LEMMA 7. *An operator of the form* (45) *with real coefficients possesses the following factorization*

$$L = (x_n + y_nT_1 + z_nT_2)(x_n + y_{n-T_1}T_1^{-1} + z_{n-T_2}T_2^{-1}) + w_n\,.$$

The proof is a simple calculation. Note that here, just as in the hyperbolic case (40), (41), the factorization is given by local algebraic formulas, which do not require the solutions of difference equations. We also have the *inverse factorization*

$$L = (x_n' + y_n'T_1^{-1} + z_nT_2^{-1})(x_n' + y_{n-T_1}'T_1 + z_{n-T_2}'T_2) + w_n'\,.$$

To any solution of $L\psi = 0$ a new solution is assigned:

$$\widetilde{L}\widetilde{\psi} = 0, \qquad \widetilde{L} = (x_n + y_{n-T_1}T_1^{-1} + y_{n-T_2}T_2^{-1})w_n^{-1}(x_n + y_nT_1 + z_nT_2) + 1,$$
$$\widetilde{\psi}_n = (x_n + y_{n-T_1}T_1^{-1} + z_{n-T_2}T_2^{-1})\psi_n\,.$$

We also have an inverse Laplace transformation induced by the inverse factorization. The two transformations are mutually inverse up to a gauge transformation.

Here the discrete analog of the two-dimensional Toda lattice, as well as those of the cyclic, semicyclic, and quasi-cyclic chains arise. They will be studied in a forthcoming paper.

REMARK. To each pair of neighboring basis periods of the same length (T_1, T_2), $(T_2, T_1^{-1}T_2)$, $(T_1^{-1}T_2, T_1^{-1})$, (T_1^{-1}, T_2^{-1}), $(T_2^{-1}, T_1T_2^{-1})$, and $(T_1, T_2T_1^{-1})$ there corresponds one Laplace transformation. The algebra of these transformations is being studied.

Introducing complex values for the fields, we also obtain the discretization of the Schrödinger operator in a magnetic field (what we considered above is a discretization of real operators only, where the physical magnetic field is equal to zero).

Appendix II (S. P. Novikov, I. A. Taimanov). **Difference analogs of the harmonic oscillator**

Let us consider the real self-adjoint second order difference operator acting in the space $\mathcal{L}_2(\mathbb{Z})$, $n \in \mathbb{Z}$, and given by

$$L\psi_n = v_n\psi_n + c_{n-1}\psi_{n-1} + c_n\psi_{n+1},$$
$$L = c_{n-1}T^{-1} + c_nT + v \cdot 1,$$

where $T\colon n \to n+1$ is the translation operator along the lattice. Here $c_n, v_n \in \mathbb{R}$, $T^+ = T^{-1}$, and $L^+ = L$.

As in the continuous case, one defines the factorization of the operator L:

$$\text{(I)} \qquad L + \alpha = (a_n + b_{n-1}T^{-1})(a_n + b_nT) \quad \text{or}$$
$$\text{(II)} \qquad L + \alpha = (p_n + q_nT)(p_n + q_{n-1}T^{-1})\,.$$

Even without requiring the coefficients to be real, such a factorization will exist for any α. In order to find the values of a_n, b_n, p_n, and q_n from v_n and c_n, we arrive at the difference analog of the Riccati equation, which appears when we factorize the continuous Schrödinger operator of the form

$$-\partial_x^2 + u(x) + \alpha = -(\partial_x + v)(\partial_x - v), \qquad v^2 + v_x = \alpha + u(x).$$

