ψ_1, ψ_2) such that ψ_1 is antiholomorphic and ψ_2 is holomorphic. Let us suppose that these functions are defined in the same simply connected domain S in the complex plane. We have a system of equations

$$\begin{cases} \psi_{1z} = 0, \\ \psi_{2\bar z} = 0. \end{cases} \tag{3.1}$$

Now, in terms of these functions let us define the mapping

$$T\colon S \to \mathbb{R}^3 \tag{3.2}$$

by the following formulas:

$$z \in S \to (X^1(z,\bar z), X^2(z,\bar z), X^3(z,\bar z)) \in \mathbb{R}^3,$$

where

$$\begin{gathered} X^1 + iX^2 = i\int_\gamma (\bar\psi_1^2\,dz' - \bar\psi_2^2\,d\bar z'), \qquad X^1 - iX^2 = i\int_\gamma (\psi_2^2\,dz' - \psi_1^2\,d\bar z'), \\ X^3 = -\int_\gamma (\psi_2\bar\psi_1\,dz' + \psi_1\bar\psi_2\,d\bar z'). \end{gathered} \tag{3.3}$$

Everywhere we suppose that the integrals are taken over any path γ which lies in the domain S and connects the point z to some initial point z_0. It follows from (3.1) that integrands are closed forms and hence the values of the integrals do not depend on the choice of the path γ.

Weierstrass showed that the *surface* $T(S)$ *is minimal in the sense that its mean curvature vanishes everywhere.*

3.2. **Generalized Weierstrass formulas.** It is natural to ask when formulas (3.3) define a surface in three-dimensional Euclidean space. As one can see, the integrands must be closed forms, and this condition is sufficient. In the case of the Weierstrass representation, this fact follows from (3.1).

It turns out that if the functions ψ_1, ψ_2 satisfy the more general system

$$\begin{cases} \psi_{1z} = U\psi_2, \\ \psi_{2\bar{z}} = -U\psi_1 \end{cases} \tag{3.4}$$

with real potential U, then the integrands in (3.3) turn out to be closed forms. Hence in this case formulas (3.3) define a surface for every solution to system (3.4).

This was shown in [**Kon**], where formulas for the induced metric and curvatures were also derived. Let us explain them here. The coordinates $(z, \bar{z})$ are conformal and in terms of them, the metric tensor takes the form

$$D(z,\bar{z})^2\, dz\, d\bar{z}, \quad \text{where } D(z,\bar{z}) = |\psi_1(z,\bar{z})|^2 + |\psi_2(z,\bar{z})|^2. \tag{3.5}$$

The Gaussian curvature is given by

$$K = -\frac{1}{D^2}\,\Delta \log D, \tag{3.6}$$

and the mean curvature takes the form

$$H = 2U/D\,. \tag{3.7}$$

This representation is not new. For instance, it appears in the survey [**Bb**], it was discussed by U. Abresch in the mid-eighties with relation to the construction of constant mean curvature surfaces; in other terms it is given in the book of Eisenhart [**E**], and, moreover, it is equivalent to the well-known Kenmotsu representation ([**Ken**], see also [**HO**]).

Notice that the convenience of this form of representation is that the operator in the linear problem (3.4) coincides with the operator L^{mNV} to which the modified Novikov–Veselov hierarchy is attached. That was the starting point for defining the local deformation of surfaces given in [**Kon**].

We shall show in 3.4 that the existence of such a representation for every regular surface "almost follows" from the definition of the second fundamental form (see Proposition 1).

3.3. **On the representation of surfaces by Weierstrass formulas.** Let us study the width of the class of surfaces represented by formulas (3.3)–(3.4). Let

$$F\colon \Sigma \to \mathbb{R}^3 \tag{3.8}$$

be a regular mapping of the domain Σ of the complex plane $\mathbb{C}$ with coordinates $(z, \bar{z})$ into three-dimensional Euclidean space, the induced metric being conformally Euclidean with respect to these coordinates, i.e., the metric tensor takes the form $D(z,\bar{z})^2\, dz\, d\bar{z}$.

