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SOME QUESTIONS CONCERNING THE TOPOLOGY OF MANIFOLDS OF POSITIVE SECTIONAL CURVATURE

I. A. Taĭmanov *

Abstract

The article contains a review of results and problems on the topology of Riemannian manifolds of positive sectional curvature.

Key words and phrases: topology, Riemannian manifold, sectional curvature.

1. Positivity of sectional curvature and topology

1.1. *Definition of curvatures.* Let M^n be a smooth manifold with a Riemannian metric $\langle\cdot,\cdot\rangle_x\colon \mathrm{T}_xM^n \times \mathrm{T}_xM^n \to \mathbb{R}$. All manifolds are supposed to be complete.

To the Riemannian metric there corresponds a unique Levi–Civita connectedness ∇ (which is symmetric and preserves the metric tensor), defining differentiation of vector fields. The *Riemannian curvature* tensor $R(X,Y)Z$ is defined as

$$R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$$

and its value at a point $x \in M^n$ depends only on the values of the vector fields X, Y, and Z at this point. Therefore, at every point $x \in M^n$, the following operator is defined:

$$R\colon \mathrm{T}_xM^n \times \mathrm{T}_xM^n \times \mathrm{T}_xM^n \to \mathrm{T}_xM^n.$$

For every pair $U, V \in \mathrm{T}_xM^n$, consider the trace of the linear transformation

$$W \to R(U,W)V.$$

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* Sobolev Institute of Mathematics, Novosibirsk, RUSSIA.
E-mail address: taimanov@math.nsc.ru.

It is a bilinear function

$$\mathrm{T}_xM^n \times \mathrm{T}_xM^n \to \mathbb{R}$$

and called the *Ricci tensor*: $(U,V) \to R_{ij}U^iV^j$.

The trace of the Ricci tensor

$$R = R_i^i$$

is called the *scalar curvature.*

Let σ be a two-dimensional plane in the tangent space T_xM^n with an orthonormalized basis U, V. The *sectional curvature* along σ is the value $K(\sigma) = \langle R(U,V)U, V\rangle_x$.

We arrive at the following definitions:

1) a manifold has positive sectional curvature if $K > 0$ everywhere;
2) a manifold has positive Ricci curvature if the bilinear form $R_{ij}U^iV^j$ is positive;
3) a manifold has positive scalar curvature if $R > 0$ everywhere.

1.2. *The geometric meaning of sectional curvature.* The definition of sectional curvature for many-dimensional case agrees with the classical definition of the Gaussian curvature of a two-dimensional surface. Let $\sigma \subset \mathrm{T}_xM^n$ be a two-dimensional plane in the tangent space at a point $x \in M^n$. Emit all geodesics that are tangent to σ from x. In a neighborhood about x, this family of geodesics forms an embedded two-dimensional surface whose Gaussian curvature is equal to $K(\sigma)$ in the induced metric.

Therefore, locally, along the two-dimensional plane σ with $K(\sigma) = 1/r^2 > 0$, a Riemannian manifold looks like a two-dimensional sphere of radius r. This leads to the following qualitative description.

Take a pair of unit vectors u_1 and u_2 tangent at a point $x \in M^n$ and emit the naturally parametrized geodesics γ_1 and γ_2 along them:

$$\frac{d\gamma_j(0)}{dt} = u_j, \qquad \int_0^t \left|\frac{d\gamma_j(\tau)}{d\tau}\right| d\tau = t.$$

Define the function $f(t) = \mathrm{dist}(\gamma_1(t), \gamma_2(t))$, the distance between the points $\gamma_1(t)$ and $\gamma_2(t)$. If this function is convex in a neighborhood about zero $(f(\lambda t) > \lambda f(t)$ for $\lambda > 0)$ then the curvature along the two-dimensional direction generated by u_1 and u_2 is positive, and it is negative if the function is concave $(f(\lambda t) < \lambda f(t)$ for $\lambda > 0)$.

This qualitative definition can be extended to the more general case of Alexandrov spaces [6].

Let $\gamma\colon (0,T) \to M^n$ be a geodesic. Here and in the sequel, we suppose that geodesics are naturally parametrized. A family of curves $\gamma_\varepsilon\colon (0,T) \to M^n$, $-\varepsilon_0 < \varepsilon < \varepsilon_0$, is called a *variation* of the curve γ if $\gamma_0 = \gamma$. If, moreover, for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the curve γ_ε is geodesic, then this family is called a *geodesic variation.* Given a variation, we define the *variation field* W as

$$W(t) = \left.\frac{\partial \gamma_\varepsilon(t)}{\partial \varepsilon}\right|_{\varepsilon=0} \in \mathrm{T}_{\gamma(t)}M^n.$$

The following fact is valid:

1.2.1. *The geodesic variation field satisfies the Jacobi equation*

$$\frac{D^2W}{\partial t^2} + R(V,W)V = 0, \qquad (1)$$

where $V = \dot\gamma$ *and* $\frac{D}{\partial t} = \nabla_V$.

Consider a very simple example. Let γ be a geodesic and take $u_1, u_2 \in \mathrm{T}_xM^n$. Let the curvature along the plane generated by $u_1 - u_2$ and $\dot\gamma(0)$ be equal to K. Consider the solution of equation (1) along γ with initial data $W(0) = 0$ and $DW(0)/\partial t = u_1 - u_2$. Then the length of the vector $W(t)$ has the form

$$|W(t)| = |u_1 - u_2|\left(t - \frac{Kt^3}{6} + o(t^3)\right)$$

and $f(t) \approx |W(t)|$.

The claim of Lemma 1.2.1 is based on the formula of second variation for the energy functional. Namely, geodesics with natural parametrization (or with parametrization proportional to that natural) are extremals of the energy functional

$$E(\gamma) = \int_\gamma \langle \dot\gamma, \dot\gamma\rangle\, dt, \qquad (2)$$

i.e., are solutions to the equation

$$\delta E(\gamma) = 0.$$

Functional (2) is connected with the length functional L by the simple inequality

$$L^2(\gamma) \le TE(\gamma), \quad \text{where } \gamma\colon [0,T] \to M^n, \qquad (3)$$

moreover, equality holds only if $|\dot\gamma| = \text{const}$.

1.2.2. The formula of second variation. *Let smooth vector fields* W_1 *and* W_2 *define two-parametric variation of a geodesic* γ *(in the space of closed curves or curves with fixed endpoints). Then the value of* $\delta^2 E$ *on them has the form*

$$\delta^2 E(\gamma)(W_1, W_2) = -\int_\gamma \left\langle W_1, \frac{D^2W_2}{\partial t^2} + R(\dot\gamma, W_2)\dot\gamma\right\rangle dt. \qquad (4)$$

In the case covariant derivatives of the fields W_1 and W_2 along γ have discontinuities, formula (4) becomes more complicated.

1.3. *Topological obstructions to existence of metrics of positive sectional curvature.* The formula of second variation, 1.2.2, immediately leads to the important corollaries 1.3.1–1.3.3.