In the discrete case we have

(I) $$v_n + \alpha = a_n^2 + b_{n-1}^2, \qquad c_n = a_n b_n,$$

(II) $$v_n + \alpha = p_n^2 + q_{n-1}^2, \qquad c_n = q_n p_n.$$

If all the coefficients a_n and b_n are real, then

(I) $$L + \alpha = QQ^+, \qquad Q^+ = a_n + b_n T.$$

Similarly, if all the coefficients p_n and q_n are real,

(II) $$L + \alpha = RR^+, \qquad R^+ = p_n + q_n T^{-1}.$$

The Darboux–Bäcklund transformations B_α, by definition, are given by

(I) $$L \to Q^+Q = \widetilde{L} = B_\alpha^{(I)} L,$$

(II) $$L \to R^+R = \widetilde{\widetilde{L}} = B_\alpha^{(II)} L.$$

It is obvious that these transformations can be taken to be inverse to each other after an appropriate factorization:

$$B_0^{(II)}(B_\alpha^{(I)} L) = L + \alpha, \qquad B_0^{(I)}(B_\alpha^{(II)} L) = L + \alpha.$$

(Recall that in contrast with the two-dimensional case, the factorization here requires solving a Riccati type equation and is therefore nonunique.) The cyclic chains, just as in the continuous case, are determined from the condition:

$$L_N = B_{\alpha_N} \cdots B_{\alpha_0} L, \qquad L_j = B_{\alpha_j} L_{j-1}, \qquad L_0 = L_N.$$

It is not difficult, by following [**VS1**], to prove the following

PROPOSITION 1. *Cyclic chains satisfying the condition* $\sum_{j=1}^N \alpha_j = 0$ *consist of finite-zone difference operators* L_j *corresponding to one and the same Riemann surface of genus* $g \leqslant [N/2]$ *(see* [**DMN, DKN2**]*).*

More interesting are the chains for which $\sum_{j=1}^N \alpha_j = h > 0$.

The first difference analog of the harmonic oscillator. Already for $N = 1$ nontrivial phenomena arise. Consider the cyclicity conditions

(1) $$L = QQ^+ - h, \quad \widetilde{L} = Q^+Q = B_h L, \quad Q^+Q = QQ^+ + h, \quad Q^+ = a_n + b_n T.$$

From (1) we obtain an equation of Riccati type, which implies

(2) $$a_n = a = \text{const}, \qquad b_n^2 - b_{n-1}^2 = h$$

or $b_n = \sqrt{nh + b_0^2}$. Suppose that further

(3) $$a = 1.$$

We come to the following conclusion: *The operator* $L = QQ^+$ *is not defined in* $\mathcal{L}_2(\mathbb{Z})$ *as a real operator.*

PROPOSITION 2.[1] *The operator $L = QQ^+$ described by formulas* (1)–(3) *determines a real selfadjoint operator L in the space $\mathcal{H}_l \subset \mathcal{L}_2(\mathbb{Z})$ if and only if $l \in \mathbb{Z}$* (*the* quantization condition).

(*Here $\psi_n \in \mathcal{H}_l$ if and only if $\psi_n = 0$ for $n \leqslant -b_0^2/h = l$.*) *The space $\mathcal{H}_l$ is isomorphic to $\mathcal{L}_2(\mathbb{Z}^+)$ with zero boundary condition for $k = 0$.*

PROOF. If the number l is an integer, then

$$Q_\pm^+ = (1 \pm \sqrt{(n-l)h}\,T).$$

Setting $n = l + k$, we come to $\mathbb{Z}^+$. It is readily verified by substitution that the operators $L_\pm = Q_\pm Q_\pm^+$ take the space $\mathcal{H}_l$ to itself if and only if l is an integer. We come to the following operators in $\mathcal{L}_2(\mathbb{Z}^+)$:

$$L_\pm = Q_\pm Q_\pm^+, \qquad Q_\pm^+ = (1 \pm \sqrt{hk}\,T).$$

The spectrum of the operator $L_\pm = Q_\pm Q_\pm^+$ in the space $\mathcal{L}_2(\mathbb{Z}^+)$ is the same as that of the ordinary harmonic oscillator. The ground state will be

$$L_\pm \psi_0^\pm = 0, \qquad Q_\pm^+ \psi_0^\pm = 0, \qquad \psi_{0k}^\mp = \begin{cases} (\pm 1)^{k-1}/\sqrt{h^{k-1}(k-1)!}, & k \geqslant 1, \\ 0, & k \leqslant 0. \end{cases}$$