In this case the vector

$$G(z) = \left(\frac{\partial F^1}{\partial z}, \frac{\partial F^2}{\partial z}, \frac{\partial F^3}{\partial z}\right)$$

satisfies the evident equation

$$\left(\frac{\partial F^1}{\partial z}\right)^2 + \left(\frac{\partial F^2}{\partial z}\right)^2 + \left(\frac{\partial F^3}{\partial z}\right)^2 = 0. \tag{3.9}$$

This immediately follows from the formula

$$G(z) = \frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial F}{\partial x} - i\,\frac{\partial F}{\partial y}\right) \tag{3.10}$$

and the condition that the metric is conformally Euclidean:

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial x}\right) = \left(\frac{\partial F}{\partial y}, \frac{\partial F}{\partial y}\right), \qquad \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) = 0.$$

The subvariety $Q \subset \mathbb{C}P^2$ defined in terms of the homogeneous coordinates $(\varphi_1, \varphi_2, \varphi_3)$ by the equation

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$$

is diffeomorphic to the Grassmann manifold $G_{3,2}$ formed by two-dimensional subspaces of $\mathbb{R}^3$. This diffeomorphism is given by the mapping

$$G_{3,2} \to Q,$$

which to the plane generated by the pair of orthogonal unit vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) assigns the point $(a_1 + ib_1, a_2 + ib_2, a_3 + ib_3) \in Q$.

Thus we can regard this mapping G as the Gauss map.

The Gauss map defined in this way for the surface (3.3) takes the form

$$G(z) = (i(\bar{\psi}_1^2 + \psi_2^2)/2, (\bar{\psi}_1^2 - \psi_2^2)/2, -\psi_2\bar{\psi}_1). \tag{3.11}$$

This formula suggests the proof of the following proposition.

PROPOSITION 1. *Every regular conformally Euclidean immersion of a surface in three-dimensional Euclidean space is locally defined by the generalized Weierstrass formulas* (3.3)–(3.4).

For brevity, we did not mention the fact that formulas (3.3)–(3.4) represent every surface locally up to translation in $\mathbb{R}^3$. This is easy to see from (3.3).

PROOF OF PROPOSITION 1. We assume that $F_z^3 \neq 0$; otherwise we change coordinates in $\mathbb{R}^3$ to ensure that condition. Let us compare (3.10) and (3.11) and define the functions

$$\begin{cases} \varphi_1 = \sqrt{\overline{F_z^2 + iF_z^1}}, \\ \varphi_2 = \sqrt{-(F_z^2 + iF_z^1)}. \end{cases} \tag{3.12}$$

It follows from (3.9) that

$$F_z^3 = -\bar{\varphi}_1\varphi_2.$$

Now recall the definition of the second fundamental form h_{ij}. Let $D(z, \bar{z})^2\,dz\,d\bar{z}$ be the metric tensor on the surface (3.8). In the tangent plane (at the point z) we take an orthonormal basis

$$e_1 = \frac{1}{D}\frac{\partial F}{\partial x}, \qquad e_2 = \frac{1}{D}\frac{\partial F}{\partial y}$$

and extend it to a basis in $\mathbb{R}^3$ by adding the unit normal vector $e_3 = e_1 \times e_2$. Components of the curvature tensor are defined by the well-known decomposition formulas (see, for instance, [**Ken**]):

$$\begin{aligned}
\frac{\partial^2 F}{\partial x^2} &= \frac{\partial D}{\partial x} e_1 - \frac{\partial D}{\partial y} e_2 + D^2 h_{11} e_3, \\
\frac{\partial^2 F}{\partial x \partial y} &= \frac{\partial D}{\partial y} e_1 + \frac{\partial D}{\partial x} e_2 + D^2 h_{12} e_3, \\
\frac{\partial^2 F}{\partial y^2} &= -\frac{\partial D}{\partial x} e_1 + \frac{\partial D}{\partial y} e_2 + D^2 h_{22} e_3 .
\end{aligned}$$