1.3.1. The Meyers theorem. *If, for every nonzero vector* U *tangent to an* n*-dimensional Riemannian manifold* M^n*, the inequality*

$$\frac{R_{jk}U^jU^k}{\langle U,U\rangle} \ge \frac{n-1}{r^2} \qquad (5)$$

holds, where r *is a positive constant; then geodesics of length greater than* πr *are not minimal and therefore the diameter of* M^n *does not exceed* πr.

Note that (5) contains the Ricci curvature rather than sectional curvature. A proof of Theorem 1.3.1 is based on the fact that, along geodesics of sufficiently large length, there are smooth vector fields W vanishing at a pair of points and such that $\delta^2(\gamma)(W,W) < 0$. If the sectional curvature of the metric is positive, the proof is simple.

Let $K \ge \tau > 0$. Consider a parallel field W along γ ($DW/\partial t = 0$) and orthogonal to $\dot\gamma$. Construct the corresponding field $W^*(t) = \sin(\pi t/T)W(t)$ along γ_0^T, the segment of the geodesic covered when the parameter varies from 0 to T. Then

$$\delta^2(\gamma_0^T)(W^*, W^*) = \int_0^T \sin^2(\pi t/T)\left(\frac{\pi^2}{T^2} - \langle R(\dot\gamma, W)\dot\gamma, W\rangle\right) dt. \qquad (6)$$

For $T > \pi/\sqrt{\tau}$, the right-hand side of (6) is negative and the field W^* defines the variation of γ which decreases the value of the energy functional. In view of (3), this variation decreases the value of the length functional too.

Theorem 1.3.1, more precisely, finiteness of the diameter, implies the following assertion:

1.3.2. *If* (5) *is satisfied (in particular, if* $K \ge \tau > 0$*) then the manifold is compact and its fundamental group is finite.*

In the case of even-dimensional manifolds, assertion 1.3.2 on finiteness of the fundamental group can be strengthened:

1.3.3. The Singe theorem. *Let* $K \ge \tau > 0$ *and let* M^n *be an even-dimensional manifold. Then* $\pi_1(M^n) = 0$ *whenever the manifold is orientable and* $\pi_1(M^n) = \mathbb{Z}_2$ *otherwise.*

Proof. It is sufficient to prove that if a manifold is orientable then it is simply connected.

Assume the contrary. Let η be a nontrivial homotopic class of maps from S^1 into M^n and let γ be a closed geodesic that is the shortest contour in this class. Let T be the period of γ. Choose a point $x \in \gamma$ and take an $(n-1)$-dimensional subspace V in $\mathrm{T}_x M^n$ which is the orthogonal complement of $\dot\gamma$. Consider all possible parallel vector fields $v(t)$ that are solutions to the linear equation $Dv/\partial t = 0$ with $v(0) \in V$. Since $v(T) \in V$, we obtain an orthogonal map $A\colon V \to V$ of the form $v(0) \to v(t)$.

Since the dimension of V is odd and M^n is orientable, A has an eigenvector $\tilde v$ such that $A(\tilde v) = \tilde v$. It generates a parallel periodic vector field $\tilde v(t)$. By the formula of second variation,

$$\delta^2(\gamma)(\tilde v, \tilde v) = -\int_\gamma \langle R(\dot\gamma, \tilde v)\dot\gamma, \tilde v\rangle\, dt < 0,$$

which contradicts minimality of the closed geodesic γ. The Singe theorem is proven.

The proof of the following Gromov theorem now uses facts concerning global geometry.

1.3.4. The Gromov theorem [19, 21]. *There exist constants $c_1(n) \le 2(\sqrt5)^n$ and $c_2(n)$ such that if an n-dimensional compact manifold M^n has nonnegative sectional curvature then*

1) *the minimal number of generators $\pi_1(M^n)$ does not exceed $c_1(n)$;*
2) *the sum of its Betti numbers does not exceed $c_2(n)$.*

The best estimate for $c_2(n)$ was obtained by Abresch [1]:

$$c_2(n) \le \exp(6n^3 + 9n^2 + 4n + 4) \cdot \exp\left(\frac{15n-13}{4} b_1(M^n)\right),$$

where $b_1(M^n)$ is the first Betti number that is equal to zero in the case $K > 0$.

Sha and Yang [37] proved that if we take a product of spheres $\Sigma = S^m \times S^n$, where $m, n > 2$ are odd, then, on a connected sum of every number of Σ's, there exists a metric of positive Ricci curvature. Now, with the help of the Gromov theorem, we deduce the following:

1.3.5. *There exist closed simply connected manifolds admitting metrics of positive Ricci curvature and not admitting metrics of positive sectional curvature (these are connected sums of sufficiently many copies of Σ).*

The listed facts almost exhaust topological restrictions, known by now, on manifolds admitting metrics of positive sectional curvature. For completeness, it is necessary to add Hitchin's theorem that we present separately in 1.4.

1.4. *The Weitzenböck formulas and their application.* Let M^n be a closed Riemannian manifold. On completed spaces of k-forms, the structure of Hilbert spaces can be naturally introduced and, to operators, their adjoints can be assigned.

Hodge defined the operator δ that is adjoint to the exterior derivative d, with the help of which the *Laplace–Hodge operator*

$$\Delta = d\delta + \delta d$$

is introduced, with the same main symbol as the *Laplace–Beltrami operator* $\nabla^*\nabla$.

Define the operator Ric on vector fields by the formula

$$W \to \mathrm{Ric}(W) = \sum_{j=1}^n R(e_j, W)e_j,$$

where $\{e_j\}$ is an orthonormalized basis in $\mathrm{T}_x M^n$. The Laplace–Hodge operator and the Laplace–Beltrami operator on the space of 1-forms are connected by the following formula.

1.4.1. The Weitzenböck formula. *The equality*

$$\Delta = \nabla^*\nabla + \mathrm{Ric}$$

holds, where $\mathrm{Ric}(\omega)_k = g_{jk}\mathrm{Ric}(\omega^*)^j$ and $\omega^{*k} = g^{kj}\omega_j$.

Important corollaries to formulas of Weitzenböck type were found by Bochner:

1.4.2. The Bochner theorem. *If a manifold has nonnegative Ricci curvature then every harmonic 1-form is parallel:*

$$\Delta\omega = 0 \;\Rightarrow\; \nabla\omega = 0.$$

Since, by the Hodge theorem, the space of harmonic 1-forms is isomorphic to $H^1(M^n; \mathbb{R})$ and the k-dimensional space of parallel forms defines a free action of the k-dimensional torus T^k on M^n, we obtain:

1.4.3. *If* $\mathrm{Ric} \ge 0$ *then*

1) $b_1 = \dim H^1(M^n; \mathbb{R}) \le n = \dim M^n$;
2) *if $b_1 = n$ then M^n is isometric to the flat n-dimensional torus T^n.*

Although these results do not concern manifolds with positive sectional curvature, we present them because their extensions to the case of spinor manifolds give obstructions to existence of metrics of positive scalar curvature. Namely,

1.4.4. The Lichnerowicz theorem*. *If M^n is a connected compact spinor manifold, ∇ is a Levi–Civita connectedness on the spinor bundle, and*

* For four-dimensional manifolds, there exists a wide generalization of this theorem, namely: if a closed four-dimensional manifold admits a metric of positive scalar curvature, then its Seiberg–Witten invariants are equal to zero (Witten).