The higher eigenfunctions have the form

$$\psi_m^\pm = Q_\mp^m \psi_0^\pm, \quad L_\pm \psi_m^\pm = mh\psi_m^\pm, \qquad m \in \mathbb{Z}^+.$$

LEMMA 1. *The eigenfunctions $\psi_m^\pm$ of the operator $L_\pm$ have the form*

$$\psi_{mk}^\pm = \frac{(\mp 1)^{k-1}}{\sqrt{h^{k-1}(k-1)!}} \cdot P_m(k) \cdot \Theta(k).$$

Here the $P_j(k)$ are polynomials such that

$$P_j(k) = (1 - h(k-1)T^{-1})P_{j-1}(k), \quad P_0 \equiv 1, \quad \text{and} \quad \Theta(k) = \begin{cases} 1, & k \geqslant 1, \\ 0, & k \leqslant 0. \end{cases}$$

The expression ψ_0^2 is a Poisson distribution, while the polynomials $P_j(k)$ are orthogonal with Poisson weight $\psi_0^2(k)$ on $\mathbb{Z}^+$. Undoubtedly, they are known, although their relationship with harmonic oscillators and bosonic commutation relations, most probably, was never discussed.[2]

The second difference analog of the harmonic oscillator. Let $N = 1$. Put $L = QQ^+$ and $\widetilde{L} = Q^+Q$. We shall introduce a family of operators L depending on two constants $c, a \in \mathbb{R}$:

$$L(c,a) = Q(c,a)Q^+(c,a), \qquad \widetilde{L}(c,a) = Q^+(c,a)Q(c,a)$$

so as to have

$$a^2 Q^+(c,a)Q(c,a) = Q(ca^2,a)Q^+(ca^2,a) + D, \qquad D = a^2 - 1. \tag{4}$$

[1] Similar operators were constructed in [**AS90**]. However, the analysis of the corresponding formulas does not reveal if their authors have the same thing in mind as we do. Apparently our results are new, at least methodically.

[2] These polynomials are known as the Charlet polynomials.

Let us put

$$Q^+ = 1 + ca^n T, \quad Q = 1 + ca^{n-1} T^{-1}, \qquad a \neq 0,\ c \neq 0.$$

Consider the transformation $\tau: n \to 1 - n$. We have the formula

$$\tau Q(c,a) = Q^+(c, a^{-1})\tau. \tag{5}$$

The relations (4)–(5) are extremely interesting. Apparently, they have not appeared previously.

THEOREM. *The spectrum of the operator $L(c,a)$ for λ in the semi-interval $[0,1)$ in the Hilbert space $\mathcal{L}_2(\mathbb{Z})$ has the form*

$$\begin{aligned} &1)\ a > 1, \qquad \lambda_n = 1 - a^{-2n}, \qquad n \geqslant 0;\\ &2)\ a < 1, \qquad \lambda_n = 1 - a^{2n}, \qquad n \geqslant 1. \end{aligned}$$

The eigenfunctions are the following:

$$\begin{aligned} 1)\ a > 1, \qquad &\psi_{0k}(c,a) = (-1)^k c^{-k} a^{-(k-1)k/2}, \qquad k \in \mathbb{Z},\\ &\psi_n(c,a) = Q(c,a)\,Q(ca^2,a)\cdots Q(ca^{2n-2},a)\,\psi_0(ca^{2n},a),\\ &\psi_{nk}(c,a) = P_n(k,c,a)\,\psi_{0k}(c,a),\\ &\qquad P_n(k,c,a) = (a^{-2k} - ca^{2n-2})^n, \qquad n \geqslant 1;\\ 2)\ a < 1, \qquad &\psi_1(c,a) = \tau\psi_0\Big(\frac{c}{a^2}, \frac{1}{a}\Big),\\ &\psi_n(c,a) = Q^+\Big(\frac{c}{a^2}, a\Big)\cdots Q^+\Big(\frac{c}{a^{2n-2}}, a\Big)\psi_1\Big(\frac{c}{a^{2n-2}}, a\Big), \quad n \geqslant 2. \end{aligned}$$

REMARKS. 1. For $\lambda \geqslant 1$ the spectrum of the operators $L(c,a)$ is not known to the authors. We conjecture that it is continuous.