Substituting these expressions for the second derivatives of F into the formulas for $\varphi_{1z}, \varphi_{2\bar{z}}$, derived from (3.12), by direct computations we obtain

$$\begin{cases} \varphi_{1z} = \frac{1}{2} D H \varphi_2, \\ \varphi_{2\bar{z}} = -\frac{1}{2} D H \varphi_1, \end{cases}$$

where H is the mean curvature.

Proposition 1 is proved.

An important corollary of Proposition 1 is the following statement.

PROPOSITION 2. *Every regular analytic surface is represented by formulas* (3.3)–(3.4) *globally.*

This follows from the existence of the local representation and the uniqueness of the analytic continuation.

3.4. **Examples of surfaces represented by Weierstrass formulas.** Let us consider the simplest examples of surfaces represented by formulas (3.3)–(3.4).

1) *Surfaces of revolution.*

We assume, without loss of generality, that the axis OX^3 is the axis of revolution. In this case the functions ψ_1 and ψ_2 are given by

$$\psi_1 = r_1(x) \exp \frac{iy}{2}, \qquad \psi_2 = r_2(x) \exp \frac{iy}{2},$$

and system (3.4) takes the form

$$\left(\frac{\partial}{\partial x} - \frac{1}{2} \begin{pmatrix} -1 & 4U \\ -4U & 1 \end{pmatrix} \right) \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0 . \tag{3.13}$$

Here the potential U depends only on one space variable x and it is easy to see that the matrix differential operator from linear problem (3.13) coincides with the operator L^{mKdV} of form (2.10). Hence, in terms of the generalized Weierstrass representation, the reduction of L^{mNV} to L^{mKdV} has a natural geometrical meaning.

2) *Closed surfaces of genus* $\geqslant 1$.

Let $F\colon \Sigma \to \mathbb{R}^3$ be the immersion of a surface of genus $g \geqslant 1$ given by formulas (3.3)–(3.4).

It is well known that every closed oriented surface Σ of positive genus is uniformizable; this means that there exists a mapping $p\colon M \to \Sigma$ of a simply connected surface M of constant curvature (the Euclidean plane for $g = 1$ and the Lobachevskiĭ plane for $g > 1$) and this mapping is a conformal covering.

In other words, there exists a discrete subgroup Γ of the isometry group of M such that the quotient space M/Γ is conformally equivalent to the surface Σ.

We consider the cases $g = 1$ and $g > 1$ separately.

2.1) *Tori* $(g = 1)$.

In this case the subgroup Γ is isomorphic to a free Abelian group of rank 2 (i.e., a two-dimensional lattice) generated by a pair of independent shifts.

If $\gamma \in \Gamma$, then

$$\gamma^*(dz\, d\bar{z}) = dz\, d\bar{z}$$

and hence the following proposition holds.

PROPOSITION 3. *Let Σ be a two-dimensional torus immersed in $\mathbb{R}^3$ by the formulas* (3.3)–(3.4). *Then there exists a lattice of periods Γ of rank* 2 *such that the potential $U(z)$, the metric tensor $D(z)^2$, and the mean curvature are invariant with respect to the action of Γ. At the same time the functions ψ_1, ψ_2 are transformed as follows*

$$\psi_1(z+\gamma) = (\pm 1)\,\psi_1(z), \qquad \psi_2(z+\gamma) = (\pm 1)\,\psi_2(z),$$
$$z \to z+\gamma, \qquad \gamma \in \Gamma.$$

2.2) *Surfaces of genus* $g > 1$.