$\mathcal{D}$ is the Dirac operator on it ($\mathcal{D} = \mathcal{D}^*$), then

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{1}{4}R.$$

Whence, by Bochner's scheme, we obtain:

1.4.5. 1) *If $R \geq 0$ and R is not identically equal to zero, then there are no nonzero harmonic spinors (those are 1-forms ω such that $\mathcal{D}\omega = 0$).*

2) *If $\dim M^n = 4k$ and $R > 0$ then the $\widehat{A}$-genus of the manifold M^n is equal to zero.*

Hitchin found conditions under which harmonic spinors exist and showed the following (see [26]):

1.4.6. *If M^n is a compact spinor manifold with $R > 0$ then $\widehat{\mathcal{A}}(M^n) = 0$, where $\widehat{\mathcal{A}}\colon \Omega^*_{\text{spin}} \to \mathrm{KO}^{-*}(pt)$ is the Milnor homomorphism from the ring of spinor cobordisms into KO-groups of the point that coincides with the $\widehat{A}$-genus for $4k$-dimensional manifolds.*

In particular, the result implies existence of 9- and 10-dimensional exotic spheres that do not admit even a metric of positive scalar curvature.

1.5. *Sphere theorems.* Since we will discuss pinchings of various manifolds below, we briefly recall theorems on a sphere. For a detailed and thorough review, see [2].

The pinching of a manifold is equal to δ if $\min K/\max K = \delta$.

1.5.1. The Berger–Klingenberg theorem. *If the pinching of a simply connected manifold is greater than 1/4 then the manifold is homeomorphic to a sphere.*

This theorem is not improvable for even dimensions, since the pinchings of the complex and quaternionic projective spaces $\mathbb{C}P^n$ and $\mathbb{H}P^n$ and the projective Cayley plane $\mathbb{C}aP^2$ with Fubini–Study metrics are equal to 1/4. But the following holds:

1.5.2. The Berger rigidity theorem. *If the pinching of a simply connected manifold M is equal to 1/4 and M is not homeomorphic to a sphere then it is isometric to one of the spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$, or $\mathbb{C}aP^2$ with the Fubini–Study metric.*

Furthermore, for every n, there exists a constant δ_n such that $\delta_n < 1/4$ and δ_n-pinching of an n-dimensional simply connected manifold implies its homeomorphy to one of the spaces mentioned above.

For odd dimensions, we have

1.5.3. The Abresch–Meyer theorem. *If the pinching of a simply connected odd-dimensional manifold is greater than $1/4 \cdot (1 + 10^{-6})^{-2} < 1/4$ then the manifold is homeomorphic to a sphere.*

Diffeomorphy is guaranteed by a stronger pinching (see [23]):

1.5.4. *If the pinching of a simply connected manifold is greater than ≈ 0.68 then the manifold is diffeomorphic to the standard sphere.*

2. On problems concerning the topology of manifolds of positive sectional curvature

2.1. *Problems and conjectures.* The most known is the H. Hopf problem:

2.1.1. H. Hopf's problem. *Does $S^2 \times S^2$ admit existence of a metric of positive sectional curvature?*

It is generalized as follows:

2.1.2. *Can a metric of positive sectional curvature exist on a direct product of closed simply connected manifolds?*

The following conjecture is due to H. Hopf too.

2.1.3. The conjecture on the Euler characteristic. *The Euler characteristic $\chi(M^n)$ of an even-dimensional closed manifold M^n of positive sectional curvature is nonnegative.*

Sometimes, this conjecture is formulated in a stronger form:

2.1.4. *The Euler characteristic of an even-dimensional closed manifold M^n of positive sectional curvature is positive.*

Gromov stated the following conjecture in [21]:

2.1.5. Gromov's conjecture. *The sum of Betti numbers $b_k(M^n)$ of a closed connected n-dimensional Riemannian manifold M^n of nonnegative sectional curvature does not exceed 2^n, the sum of Betti numbers of the n-dimensional torus T^n.*

It can be strengthened too:

2.1.6. *The $b_k(M^n)$ th Betti number of an even-dimensional closed manifold M^n of positive sectional curvature does not exceed $n!/k!(n-k)! = b_k(\mathrm{T}^n)$.*

There is a close connection between the conjecture on the Euler characteristic and Gromov's conjecture with the Bott conjecture which was first stated in the present form by Grove and Halperin in [22].

2.1.7. Bott's conjecture. *A closed simply connected manifold of nonnegative sectional curvature is rationally elliptic, that is, the sum of ranks of its homotopic groups is finite.*

Relying on Zassenhaus's classification of finite groups acting by orthogonal transformations on spheres, Chern stated the following conjecture.

2.1.8. Chern's conjecture. *Abelian subgroups of fundamental groups of manifolds of positive sectional curvature are cyclic.*

Apparently, the known stock of examples was the base for the following problem posed by Yau [42]:

2.1.9. Yau's problem. *Is it true that every manifold of positive sectional curvature admits an effective smooth action of* S^1?

2.2. *On Bott's conjecture and the entropy of a geodesic flow.* Important properties of rationally elliptic simply connected manifolds were presented by Friedlander and Halperin [17, 24], namely:

2.2.1. *If* M^n *is an* n*-dimensional rationally elliptic manifold then*

1) $b_k(M^n) \leq b_k(T^n)$ *and, as a consequence,* $\sum_k b_k(M^n) \leq 2^n$;
2) $\chi(M^n) \geq 0$ *and, furthermore,* $\chi(M^n) > 0$ *if and only if all odd-dimensional homology groups are finite;*
3) $\dim(\pi_*(M^n) \otimes \mathbb{R}) \leq n$.

The following assertion ensues immediately from 2.2.1.

2.2.2. *If Conjecture* 2.1.7 (*due to Bott*) *is true then, for manifolds admitting metrics of nonnegative sectional curvature, Conjectures* 2.1.3 (*on the Euler characteristic*), 2.1.5 (*due to Gromov*), *and* 2.1.6 *are true too.*

Therefore, we dwell upon this conjecture especially. The only approach to it we know is based on the articles by Berger and Bott [8], Gromov [20], and Yomdin [43] and relates to using the concept of entropy.

According to Gromov's theorem [20], for every closed simply connected Riemannian manifold M^n, there exists a constant $\lambda > 0$ such that

$$n(x, y, t) \geq \sum_{j=1}^{[\lambda t]} \dim H_j(\Omega M^n; \mathbb{R}),$$

where $n(x, y, t)$ is the number of geodesics of length $\leq t$ connecting the pair of points $x, y \in M^n$ in general position and ΩM^n is the space of loops on M^n.

Results of [8, 43] imply that, if the metric is of class C^∞, then

$$\overline{\lim_{m\to\infty}} \frac{1}{m} \log\left(\sum_{j=1}^{m} \dim H_j(\Omega M^n; \mathbb{R})\right) \leq \frac{h_{top}}{\lambda},$$

where h_{top} is the topological entropy of a geodesic flow. At the same time, it is known that the sub-exponential increase of

$$\sum_{j=1}^{m} \dim H_j(\Omega M^n; \mathbb{R})$$

is equal to the rational ellipticity [22].