2. If $a > 1$ and $c < 0$, then the operator L may be considered in $\mathcal{L}_2(\mathbb{R})$ so that its restriction to the family of lattices $\delta + \mathbb{Z} \subset \mathbb{R}$ yields our family $L(c,a)$:

$$L_-(m,\gamma) = (1 - \gamma^{x-m-1}T^{-1})(1 - \gamma^{x-m}T), \qquad \gamma > 0,\ m \in \mathbb{Z}.$$

The ground state $\Phi_0(x)$ ($L\Phi_0(x) = 0$) is a function satisfying

$$(1 - \gamma^{x-m}T)\Phi_0(x) = 0 \quad \text{or} \quad \Phi_0(x) = \gamma^{x-m}\Phi_0(x+1).$$

It has the form

$$\Phi_0(x) = g(x)\gamma^{x(2m+1-x)/2},$$

where $g(x) = g(x+1)$ is any function of period 1.

Let us normalize the ground state by defining $g(x)$ from a continuum of normalizing conditions so that the expression $\Phi_0^2(x)$ is a normalized probability distribution on any lattice of the form $\delta + \mathbb{Z} \subset \mathbb{R}$:

$$g^{-1}(x) = \exp\left(\frac{ax(m+1)}{2} - \frac{ax^2}{2}\right)\Theta\left[\frac{a(m+1)}{2} - ax \,\middle|\, a\right],$$

where $a = 2\ln\gamma > 0$ and $\Theta[u \mid a]$ is a theta function:

$$\Theta[u \mid a] = \sum_{n\in\mathbb{Z}} \exp\left(-\frac{an^2}{2} + nu \right).$$

It follows from the analytical properties of theta functions ([**BE**]) that $g(x)$ is a smooth function with period 1. The normed ground state is given by the formula

$$\Phi_0(x) = \frac{\gamma^{x(x-1)/2}}{\Theta[(m+1-2x)\ln\gamma \mid 2\ln\gamma]}.$$

Thus the cyclic chains are of two types:

1) $L = B_{\alpha_N} \cdots B_{\alpha_1} L = L'$, where L acts in $\mathcal{L}_2(\mathbb{Z}^+)$ for an appropriate "quantization" of the parameters;

2) the operator L' coincides with L after multiplication by a constant and a translation along x in the natural realization in $\mathcal{L}_2(\mathbb{R})$, where the restriction to the lattices $(\delta + \mathbb{Z}) \subset \mathbb{R}$ generates a family of discrete operators participating in the definition of the cyclic chain similarly to the case $N = 1$ considered above.

ADDED IN PROOF. In the paper [**SVZ**], the problem was essentially posed already. However, that paper contains unmotivated restrictions. For example, the constant $\delta = \ln|c|/\ln a$ is assumed rational in [**SVZ**]. Further, the assertion in [**SVZ**] according to which the relation (4) above or the corresponding relation (14) in [**SVZ**] and its consequences (15)–(17) "clearly define a spectrum generating algebra" is incorrect. This assertion is certainly false, for the case in which ψ_0 satisfies the equation $Q^+\psi_0 = 0$ and growth exponentially. This is indeed the case when $q > 1$, a situation omitted in the papers [**AS91, SVZ**]. In this situation the authors have found a different ground state with eigenvalue $1 - q$ $(q^{-1} = a^2)$ by using the additional symmetry τ (see Theorem 2).

Moreover, the assertion that the relations mentioned above "clearly define a spectrum generating algebra" is incorrect for another reason: it gives no information on the spectrum of the Schrödinger operator $L = QQ^+$ with $\lambda \geqslant 1$ in the Hilbert space $\mathcal{L}_2(\mathbb{Z})$. According to our conjecture (see Remark 1 above) this spectrum is continuous and occupies the entire strip $\lambda \geqslant 1$.

References

[AC] Y. Aharonov and A. Casher, *Ground state of a spin 1/2 charged particle*, Phys. Rev. A **19** (1979), no. 6, 2461–2463.