In this case the space M is isometric to the upper half plane

$$\mathcal{H} = \{(x+iy) \in \mathbb{C} \mid y > 0\}$$

endowed with the metric $(dx^2 + dy^2)/y^2$. The isometry group of $\mathcal{H}$ is the group $PSL(2,\mathbb{R})$, which acts by fractional linear transformations

$$z \to \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{R},\ ad-bc = 1.$$

The action of elements of $PSL(2,\mathbb{R})$ on differentials has the form

$$\gamma^*(dz) = \frac{dz}{(cz+d)^2}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We conclude that

$$D(\gamma(z)) = |cz+d|^2 D(z). \tag{3.14}$$

Since the mean curvature is invariant $(H(z) = H(\gamma(z)))$, it follows from (3.7) that

$$U(\gamma(z)) = |cz+d|^2 U(z). \tag{3.15}$$

Now we are able to reach the following conclusion.

PROPOSITION 4. *Let a surface Σ of genus $g > 1$ be immersed in $\mathbb{R}^3$ by the formulas* (3.3)–(3.4) *and be conformally equivalent to the surface $\mathcal{H}/\Gamma$, where Γ is a discrete subgroup of $PSL(2,\mathbb{R})$. Then the metric tensor $D(z)^2$ and the potential $U(z)$ are transformed by elements of Γ via the formulas* (3.14) *and* (3.15), *respectively, and Γ acts on the functions ψ_1 and ψ_2 as follows*

$$\psi_1(\gamma(z)) = (c\bar{z}+d)\,\psi_1(z), \qquad \psi_2(\gamma(z)) = (cz+d)\,\psi_2(z).$$

§4. Deformation of surfaces by the modified Novikov–Veselov equation

4.1. **Definition of deformation.** In [**Kon**], Konopelchenko defined a new class of deformations of surfaces by using representation (3.3)–(3.4). The main observation of this paper is that the operator from linear problem (3.4) coincides with the operator L^{mNV} to which the modified Novikov–Veselov hierarchy is attached. Hence the following deformation is naturally defined:

1) let $F\colon S \to \mathbb{R}^3$ be a surface immersed by formulas (3.3)–(3.4);

2) assume that the potential $U(z, \bar{z}, t)$ is transformed in t via the modified Novikov–Veselov equation (2.11); at the same time the eigenfunctions ψ_1, ψ_2 are transformed via equation (2.17) and this generates a deformation of the surface F.

We call this deformation the *modified Novikov–Veselov deformation* (mNV deformation).

Moreover it is stated in [**Kon**] that every equation from the mNV hierarchy generates deformation of this type. Minimal surfaces correspond to zero potentials and hence are stationary with respect to these flows.

An interesting observation was noted in [**KT**], where the following proposition was proved.

PROPOSITION 5 ([**KT**]).[1] 1) *The integral of the squared mean curvature over a closed immersed (via* (3.3)–(3.4)) *surface S, i.e., the value of the Willmore functional on the surface S is equal to*

$$W(S) = 4 \int_{F(S)} U(z,\bar{z})^2 \, dz \, d\bar{z}; \tag{4.1}$$

2) *If the closed surface $F(S)$ is deformed by the mNV flow into closed surfaces and the lattices of periods of the functions U and V are preserved, then the value of the Willmore functional is also preserved.*

The proof of the first statement follows immediately from formulas (3.5) and (3.7). The second statement is derived from the formula

$$UU_t = \left(UU_{zz} - \frac{U_z^2}{2} + \frac{3}{2}U^2V\right)_z + \left(UU_{\bar{z}\bar{z}} - \frac{U_{\bar{z}}^2}{2} + \frac{3}{2}U^2\overline{V}\right)_{\bar{z}}, \tag{4.2}$$

which itself follows from (2.11).

In the spirit of this proposition, it is natural to study the global properties of the mNV flow and its relation to the theory of Willmore surfaces. In [**KT**] such an investigation was initiated for tori of revolution.