Mañé [31] proved that, for metrics of class C^∞, the relation

$$h_{top} = \lim_{t\to\infty} \frac{1}{t} \log \int n(x, y, t)\, dx\, dy$$

holds.

It is quite natural to assume that the topological entropy of a geodesic flow of a metric of positive sectional curvature is equal to zero. This would imply Bott's conjecture for manifolds admitting such metrics (apparently, this scheme of proving rational ellipticity was proposed first in [33]). However, it turned out that even on a convex two-dimensional sphere there exist metrics such that the entropy of their geodesic flows is positive [28], see also [12, 34].

An example of such a metric can be obtained in the following way. Consider the ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$$

with different semiaxes: $a_1 < a_2 < a_3$. Its sections by the coordinate planes are closed geodesics. In particular, there exists a homoclinic trajectory to the middle of them in length, γ (to the section by the plane $x_2 = 0$), that is a trajectory which, in both directions $t \to -\infty$ and $t \to \infty$, winds asymptotically around γ. For small perturbation of the metric, we can get the effect of "separatrix decomposition" that implies positivity of the topological entropy connected with rise of a chaotic regime on a small-dimensional (!) set. But the following question remains open.

2.2.3. *Is the entropy in Liouville measure of the geodesic flow of a metric of positive sectional curvature equal to zero?*

Recently, Burns and Paternain showed that, by using infinitesimal C^∞-perturbations of the metric on the unit two-dimensional sphere, we can obtain metrics for which there exists a point x such that $n(x, x, t)$ increases with an arbitrary speed for $t \to \infty$ [10].

Although the above approach to Bott's conjecture is not suitable, it is possible that variations of ideas and methods used in it (they are rather wide) can lead to a success.

2.3. *On analogs of the conjectures for Alexandrov spaces.* For Alexandrov spaces, analogs of the above conjectures are not true in general, which indicates their differential nature [35].

Let X be an Alexandrov space with $K > 0$. The suspension over X is defined as $\Sigma X = [0,\pi] \times X/(0 \times X \sim N,\ \pi \times X \sim S)$, i.e., in the direct product of X and the segment $[0,\pi]$, the boundaries $0 \times X$ and $\pi \times X$ are glued into points that are the poles N and S. If we define a metric on ΣX by the formula

$$d_{\Sigma X}\big((s,x),(t,y)\big) = \cos^{-1}\big(\cos s \cdot \cos t + \sin s \cdot \sin t \cdot \cos d_X(x,y)\big),$$

where d_X is the distance function on X, then the resulting Alexandrov space has positive curvature too.

This construction is generalized on joins of spaces. The join $X_1 * X_2$ of spaces X_1 and X_2 is the quotient space of the direct product $[0,\pi] \times X_1 \times X_2$ with the identifications $(0, x_1, y_1) \sim (0, x_1, y_2)$ and $(\pi, z_1, x_2) \sim (\pi, z_2, x_2)$, where $x_1, z_1, z_2 \in X_1$ and $x_2, y_1, y_2 \in X_2$.

2.3.1. 1) *The suspension* $\Sigma\mathbb{C}P^n$ *for* $n \geq 2$ *is not rationally elliptic.*

2) *The group* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *acts freely and isometrically on* $SO(3) * SO(3)$ *and, therefore, it is a fundamental group of an Alexandrov space with* $K > 0$ (see [35]).

Suspensions over projective spaces also give counterexamples to the conjecture on the Euler characteristic that should be modified in this case.

2.3.2. *If an Alexandrov space* X *has positive curvature, is it true that* $\chi(X) \geq 1$ *for an even-dimensional* X *and* $\chi(X) \leq 1$ *for an odd-dimensional* X?

Alexandrov spaces arise naturally as limits of smooth Riemannian manifolds. However, the above examples cannot be "smoothed," since the given CW-complexes are not Poincaré complexes (i.e., the Poincaré duality is not valid for them).

3. Known examples of manifolds of positive sectional curvature (experimental data)

3.1. In this section, we list all known examples of simply connected manifolds admitting metrics of positive sectional curvature and discuss methods of constructing them and their properties. Apparently, it is necessary to study thoroughly the known examples considering them as natural objects for better understanding of the situation.

3.2. *G-invariant metrics on spaces* G/H. Let G be a compact Lie group and let H be a subgroup of it acting on G by right multiplications: $g \to gh$, where $g \in G$ and $h \in H$. Denote the corresponding Lie algebras by $\mathbf{g}$ and $\mathbf{h}$. Denote by $M = G/H$ the quotient space of this action. On M, the group G acts by left "multiplications." Defining a G-invariant metric on M is equivalent to defining an Ad H-invariant scalar product on the quotient algebra $\mathbf{g}/\mathbf{h}$.

3.3. *Normally homogeneous spaces (Berger's classification).* On a Lie group G (with Lie algebra $\mathbf{g}$), the curvature of the plane generated by vectors $X, Y \in T_eG = \mathbf{g}$ is equal to

$$K(\sigma) = \frac{1}{4}|[X,Y]|^2$$

in the bi-invariant metric. This implies that, among simply connected Lie groups with Killing metrics, only $SU(2) \approx S^3$ has positive sectional curvature.

If we consider the quotient spaces G/H of such groups by subgroups of sufficiently large rank then

3.3.1. *The list of all normally homogeneous spaces (i.e. quotient spaces of Lie groups with bi-invariant metrics) with* $K > 0$ *is as follows* (see [7, 41]):

(a) *compact symmetric spaces of rank* 1 (CROSSes):

the spheres $S^n = SO(n+1)/SO(n)$,
the complex projective spaces $\mathbb{C}P^n = SU(n+1)/SU(n)$,
the quaternionic projective spaces $\mathbb{H}P^n = Sp(n+1)/Sp(n)$,
the projective Cayley plane $\mathbb{C}aP^2 = F_4/Spin(9)$;

(b) *Berger spaces:* $B^7 = Sp(2)/SU(2)$ *and* $B^{13} = SU(5)/(Sp(2) \times S^1)$;

(c) *the space* $(SU(3) \times SO(3))/U^*(2)$.

Note that the embedding $SU(2) \subset Sp(2)$ is not canonical. Pinchings of normally homogeneous metrics on CROSSes that are not spheres are equal to $1/4$ (these are Fubini–Study metrics, see 1.5.2).

Describe the space $(SU(3) \times SO(3))/U^*(2)$. The embedding of the subgroup $U(2) \subset SU(3) \times SO(3)$ is diagonal:

1) it embeds into $SU(3)$ by the formula

$$A \to \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix};$$

2) the mapping from $U(2)$ onto the second component is the projection $U(2) \to U(2)/S^1 = SO(3)$, where $S^1 \subset SO(3)$ is the center of the group $SO(3)$.

Note that the space $(SU(3) \times SO(3))/U^*(2)$ is isometric to the Aloff–Wallach space $N_{1,1}$ with normally homogeneous metric (see 3.5) and was omitted in the list of Berger. Recently, Wilking [41] indicated this gap. Furthermore, there exists a one-parameter family of bi-invariant metrics on the space $SU(3) \times SO(3)$ and, consequently, by [41], there exists a one-parameter family of pairwise nonhomothetic normally homogeneous metrics on $N_{1,1}$.

3.3.2. 1) *A normally homogeneous metric on B^7 is $1/37$-pinched* (see [13]).