[AFW] N. M. Atakishiev, A. Frank, and K. B. Wolf, *A simple difference relation of the Heisenberg q-algebra*, J. Math. Phys. **35** (1994), no. 7, 3253–3260.

[AS90] N. M. Atakishiev and S. K. Suslov, *Difference analogs of the harmonic oscillator*, Teoret. Mat. Fiz. **85** (1990), no. 1, 64–73; English transl. in Theoret. and Math. Phys. **85** (1990).

[AS91] ———, *On a difference scheme realization of the Q-harmonic oscillator*, Teoret. Mat. Fiz. **87** (1991), no. 1, 154–156; English transl. in Theoret. and Math. Phys. **87** (1991).

[B] A. I. Bobenko, *Surfaces of constant mean curvature and integral equations*, Uspekhi Mat. Nauk **46** (1991), no. 4, 3–42; English transl. in Russian Math. Surveys **46** (1991), no. 4.

[BE] H. Bateman and A. Erdelyi, *Higher transcendental functions*, vol. 1, McGraw-Hill, New York, 1953.

[Bul] S. Bulgadaev, *Two-dimensional integrable field theories related with simple Lie algebras*, Phys. Lett. B **96** (1980), 151–159.

[Dar] G. Darboux, *Leçons sur la théorie générale des surfaces et des applications géométriques du calcul infinitésimal*, Paris, 1887–1896.

[DS] V. G. Drinfeld and V. V. Sokolov, *Lie algebras and equations of the Korteweg-de Vries type*, Sovremennye Problemy Matematiki. Noveishie Dostizheniya, vol. 24, VINITI, Moscow, 1984; English transl. in J. Soviet Math. **30** (1985), no. 2.

[D75] B. A. Dubrovin, *Periodic problem for the Korteweg-de Vries equation*, Funktsional. Anal. i Prilozhen. **9** (1975), no. 3, 41–51; English transl. in Functional Anal. Appl. **9** (1975), no. 3.

[D81] ———, *Theta-functions and nonlinear equations*, Uspekhi Mat. Nauk **36** (1981), 11–80; English transl. in Russian Math. Surveys **36** (1981).

[DKN1] B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, *Schrödinger equation in a periodic field and Riemann surfaces*, Dokl. Akad. Nauk SSSR **229** (1976), 15–18; English transl. in Soviet Math. Dokl. **17** (1976).

[DKN2] ———, *Integrable systems I*, Sovremennye Problemy Matematiki. Fundamental′nye Napravleniya, vol. 4, VINITI, Moscow, 1985; English transl., Encyclopedia of Math. Sci., vol. 4, Springer-Verlag, Berlin, Heidelberg, and New York, 1990.

[DMN] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, *Nonlinear equations of the KdV type, finite zone linear operators and Abelian varieties*, Uspekhi Mat. Nauk **31** (1976), 55–136; English transl. in Russian Math. Surveys **31** (1976).

[DN74] B. A. Dubrovin and S. P. Novikov, *Periodic and conditionally periodic analogs of the multi-soliton solutions of the Korteweg-de Vries equation*, Zh. Èksper. Teoret. Fiz. **67** (1974), no. 12, 2131–2143; English transl. in Soviet Phys. JETP **67** (1974).

[DN80] ———, *Ground states of the two-dimensional electron*, Zh. Èksper. Teoret. Fiz. **79** (1980), no. 3, 1006–1016; English transl. in Soviet Phys. JETP **79** (1980).

[GGKM] C. S. Gardner, J. Green, M. Kruskal, and R. Miura, *A method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett. **19** (1967), 1095–1098.

[IM] A. P. Its and V. B. Matveev, *Schrödinger operators with finite-zone spectrum and N-soliton solutions of the Korteweg-de Vries equation*, Teoret. Mat. Fiz. **23** (1975), no. 1, 51–68; English transl. in Theoret. and Math. Phys. **23** (1975).

[Kr77] I. M. Krichever, *Algebraic geometry methods in the theory of nonlinear equations*, Uspekhi Mat. Nauk **32** (1977), no. 6, 180–208; English transl. in Russian Math. Surveys **32** (1977), no. 6.