We shall not dwell here on results of [**KT**] on tori of revolution and explain more general facts.

4.2. **Global deformations of closed surfaces.** In this subsection we investigate when the mNV flow transforms closed surfaces into closed ones and preserves their conformal structure.

[1] The result on the conservation of the Willmore functional under the modified Novikov–Veselov deformation already was used in the work of R. Carroll and B. G. Konopelchenko (*Generalised Weierstrass–Enneper inducing, conformal immersions, and gravity* (Theorem 3.6), Internat. J. Modern Phys. A **11** (1996), no. 7, 1183–1216) for applications in the quantum gravity.

First we believed that the nonautomorphic form of L^{mNV} and A operators (see Proposition 4) implied the nonexistence of mNV-deformations of surfaces of genus $g \geqslant 2$ that preserve the property and the conformal structure. But later we were persuaded by F. Pedit and U. Pinkall that this strongly depends on the correct understanding of constraint (2.12) and the definition of V. Probably, at least a deformation of the periodic Gauss map can be obtained in this manner. This problem is still open and thus we shall discuss the case of surfaces of higher genus elsewhere.

Thus we restrict ourselves to deformations of tori. First we prove the following proposition.

PROPOSITION 6. *There exists a procedure that uniquely assigns a function* $V(z)$ *satisfying* (2.12) *to each smooth double-periodic potential* $U(z)$.

PROOF OF PROPOSITION 6. Let $U(z)$ be a double-periodic function with a lattice of periods Γ, and let Γ^* be the lattice dual to Γ.

Any smooth function on the torus $\mathbb{C}/\Gamma = \mathbb{R}^2/\Gamma$ is decomposed into Fourier series with respect to the basis formed by eigenfunctions of the operator $\bar{\partial}$. Notice that these functions also form a basis of eigenfunctions of the operator ∂. These eigenfunctions are of the form $f(z \mid \gamma^*) = \exp 2\pi i \gamma^*(z)$, where by $\gamma^*(z)$ we denote the scalar product of γ^* and the vector $(\operatorname{Re} z, \operatorname{Im} z)$. For the sake of brevity we use only z as the argument, but it is easy to notice that these functions are not holomorphic.

It is evident that for a double-periodic function $w(z)$ there exists a double-periodic function $v(z)$ such that $v_{\bar{z}} = w$ if and only if the Fourier series for $w(z)$,

$$w(z) = \sum_{\gamma^* \in \Gamma^*} w_{\gamma^*} f(z \mid \gamma^*),$$

does not contain terms corresponding to the kernel of operator $\bar{\partial}$ (i.e., $f(z \mid 0) = 1$). In this case we can invert the operator $\bar{\partial}$ in an obvious way by using the Fourier decomposition.

If the function $w(z)$ is the derivative of a double-periodic function itself, then its Fourier decomposition does not contain such a term. Let us put $w(z) = (U^2)_z$ and take the function $V(z) = \bar{\partial}^{-1} w(z)$ uniquely determined by the additional condition

$$\int_{\mathbb{C}/\Gamma} V(z)\, dz\, d\bar{z} = 0 .$$

This condition holds if and only if the Fourier series for $V(z)$ does not contain terms lying in the kernel of $\bar{\partial}$.

Proposition 6 is proved.

Notice that if we add to $V(z,t)$ a function which depends only on t, then we do not change the geometric deformation of a surface, but only perform a linear translation of the conformal coordinates $(z, \bar{z})$.

Let us consider the two integrals

$$\frac{\partial (X^1(z,t) + iX^2(z,t))}{\partial t} = 2i \int \Omega_0 \quad \text{and} \quad \frac{\partial X^3}{\partial t} = -\int \Omega_1,$$

where

$$\Omega_0 = \tfrac{1}{2}((\psi_2^2)_t\,dz - (\psi_1^2)_t\,d\bar z),$$
$$\Omega_1 = (\psi_{2t}\bar\psi_1 + \psi_2\bar\psi_{1t})\,dz + (\psi_{1t}\bar\psi_2 + \psi_1\bar\psi_{2t})\,d\bar z\,.$$

Explicit formulas for the differentials Ω_0 and Ω_1 follow from (2.17). We omit these formulas together with the bulky computations using only formulas (2.12) and (3.3) and leading to the following result.