2) *A normally homogeneous metric on B^{13} is $16/(29\cdot 37)$-pinched* (see [25]).

According to results of Püttmann ([36], see also 3.9.2.2), among the normally homogeneous metrics on $N_{1,1}$, the maximal pinching is attained at only one of them and is equal to $1/37$.

3.4. *Wallach spaces.*

3.4.1. *If an even-dimensional homogeneous space is not homeomorphic to a normally homogeneous space and $K > 0$, then it is diffeomorphic to one of the Wallach spaces G/H* (see [40]):

1) $SU(3)/T^2$, *where $H = T^2$ is the maximal torus of the group* $SU(3)$;

2) $Sp(3)/(Sp(1)\times Sp(1)\times Sp(1))$, *where the embedding $H \subset G$ is diagonal;*

3) $F_4/Spin(8)$.

These spaces have similar structure, as the following construction shows.

3.4.2. *Duplication of algebras.* An algebra over $\mathbb{R}$ is a vector space (over $\mathbb{R}$) with a multiplication $a, b \to a \cdot b$.

Let (A, σ) be an algebra over $\mathbb{R}$ with an involutive automorphism (conjugation): $a \to \bar{a}$. Consider the algebra A^2 with multiplication

$$(a,b)\cdot(u,v) = (au - \bar{v}b, b\bar{u} + va) \in A^2.$$

Define a conjugation on A^2 by the formula

$$(a,b) \to (\bar{a}, -b).$$

The algebra A^2 called the *duplication* of the algebra A.

We have the following series:

$$A_0 = \mathbb{R} \to A_1 = \mathbb{C} \to A_2 = \mathbb{H} \to A_3 = \mathbb{C}a,$$

in which every arrow means duplication. These algebras are algebras with division, i.e., for all nonzero a and b, the equations $ax = b$ and $ya = b$ are uniquely solvable and the algebras possess a unit. The algebra A_4 is not an algebra with division. This agrees with the Adams theorem which states that the dimension of a finite-dimensional algebra with division over $\mathbb{R}$ is equal to 1, 2, 4, or 8.

3.4.3. *Wallach spaces as flag spaces over the projective spaces AP^2.* For every algebra $A = A_k$, $k = 0,1,2,3$, we define a projective straight line AP^1 diffeomorphic to the sphere $S^{2^{k+1}-1}$. For the algebras A_0, A_1, and A_2, the projective planes AP^2 are also defined naturally. The algebra A_3 is nonassociative, but a projective plane can be define for it too (in contrast to projective spaces of greater dimension).

Denote by P_k the projective plane over the algebra A_k. For every algebra A, we can also define the space of flags which are sequences of subspaces (over A) of the form $l^1 \subset l^2 \subset A^3$, where $\dim_A l^j = j$. It is easy to see that, for every algebra A_k, the flag space Q_k is a bundle over P_k with fiber $A_kP^1 = S^{2^{k+1}-1}$. These flag spaces are homogeneous:

$$Q_0 = SO(3), \quad Q_1 = SU(3)/T^2, \quad Q_2 = Sp(3)/Sp(1)^3, \quad Q_3 = F_4/Spin(8).$$

The spaces Q_1, Q_2, and Q_3 are exactly the Wallach spaces.

3.4.4. *Homogeneous metrics on Wallach spaces.* Define $SU(n)$ to be the group of complex $n\times n$-matrixes X such that $X\cdot X^* = I_n$, where I_n is the identity $n\times n$-matrix and $X^*_{ij} = -\overline{X}_{ji}$, and define $Sp(n)$ to be the subgroup of $SU(2n)$ constituted by matrices Y such that $Y\cdot \mathrm{diag}(J,\dots,J)\cdot Y^t = \mathrm{diag}(J,\dots,J)$, where $Y^t_{ij} = Y_{ji}$ and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The subgroup $T^2 \subset SU(3)$ is constituted by diagonal matrices and is a maximal commutative subgroup. The subgroup $Sp(1)^3 \subset Sp(3)$ is defined to be the range of the embedding

$$Sp(1)^3 \to \mathrm{diag}(X_1, X_2, X_3), \quad X_j \in Sp(1).$$

The group F_4 is defined in a more complicated way. Consider the algebra $\mathbf{A}l$ formed by 3×3-matrices of the form

$$\begin{pmatrix} E_1 & X_1 & X_2 \\ \overline{X}_1 & E_2 & X_3 \\ \overline{X}_2 & \overline{X}_3 & E_3 \end{pmatrix},$$

where $E_j \in \mathbb{R}$, $X_k \in \mathbb{C}a$, with multiplication

$$X \circ Y = \frac{XY + YX}{2}.$$

The group F_4 is the automorphism group of this 27-dimensional algebra.

Every Wallach space has the form G/H (see 3.2); moreover, the quotient algebras $\mathbf{g}/\mathbf{h}$ are divided into direct sums of irreducible and nonisomorphic Ad H-modules,

$$\mathbf{g}/\mathbf{h} = V_1 \oplus V_2 \oplus V_3,$$

which are isomorphic to $\mathbb{C}$ (for Q_1), $\mathbb{H}$ (for Q_2), and $\mathbb{C}a$ (for Q_3). Furthermore, these quotient algebras are identified with matrix algebras of the form

$$\begin{pmatrix} 0 & v_1 & v_2 \\ -\bar{v}_1 & 0 & v_3 \\ -\bar{v}_2 & -\bar{v}_3 & 0 \end{pmatrix},$$

where $v_i \in V_i$. Homogeneous metrics on them have the form

$$\lambda_1\langle\cdot,\cdot\rangle|_{V_1} + \lambda_2\langle\cdot,\cdot\rangle|_{V_2} + \lambda_3\langle\cdot,\cdot\rangle|_{V_3}, \qquad (7)$$

where $\langle\cdot,\cdot\rangle|_{V_j}$ are the bounded Killing metrics on V_j.

Wallach showed the following:

3.4.4.1. *There exist such values of the parameters λ_j that the metrics* (7) *have positive sectional curvature* (see [40]).

A detailed analysis of the pinchings of these metrics was made by Valiev who showed the following:

3.4.4.2. *Homogeneous metrics on Wallach spaces attain the maximal pinching* $1/64$ *at* $2\lambda_1 = \lambda_2 = \lambda_3$ (see [39]).

Along the lines of the article [38], it can be even shown that there exists a tower of completely geodesic embeddings

$$\mathrm{SU}(3)/\mathrm{T}^2 \subset \mathrm{Sp}(3)/(\mathrm{Sp}(1)^3) \subset F_4/\mathrm{Spin}(8)$$

such that the pinchings of metrics are preserved.

3.5. *Aloff–Wallach spaces.* The structure of metrics on Wallach spaces is generalized to the odd-dimensional case in the following way.

Let $\mathrm{T}_{k,l} \subset \mathrm{SU}(3)$ be the subgroup constituted by diagonal matrices of the form $\mathrm{diag}(z^k, z^l, z^{-(k+l)})$, where $|z| = 1$ and k and l are coprime. By $\mathbf{t}_{k,l}$ we denote the Lie algebra of this subgroup. Consider also the subgroup $G_1 \subset \mathrm{SU}(3)$ constituted by matrices of the form $\mathrm{diag}(A, \det A^{-1})$, where $A \in \mathrm{U}(2)$.