[Kr81] ———, *Periodic non-Abelian Toda lattice and its two-dimensional generalizations*, Uspekhi Mat. Nauk **36** (1981), no. 2, 72–80; English transl. in Russian Math. Surveys **36** (1981), no. 2.

[Kr85] ———, *Two-dimensional periodic difference operators and algebraic geometry*, Dokl. Akad. Nauk SSSR **285** (1985), no. 1, 31–36; English transl. in Soviet Math. Dokl. **32** (1985).

[KLWZ] I. Krichever, O. Lipan, P. Wiegman, and A. Zabrodin, *Quantum integrable systems and elliptic solutions of classical discrete nonlinear equations*, Preprint, April 17, 1996, Landau Institute for Theoretical Physics of Russian Academy of Sciences and James Frank Institute of the University of Chicago.

[L] P. S. Laplace, *Recherches sur le calcul integral aux différences partielles. Histoire de l'Académie Royale des Sciences*, 1777.

[L68] P. Lax, *Integrals of nonlinear equations and solitary waves*, Comm. Pure Appl. Math. **21** (1968), no. 5, 467–490.

[L75] ———, *Periodic solutions of the Korteweg-de Vries equation*, Comm. Pure Appl. Math. **28** (1975), 141–188.

[LS] A. N. Leznov and M. V. Saveliev, *Representation of zero curvature for the system of nonlinear partial differential equations $X_{\alpha,zz} = \exp(KX)_\alpha$ and its integrability*, Lett. Math. Phys. **3** (1979), no. 6, 489–494.

[M] S. V. Manakov, *Inverse scattering problem method and two-dimensional evolutional equations*, Uspekhi Mat. Nauk **31** (1976), no. 5, 245–246; English transl. in Russian Math. Surveys **31** (1976), no. 5.

[MV] H. McKean and P. van Moerbeke, *The spectrum of Hill's equation*, Invent. Math. **30** (1975), 217–274.

[Mik] A. V. Mikhailov, *On the integrability of the two-dimensional generalization of Toda lattices*, Zh. Èksper. Teoret. Fiz. **30** (1979), 443–448; English transl. in Soviet Phys. JETP **30** (1979).

[N74] S. P. Novikov, *Periodic problem for the Korteweg-de Vries equation*, Funktsional. Anal. i Prilozhen. **8** (1974), no. 3, 54–66; English transl. in Functional Anal. Appl. **8** (1974), no. 3.

[N81] ______, *Magnetic Bloch functions and vector bundles. Typical dispersion laws and their quantum numbers*, Dokl. Akad. Nauk SSSR **257** (1981), 538–543; English transl. in Soviet Math. Dokl. **23** (1981).

[N83] ______, *Two-dimensional Schrödinger operators in periodic fields*, Itogi Nauki i Tekhniki. Sovremennye Problemy Matematiki **23** (1983), VINITI, 3–23; English transl. in J. Soviet Math. **28** (1985), no. 1.

[NV1] S. P. Novikov and A. P. Veselov, *Finite-zone two-dimensional periodic Shrödinger operators: potential operators*, Dokl. Akad. Nauk SSSR **279** (1984), no. 5; English transl. in Soviet Math. Dokl. **30** (1984).

[NV2] ______, *Two-dimensional Schrödinger operator: inverse scattering and evolutional equations*, Phys. D **18** (1986), 267–273.

[NV3] ______, *Exactly solvable two-dimensional Shrödinger operators*, Uspekhi Mat. Nauk **50** (1995), no. 6, 180–181; English transl. in Russian Math. Surveys **50** (1995), no. 6.

[PS] U. Pinkall and I. Sterling, *On the classification of constant mean curvature tori*, Ann. Math. **130** (1989), 407–451.

[Sol] S. P. Novikov (ed.), *Soliton theory*, "Nauka", Moscow, 1980. (Russian)

[SVZ] V. Spiridonov, L. Vinet, and A. Zhedanov, *Difference Schrödinger operators with linear and exponential discrete spectra*, Lett. Math. Phys. **29** (1993), 63–73.