PROPOSITION 7. 1) $\Omega_0 = d(f_1 + g_1 + f_2 + g_2)$, *where*

$$f_1 = \frac{3}{2}V\psi_2^2, \qquad g_1 = \psi_2\partial^2\psi_2 - \frac{(\partial\psi_2)^2}{2},$$
$$f_2 = \frac{3}{2}\overline{V}\psi_1^2, \qquad g_1 = \psi_1\bar\partial^2\psi_1 - \frac{(\bar\partial\psi_1)^2}{2};$$

2) $\Omega_1 = d(h_1 + h_2)$, *where*

$$h_1 = \bar\psi_1\partial^2\psi_2 + \psi_2\partial^2\bar\psi_1 - \partial\psi_2\partial\bar\psi_1 + 3V\bar\psi_1\psi_2,$$
$$h_2 = \psi_1\bar\partial^2\bar\psi_2 + \bar\psi_2\bar\partial^2\psi_1 - \bar\partial\bar\psi_2\bar\partial\psi_1 + 3\overline{V}\psi_1\bar\psi_2\,.$$

Moreover, two modified Novikov–Veselov deformations generated by (L, A, B)-triples (2.14) and (2.15) satisfy the following formal equations

$$\frac{\partial(X^1(z,t^+) + iX^2(z,t^+))}{\partial t^+} = 2i\int d(f_1 + g_1),$$
$$\frac{\partial(X^1(z,t^-) + iX^2(z,t^-))}{\partial t^-} = 2i\int d(f_2 + g_2),$$

which may be useful for proving analogs of Proposition 7 for deformations generated by higher equations of the mNV hierarchy.

Now we are ready to state and prove the main theorem.

THEOREM 1. *Let Σ be a two-dimensional torus represented by formulas* (3.3)–(3.4) *with a double-periodic potential $U(z)$, and let $U(z,t)$ be a solution to equation* (2.11) *with the initial data $U(z,0) = U(z)$ and a double-periodic potential $V(z,t)$. Then the mNV flow deforms the torus Σ into tori Σ_t which are represented by* (3.3)–(3.4) *with potentials $U(z,t)$ conformally equivalent to Σ, and have the same value of the Willmore functional.*

PROOF OF THEOREM 1. By Proposition 7, the forms Ω_0 and Ω_1 are exact on the torus $\mathbb{C}/\Gamma$, being differentials of double-periodic functions. Therefore, a lattice of periods that determines the conformal class is preserved by the mNV flow.

Now it follows from Proposition 5 that the value of the Willmore functional is also preserved.

Theorem 1 is proved.

In the analytic case, we have a stronger result.

THEOREM 2. *The modified Novikov–Veselov equation induces, via formulas (2.17) and (3.3)–(3.4), a deformation of immersed analytic tori. Moreover, this deformation preserves their conformal structures and the values of the Willmore functional.*

PROOF OF THEOREM 2. It follows from Proposition 2 that every analytic torus is represented by formulas (3.3)–(3.4). Since the tori are analytic, by Proposition 6 and the Cauchy–Kowalewski theorem a solution of the modified Novikov–Veselov equation satisfying the conditions of Theorem 1 exists at least for small t.

Now Theorem 2 follows from Theorem 1.

4.3. **The Clifford torus as a stationary point of the mNV flow.** It follows from the definition of the potential $U(z)$ (see (3.7)) that geometrically stationary points of the mNV flow, i.e., surfaces transformed into images of themselves by translations in $\mathbb{R}^3$, correspond to stationary solutions of the mNV equation (2.11).