On the quotient space $N_{k,l} = \mathrm{SU}(3)/\mathrm{T}_{k,l}$, in terms of the bi-invariant Killing metric on SU(3), we define the homogeneous metrics

$$t\langle\cdot,\cdot\rangle_{V_1} + \langle\cdot,\cdot\rangle_{V_2}, \qquad (8)$$

where $\mathbf{t}_{k,l}^{\perp}$ is the orthogonal complement of $\mathbf{t}_{k,l}$ in $su(3)$ and $\mathbf{t}_{k,l} = V_1 \oplus V_2$ is an orthogonal decomposition with $V_1 = \mathbf{t}_{k,l} \cap \mathbf{g}_1$ and $V_2 = \mathbf{g}_1^{\perp}$.

3.5.1. *If $kl > 0$ and $0 < t < 1$ then metric* (8) *has positive sectional curvature* (see [3]).

This series is the first example of an infinite family of pairwise nonhomeomorphic manifolds of equal dimensions admitting metrics with $K > 0$.

Aloff–Wallach spaces form bundles over $\mathrm{SU}(3)/\mathrm{T}^2$ with stalk S^1. Indeed, they are obtained by factorization of SU(3) by the winding of the maximal torus rather than by the entire torus. The denser is the winding, the smaller are the radius of injectivity and the volume of the manifold. We can also show that, as $k, l \to \infty$ with $k/l \to 1$, the pinching of $N_{k,l}$ tends to the pinching of $N_{1,1}$ (with the parameter t fixed in (8)). This implies the following assertion.

3.5.2. *There exists an infinite series of manifolds M_n with $K > 0$ such that their pinchings are separated from zero, $0 < \varepsilon < K < 1$, and their radii of injectivity and volumes tend to zero as $n \to \infty$* (see [27]).

Other remarkable properties of these spaces were indicated by Kreck and Stolz.

3.5.3. 1) *There exist homogeneous pairwise homeomorphic but not diffeomorphic spaces with $K > 0$* (*for example*, $N_{-56788,5227}$ *and* $N_{-42652,61213}$) (see [29]);

2) *There exist homogeneous spaces such that the spaces of metrics with $K > 0$ are disconnected on them* (*for example*,

$$N_{-4638661,582656} \approx N_{-2594149,5052965};$$

metrics (8) *on these diffeomorphic manifolds lie in different connected components of the space of metrics with $K > 0$*) (see [30]).

3.6. *The Wallach–Berard Bergery theorem: completeness of the classification of homogeneous spaces with $K > 0$.* Naturally, homogeneous metrics with $K > 0$ on CROSSes and Berger spaces are not exhausted by normally homogeneous metrics and admit variations of the form (7) and (8). However, diffeomorphic types of homogeneous spaces with $K > 0$ are exhausted by such metrics:

3.6.1. *If a simply connected homogeneous manifold has positive sectional curvature then it is diffeomorphic to one of the following spaces: a CROSS, the Berger space B^7 or B^{13}, a Wallach space, or an Aloff–Wallach space.* (See [40] for even-dimensional manifolds and [5] for odd-dimensional manifolds.)

3.7. *Eschenburg spaces.* The first examples of spaces nonhomeomorphic to homogeneous spaces, with $K > 0$, were found by Eschenburg [14, 15], who used for this purpose the construction of double factors proposed by Gromoll and Meyer ([18], see also 3.10.1).

The Aloff–Wallach manifold $N_{k,l}$ endowed with metric (8) for $t = 1/2$ (denote it by $N_{k,l,1/2}$) can be constructed in the following way.

Let $H = \mathrm{U}(2) \times \mathrm{U}(1)$ be the subgroup of U(3) constituted by block matrices of the corresponding form. On U(3), consider the following homogeneous metric that is left-invariant with respect to U(3) and right-invariant with respect to H:

$$\frac{1}{2}\langle\cdot,\cdot\rangle_{\mathbf{h}} + \langle\cdot,\cdot\rangle_{\mathbf{h}^{\perp}}, \qquad (9)$$

where $\langle\cdot,\cdot\rangle$ is the Killing metric and $\mathbf{h}$ is the tangent algebra for H.

Then $N_{k,l,1/2}$ is the quotient space of U(3) (with metric (9)) by the isometric action

$$g \to g \cdot u^{-1}, \quad u \in \mathrm{U}_{k,l}Z,$$

where $\mathrm{U}_{k,l} = \{\mathrm{diag}(z^k, z^l, 1) \mid |z| = 1\}$ and $Z = \{\mathrm{diag}(z, z, \bar{z}) \mid |z| = 1\}$.

In other words, we "multiply" SU(3) by S^1 (passing to U(3)) and, in addition, "divide" by S^1 from the right. Next, we move the division by S^1 to the left side and consider the double quotient spaces

$$N'_{k,l} = Z\backslash \mathrm{U}(3)/\mathrm{U}_{k,l}.$$

3.7.1. *The Riemannian manifolds $N'_{k,l}$, for $k, l > 0$, have positive sectional curvature and are not homeomorphic to homogeneous manifolds* (see [14, 15]).

The manifolds $N'_{k,l}$ can be obtained as the double quotient spaces of SU(3) with respect to the action

$$X \to \mathrm{diag}(z^k, z^l, 1) \cdot X \cdot \mathrm{diag}(z^{-(k+l)}, z^{-(k+l)}, z^{k+l})$$

and admit the following generalizations.

3.7.2. *Let the action*

$$X \to \mathrm{diag}(z^{a_1}, z^{a_2}, z^{a_3}) \cdot X \cdot \mathrm{diag}(z^{-b_1}, z^{-b_2}, z^{-b_3})$$

of the group $\mathrm{U}_{\mathrm{a,b}}$ (*where* $\sum a_j = \sum b_j$) *be free and let the metric on* SU(3) *be induced by metric* (9) *under the embedding* SU(3) ⊂ U(3). *Then the double quotient space* $\mathrm{SU}(3)/\mathrm{U}_{\mathrm{a,b}}$ *has positive sectional curvature if and only if* $b_j \notin [a_{\min}, a_{\max}]$ *for all* $j = 1, 2, 3$ (see [15]).

In much the same way as Aloff–Wallach spaces form bundles with stalk S^1 over $\mathrm{SU}(3)/\mathrm{T}^2$, the Eschenburg spaces $N'_{k,l}$ form bundles over the space

$$F' = Z\backslash \mathrm{U}(3)/\{\mathrm{diag}(z_1, z_2, 1) \mid |z_1| = |z_2| = 1\}.$$

3.7.3. *Among even-dimensional double quotient spaces of prime compact Lie groups endowed with G-left-invariant and H-right-invariant metrics, where H includes the maximal torus, the space F' is the only space which is nondiffeomorphic to a homogeneous space and has $K > 0$* (see [15]).

3.8. *Bazaĭkin spaces.* An elegant generalization of 7-dimensional Eschenburg spaces to the 13-dimensional case was found by Bazaĭkin.