[SY] A. B. Shabat and P. I. Yamilov, *Symmetries of nonlinear chains*, Algebra i Analiz **2** (1990), no. 2, 183–208; English transl. in Leningrad Math. J. **2** (1991).

[S1] A. B. Shabat, *The infinite-dimensional dressing dynamical system*, Inverse Problems **6** (1992), 303–308.

[S2] ______, *To the theory of Laplace-Darboux transformations*, Teoret. Mat. Fiz. **103** (1995), no. 1, 170–175; English transl. in Theoret. and Math. Phys. **103** (1995).

[VS1] A. P. Veselov and A. B. Shabat, *Dressing chains and the spectral theory of Schrödinger operators*, Funktsional. Anal. i Prilozhen. **27** (1993), no. 2, 1–21; English transl. in Functional Anal. Appl. **27** (1993), no. 2.

[VS2] ______, *Doubly-periodic chains of Laplace-Darboux transformations in the theory of two-dimensional Schrödinger operators* (1996) (to appear).

[T] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49** (1982), 405.

[Tzi] G. Tzitzéica, *Géométrie différentielle projective des réseaux*, Paris–Bucharest, 1924.

[W] J. Weiss, *Period fixed points of Bäcklund transformations and the KdV equation*, J. Math. Phys. **27** (1986), 2647–2656.

University of Maryland at College Park, Department of Mathematics and IPST, College Park, MD 20742-2431 USA and Landau Institute for Theoretical Physics, Moscow 117940, Kosygin street 2, Russia.

E-mail address: novikov@ipst.umd.edu, novikov@landau.ac.ru

Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire, LE11 3TU, UK and Department of Mathematics and Mechanics, Moscow State University, 119899 Moscow Russia.

E-mail address: a.p.veselov@lboro.ac.uk, veselov@landau.ac.ru

Translated by A. SOSSINSKY

Amer. Math. Soc. Transl.
(2) Vol. **179**, 1997

Exactly Solvable Two-Dimensional Schrödinger Operators and Laplace Transformations

S. P. Novikov and A. P. Veselov

§1. Solitons and the Schrödinger operator

The modern theory of exactly solvable one-dimensional and two-dimensional Schrödinger operators

$$L = -\partial_x^2 + u(x) \quad (n = 1),$$
$$L = (\overline{\partial} + B(x,y))(\partial + A(x,y)) + 2V(x,y), \quad \partial = \partial_x - i\partial_y,\ z = x + iy \quad (n = 2)$$

cannot be separated from the theory of spatial one- and two-dimensional integrable nonlinear systems of soliton theory. This is especially so in the rapidly decaying and periodic case (when the coefficients of the operators are rapidly decaying or periodic functions of the space variables, respectively).

I. One-dimensional case ($n = 1$). In the one-dimensional case, it follows from [**GGKM, L**], where the famous KdV equation

$$u_t = 6uu_x - u_{xxx}$$

was solved in the rapidly decaying case, that it can be viewed as a kind of "symmetry" of the spectral theory of the class of operators $L = -\partial_x^2 + u(x)$ acting in the Hilbert space $\mathcal{L}_2(\mathbb{R})$. This means that the KdV equation can be written in the form

$$L_t = AL - LA = [A, L],$$

where A is the linear differential operator $A = -4\partial_x^3 + 3(u\partial_x + \partial_x u)$. Therefore, for example, the eigenvalues $L\psi = \lambda_i\psi$ of the operator L on any translation-invariant class of functions of x on the line $\mathbb{R}$ turn out to be integrals of the KdV equation, $d\lambda_i/dt = 0$. Using the previously solved inverse scattering problem, KdV was solved in [**GGKM**] for rapidly decaying functions. Beginning with [**N74**], in [**DN74, D75, IM, L75, MV**] the solution of the periodic problems for nonlinear KdV from soliton theory was obtained simultaneously with that of the inverse problems for the spectral theory of the linear operator L, which had not been

---

1991 *Mathematics Subject Classification.* Primary 58F07, 81Qxx.