It is also natural to expect that the simplest stationary solutions will be one-dimensional, i.e., stationary solutions of the modified Korteweg–de Vries equation.

We shall show that the simplest stationary solution is realized by a famous surface, the *Clifford torus.*

Let S^3 be a unit sphere in four-dimensional Euclidean space $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4). The Clifford torus (in $\mathbb{R}^4$) is the image of the following embedded torus

$$\mathbb{R}^2 \to S^4 : (x,y) \to \left(\frac{\cos y}{\sqrt{2}}, \frac{\sin y}{\sqrt{2}}, \frac{\cos x}{\sqrt{2}}, \frac{\sin x}{\sqrt{2}}\right). \tag{4.3}$$

Let us consider the stereographic projection of S^4 onto the plane $x^4 = -1$ with the pole $(0,0,0,1)$:

$$(x_1, x_2, x_3, x_4) \to \left(\frac{-2x_1}{x_4 - 1}, \frac{-2x_2}{x_4 - 1}, \frac{-2x_3}{x_4 - 1}, -1\right). \tag{4.4}$$

For the image of the Clifford torus in S^3 under this projection, we preserve the term "Clifford torus".

The variables (x, y) turn out to be conformal and the metric tensor takes the form

$$\frac{4}{(\sqrt{2} - \sin x)^2}\,(dx^2 + dy^2). \tag{4.5}$$

The Gaussian and mean curvatures are given by

$$K = \frac{\sqrt{2}\sin x - 1}{4}, \qquad H = \frac{\sin x}{2\sqrt{2}}. \tag{4.6}$$

Let us determine the potential $U(x)$ by using formula (3.7); we obtain

$$U(x) = \frac{\sin x}{2\sqrt{2}(\sqrt{2} - \sin x)}. \tag{4.7}$$

It follows from direct computations that this potential induces the Clifford torus via formulas (3.3)–(3.4). Let us also notice that the potential (4.7) satisfies the following equation:

$$U_x^2 = -4U^4 + 2U^2 + \frac{U}{\sqrt{2}} + \frac{1}{16}. \tag{4.8}$$

If a solution of the mNV equation depends only on the variable $x - \text{const}\cdot t$, then it satisfies the equation

$$(U_{xxx} + 24U^2U_x - \text{const}\cdot U_x)_x = 0. \tag{4.9}$$

It follows from (4.8) that the Clifford torus (4.7) satisfies (4.9). Hence we conclude that the Clifford torus is a geometrically stationary point of the mNV flow.

§5. Willmore functional

We already mentioned (see Proposition 5 and Theorems 1 and 2) that the mNV flow preserves values of the Willmore functional and briefly gave a definition of this functional. In the last few years this functional has attracted the attention of geometers (see the history of its investigation and the explanation of many facts about it in [**Wm**], also see [**ST, W, LY, Br, LS, Kus, FPPS, HJP, Sm, BBb, B**]).

In this section we present a brief survey of the modern history of the Willmore conjecture and consider its relationship with mNV flows.

Let $F\colon S \to \mathbb{R}^3$ be an immersed surface. The value of the Willmore functional on this surface is defined by the following formula:

$$W(S) = \int_S H^2\, d\mu. \tag{5.1}$$

Here $d\mu$ is the Liouville measure with respect to the induced metric on S.

This functional is conformally invariant, i.e., any conformal transformation of three-dimensional Euclidean space transforms any immersed surface into another one with the same value of the Willmore functional.

A surface is called *Willmore* if it is a critical point of the Willmore functional. The Euler–Lagrange equation for this functional has the form

$$\Delta H + 2H(H^2 - K) = 0, \tag{5.2}$$

where Δ is the Laplace–Beltrami operator on the surface (see [**Wm**]).

The following proposition can be obtained by direct computations.