3.8.1. *Let* U(5) *be endowed with a metric of the form* (9) *for* $H = \mathrm{U}(4) \times \mathrm{U}(1)$. *Let* $M_{\bar{p}}$ *be the double quotient space by the action*

$$X \to \mathrm{diag}(z_1^{p_1}, z_1^{p_2}, z_1^{p_3}, z_1^{p_4}, z_1^{p_5}) \cdot X \cdot \begin{pmatrix} A^* \bar{z}_2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $z_1, z_2 \in S^1$ *and* $A \in \mathrm{Sp}(2)$. *Assume that, for every permutation* σ, *the following conditions hold:*

a) $p_{\sigma(1)} + p_{\sigma(2)} - p_{\sigma(3)} - p_{\sigma(4)}$ *is coprime to* $p_{\sigma(5)}$;
b) $p_{\sigma(1)} + p_{\sigma(2)} + p_{\sigma(3)} > p_{\sigma(4)} + p_{\sigma(5)}$;
c) $p_{\sigma(1)} + p_{\sigma(2)} + p_{\sigma(3)} + p_{\sigma(4)} > 3p_{\sigma(5)}$;
d) $3(p_{\sigma(1)} + p_{\sigma(2)}) > p_{\sigma(3)} + p_{\sigma(4)} + p_{\sigma(5)}$.

Then the manifold $M_{\bar{p}}$ *has positive sectional curvature* (see [4]).

Since, in contrast to the preceding examples, the above examples are rather new and not well studied, we present their topological characteristics.

3.8.2. *The space* $M_{\bar{p}}$ *has the following cohomology groups* (see [4]):

$$H^i = \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, 9, 11, 13, \\ 0 & \text{for } i = 1, 3, 5, 7, 10, 12; \end{cases}$$

the groups H^6 *and* H^8 *are finite with order* $|\sigma_1^3 - 4\sigma_1\sigma_2 + 8\sigma_3|$, *where* σ_k *is the value of the elementary symmetric polynomial of degree k in five variables at the point* $(p_1, \ldots, p_5)$.

The series $p_1 = 1$, $p_2 = p_3 = p_4 = p_5 = q^n$, where q is a prime number, gives us an example of an infinite family of pairwise nonhomeomorphic 13-dimensional manifolds with $K > 0$.

It remains an open problem to find analogs of examples 3.5.2 and 3.5.3 among $M_{\bar{p}}$. Note only that distinguishing smooth structures within dimension 13 is a very difficult problem because of the structure of the group of smooth structures on S^{13}: it is the group $\mathbb{Z}_3$ that has a prime odd order.

3.9. *Completely geodesic embeddings of 7-dimensional spaces into 13-dimensional spaces and pinchings of metrics with* $K > 0$.

3.9.1. *Isometric involutions and completely geodesic embeddings.* A connection between infinite series of 7- and 13-dimensional examples was established in [38]. The main purpose of that article was to explain coincidence of the pinchings of the 13-dimensional normally homogeneous Berger manifold B^{13} and the 7-dimensional Aloff–Wallach space $N_{1,1,1/2}$: they are equal to $16/(29 \cdot 37)$ (see [25, 27]).

It turned out that the simplest involution on U(5) (and on SU(5)),

$$\sigma\colon \mathrm{U}(5) \to \mathrm{U}(5), \quad A \to S \cdot A \cdot S^{-1}, \tag{10}$$

where $S = \mathrm{diag}(-1,-1,1,1,1)$, induced isometric involutions on $B^{13} = SU(5)/(Sp(2)\times S^1)$ and $M_{\bar p}$ with the following properties.

3.9.1.1. 1) *The submanifold $W^7 \subset B^{13}$, defined to be the component of the set of fixed points of the involution $\sigma\colon B^{13}\to B^{13}$ containing the orbit of the unit of the group* SU(5), *is isometric to $N_{1,1,-1/2}$. Moreover, the sectional curvature of B^{13} attains its minimal and maximal values on planes tangent to W^7.*

2) *The component of the set of fixed points of the involution $\sigma\colon M_{\bar p}\to M_{\bar p}$ containing the $T_{\bar p}\times(Sp(2)\times T_0)$-orbit of the unit of the group* U(5) *is a completely geodesic submanifold $W^7_{\bar p}$ that is isometric to the Eschenburg space $T_{2p_3,2p_4,2p_5}\backslash SU(3)/T_{2p_1-P,2p_2-P,1}$, where $P = p_1+\dots+p_5$* (see [38]).

Here

$$T_{\bar p} = \{\mathrm{diag}(z^{p_1},z^{p_2},z^{p_3},z^{p_4},z^{p_5}) \mid |z|=1\},$$
$$T_0 = \{\mathrm{diag}(z^1,z^1,z^1,z^1,1) \mid |z|=1\},$$
$$T_{k,l,m} = \{\mathrm{diag}(z^k,z^l,z^m) \mid |z|=1\}.$$

These spaces are also similar from a topological point of view.

3.9.1.2. *There exists a bundle*

$$W^7 \xrightarrow{\mathbb{R}P^3} \mathbb{C}P^2$$

such that the transgression d_4 in its spectral sequence has the form

$$d_4\colon E_4^{0,3} = \mathbb{Z} \xrightarrow{\times 3} E_4^{4,0} = \mathbb{Z}$$

and

$$H^4(W^7) = \mathbb{Z}_3.$$

Furthermore, there exists a bundle

$$B^{13} \xrightarrow{\mathbb{R}P^5} \mathbb{C}P^4$$

such that the transgression d_6 in its spectral sequence has the form

$$d_6\colon E_6^{0,5} = \mathbb{Z} \xrightarrow{\times 5} E_6^{6,0} = \mathbb{Z}$$

and

$$H^6(B^{13}) = \mathbb{Z}_5$$

(see [38]).

In [38], sketches are proposed of the proofs of an analog of 3.9.2 for the spaces $M_p = M_{(p,p,p,p,1)}$ and W_p, the double quotient space of U(3) by the action

$$X \to \mathrm{diag}(z^{2p},z^{2p},z^2)\cdot X\cdot \mathrm{diag}\left(uz^{-(2p+1)},uz^{-(2p+1)},1\right),$$

where $|z| = |u| = 1$. It is W_p what is extracted in M_p as a completely geodesic component of the set of fixed points of the involution σ (see 3.9.1). Namely, the analog of 3.9.2 has the following form.

3.9.1.3. *There exist bundles*

$$W_p \xrightarrow{\mathbb{R}P^3} \mathbb{C}P^2$$

and

$$M_p \xrightarrow{\mathbb{R}P^5} \mathbb{C}P^4;$$

moreover, the orders of the finite cohomology groups equal $r_p = (4p-1)$ for $H^4(W_p)$ and $s_p = (8p^2-4p+1)$ for $H^6(M_p)$ and are connected by the relation

$$s_p = \frac{r_p^2+1}{2}. \qquad (11)$$

Apparently, the study of the interrelation (that is represented by completely geodesic embeddings) between various manifolds with $K>0$ will help to clarify some questions concerning their structure. The following question seems to be interesting.

3.9.1.4. *Which manifolds are represented as completely geodesic submanifolds of a given space X with $K>0$? Which of these submanifolds have pinching equal to the pinching of X?*

In [38], we posed the following problem.

3.9.1.5. *Is it true that, for every positive integer k, there exists a space Γ_k such that*

1) *there exists a bundle*

$$\Gamma_k \xrightarrow{\mathbb{R}P^{2k+1}} \mathbb{C}P^{2k}; \qquad (12)$$

2) *the transgression d_{2k+2} in the spectral sequence of the bundle* (12) *has the form*

$$d_{2k+2}\colon E_{2k+2}^{0,2k+1} = \mathbb{Z} \xrightarrow{\times(2k+1)} H^{2k+2}(\mathbb{C}P^{2k})$$

and

$$H^{2k+2}(\Gamma_k) = \mathbb{Z}_{2k+1};$$

3) *the manifold* Γ_k *has positive sectional curvature;*
4) $\Gamma_1 = W^7$ *and* $\Gamma_2 = B^{13}$;
5) *the spaces* Γ_k *form a tower*

$$\Gamma_1 \to \Gamma_2 \to \cdots \to \Gamma_n \to \Gamma_{n+1} \to \cdots$$

of completely geodesic embeddings preserving the pinchings of metrics?

Relations (11) also suggest a possible statement of an analog to this question for other series of manifolds.

In item 5 of 3.9.1.5, we required that the pinching of Γ_n was equal to $16/(29 \cdot 37)$; however, recent Püttmann's results, which we now proceed to, obliges us to make corrections.

3.9.2. *The pinchings of homogeneous metrics.* Homogeneous metrics on the 13-dimensional Berger space B^{13} form one-dimensional family. One of them is normally homogeneous and its pinching is $16/(29 \cdot 37)$, as was shown by Heintze [25]. Another example of a homogeneous metric is a metric on the Bazaĭkin space $M_{\bar{p}}$, where $\bar{p} = (1,1,1,1,1)$.

3.9.2.1. *The pinching of the metric on* $M_{\bar{p}}$ *for* $\bar{p} = (1,1,1,1,1)$ *is* $1/64$, *and the greatest pinching of a homogeneous metric on* B^{13} *is* $1/37$ (see [36]).

Since involution (10) generates an isometric involution on B^{13} with an arbitrary homogeneous metric, 3.9.1.1 implies existence of a homogeneous metric with pinching not less than $1/37$ on $N_{1,1}$ (this contradicts the assertion of [27] that a metric on $N_{1,1,1/2}$ has the greatest pinching equal to $16/(29 \cdot 37)$). Furthermore, the following holds.

3.9.2.2. *The greatest pinching of a homogeneous metric on* $N_{1,1}$ *is* $1/37$ (see [36]).

We indicate once more the strange coincidence of constants that we need to explain: the pinching of the metric on B^7 is $1/37$ and the greatest pinching of a homogeneous metric on an even-dimensional Wallach space is $1/64$.

3.10. *Some examples of manifolds with* $K \geq 0$.

3.10.1. *The Cheeger and Gromoll–Meyer constructions.* The set of all known manifolds admitting metrics with $K \geq 0$ is richer than that of manifolds known to have a metric with $K > 0$.

A large family is obtained by a construction due to Cheeger who showed the following:

3.10.1.1. *On the connected sum of two arbitrary* CROSSes *(of the same dimension) there exists a metric with* $K \geq 0$ (see [11]).

At the same time, from the theory of rationally elliptic spaces it is known that, among connected sums, connected sums of two CROSSes are only rationally elliptic [16].

Another famous example was obtained by Gromoll and Meyer. Let Sp(2) be represented as the group of 2×2-matrices over $\mathbb{H}$ which define orthogonal transformations $\mathbb{H}^2 \to \mathbb{H}^2$, and let the group Sp(1) be represented as the group of unit quaternions. Consider the action of Sp(1) on Sp(2):

$$X \to \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \cdot X \cdot \begin{pmatrix} \bar{q} & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

If the Killing metric is chosen on Sp(2) then action (13) is isometric and the following assertion holds.

3.10.1.2. *The double quotient space of* Sp(2) *by action* (13) *is an exotic (nondiffeomorphic to a standard sphere) seven-dimensional sphere* Σ^7 *with metric of nonnegative sectional curvature* (see [18]).

So far this example of metrics with $K \geq 0$ on an exotic sphere remains the only one. Note that the sphere Σ^7 is a generator of the group of smooth structures on S^7 (this group is isomorphic to $\mathbb{Z}_{28}$).

3.10.2. *On deformation of metrics with* $K \geq 0$ *into metrics with* $K > 0$. The existence problem for such deformations has been under study for a long time due to its connection with Hopf's problem (see 2.1.1): on $S^2 \times S^2$ there exists a unique metric of the direct product of unity spheres in $\mathbb{R}^3$ for which $K \geq 0$. With respect to this problem, the following result was obtained.

3.10.2.1. *The product metric* g_0 *on* $S^2 \times S^2$ *does not admit analytic deformations* $g_0 \to g_t$ *such that, for* $0 < t < \varepsilon$, *the sectional curvatures of the metrics* g_t *are positive* (see [9]).

Apparently, the main obstruction is existence of a residual family of two-dimensional planes on which $K = 0$ and which are tangent to the set of points of the manifold where the curvature vanishes along some two-dimensional directions. An analysis of this situation has led us to the following conjecture that has been proven by Bazaĭkin.

3.10.2.2. *Let* M *be a Riemannian manifold with* $K \geq 0$ *and let* N *be a smooth compact submanifold satisfying the following assumptions:*

1) $K > 0$ *outside* N;
2) $K > 0$ *in two-dimensional directions tangent to* N.

Then there exists a Riemannian metric on M *with* $K > 0$ *conformally equivalent to the initial metric.*

Moreover, the conformal factor (the ratio of metric tensors) can be chosen arbitrarily close to the identity unit together with all its derivatives.

Unfortunately, at present, we cannot effectively apply this lemma to constructing new examples.

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THE SHAPLEY FUNCTIONAL AND POLAR FORMS OF HOMOGENEOUS POLYNOMIAL GAMES

V. A. Vasil'ev *

Abstract

In the present article, we study the generalized Owen extension for infinite cooperative games of bounded polynomial variation. The study of this extension and the corresponding polar forms is carried out in the framework of the theory of semiordered K-spaces. The main result of the article consists in establishing interrelations between the Shapley functional and the polar forms of homogeneous games.

Key words and phrases: Owen's extension, nonatomic measure, cooperative game, Shapley value, polar form.

The present article is an extended presentation of the results announced in [20] and devoted to interrelations between the polar forms and the Shapley functional. The emphasis is on studying the generalized Owen extension and the Shapley functional for polynomial games with infinitely many participants. An analysis of the structural properties of the Owen extension and the corresponding polar forms is realized in the framework of the theory of semiordered K-spaces. The main result of the article is a formula for representation of the Shapley value of a homogeneous game v with the help of its polar form ψ_v.

It should be noted that, in contrast to [2], the spaces of games in question contain both nonatomic and mixed set functions. This fact allows us to use appropriate analogs of the well-known Kantorovich–Rubinshteĭn norm for a finite-dimensional approximation of infinite games and their solutions

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* Sobolev Institute of Mathematics, Novosibirsk, RUSSIA.
E-mail address: vasil@math.nsc.ru.