PROPOSITION 8. *If a surface is represented by formulas* (3.3)–(3.4), *then it is Willmore if and only if the following equality holds*

$$\Delta U \cdot D - 2(U_xD_x + U_yD_y) + U \cdot \Delta D + 8U^3D = 0, \tag{5.3}$$

where $z = x + iy$ *and* $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The simplest examples of Willmore surfaces are stereographic projections of minimal surfaces M in S^3. Moreover, the area of a minimal surface M in S^3 is equal to the value of the Willmore functional on its projection. All this was known to Thomsen and Blaschke in the twenties. The conformal properties of this functional and its relation to minimal surfaces gave Blaschke a reason to call such surfaces *conformally minimal* ([**Bl**]).

But these examples do not cover the class of Willmore surfaces (see, for instance, [**P**] where the first examples of compact embedded Willmore surfaces that are not stereographic projections of minimal surfaces in S^3 were obtained).

All Willmore spheres were classified by Bryant ([**Br**]).

Of fundamental interest is the following conjecture stated by Willmore in the mid-sixties.

WILLMORE CONJECTURE. *For immersed tori, the Willmore functional satisfies the following inequality*

$$W \geqslant 2\pi^2, \tag{5.4}$$

which is attained only on the Clifford torus and its images under conformal transformations of $\mathbb{R}^3$.

Analogs of this conjecture for arbitrary genera were proposed in [**Kus**], but the conjecture is still open.

Simon proved that the minimum is attained on an analytic minimal torus ([**Sm**]).

The following list contains all known classes of tori for which the Willmore conjecture has been proved.

1) In the early seventies, Willmore and independently Shiohama and Takagi ([**ST**]) proved this conjecture for tube tori with constant radii. Here we call a torus a *tube* if it is formed by carrying a small circle around a closed space curve so that the center moves along the curve and the plane of the circle is the normal plane to the curve at each point.

2) Hertrich–Jeromin and Pinkall ([**HJP**]) generalized the result of Willmore and Shiohama–Takagi to tube tori with arbitrary radii, i.e., for the case in which the radius of a circle can vary along the curve.

3) Langer and Singer ([**LS**]) proved the Willmore conjecture for tori of revolution.

4) In [**LY**] the authors brought together the spectral theory of the Laplace–Beltrami operator with the theory of conformal invariants. Li and Yau proved this conjecture for tori whose conformal structures are defined by lattices generated by vectors $(1,0)$ and (a,b), where

$$0 \leqslant a \leqslant 1/2, \qquad \sqrt{1-a^2} \leqslant b \leqslant 1\,.$$

In terms of theta-functions, all Willmore tori are described in [**BBb, B**] (see also [**FPPS**]). Regretably, theta-functional formulas are rather complicated and not too efficient for applications.

We propose the following conjecture.

CONJECTURE. *A nonstationary (with respect to the mNV flow) torus cannot be a local minimum of the Willmore functional.*

To us, this conjecture looks convincing, because it is strange to expect that a minimum of this variational problem, taken up to conformal transformations of

$\mathbb{R}^3$, can be degenerate. Probably the methods developed in [**W, Pm**] will help to prove it.

If this conjecture is true, then the Willmore conjecture is reduced to the investigation of stationary points of the mNV flow. It is known from soliton theory that stationary solutions are simpler than general ones. For instance, stationary solutions of equations from the KdV hierarchy are described by very simple hyperelliptic functions. Of course the mNV equation is a $(2+1)$-equation and we cannot expect such a simple description for it.

We also would like to pose the following question.

QUESTION. *Higher equations of the mNV hierarchy also have first integrals. What is the geometric meaning of the critical points of these functionals?*

The similarity of formulas for mNV and mNV2 equations shows that one can expect that these flows will deform tori into tori. Thus these deformations should have a geometric meaning. Most probably these flows preserve conformal structures and have their origin in conformal geometry.

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