A remark on positively curved manifolds of dimensions 7 and 13

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1 Imtroduction

In this paper we construct totally geodesic embeddings of some 7-dimensional manifolds into 13-dimensional manifolds with positive sectional curvature and explain the strange coincidence of pinching constants of the normally homogeneous Berger space [1] and the homogeneous Aloff–Wallach space $N_{1,1,-1/2}$ [2]. This constant is equal to $16/(29 \times 37)$, as was established by Heintze for the Berger space [3] and by Huang [4] for the Aloff–Wallach space. Moreover, these totally geodesic embeddings shed light on a relation of the well-known 7-dimensional manifolds constructed by Aloff, Wallach and Eschenburg [5] to the series of 13-dimensional positively curved manifolds found recently by Bazaikin [6].

We identify the Lie groups $U(n)$ with the groups formed by $n \times n$-matrices A such that

$$A \cdot I_n \cdot A^* = I_n,$$

where I_n is the unit $n \times n$-matrix and $A^*_{ij} = \bar{A}_{ji}$. We consider the Lie groups $SU(n)$ as the subgroups of $U(n)$ formed by matrices with $\det = 1$. We mean by $Sp(2)$ the subgroup of $SU(4)$ formed by matrices A that satisfy the equality

$$A \cdot \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \cdot A^t = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

where $A^t_{ij} = A_{ji}$. Moreover, we shall consider $Sp(2)$ as the subgroup of $SU(5)$ formed by matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in SU(5), \quad A \in Sp(2) \subset SU(4).$$

We consider the Lie algebras of these groups as realized by matrices in the usual manner.

We denote by T^1, T^1_p and $T_{(0)}$ the subgroups of $U(5)$ formed by diagonal matrices $\mathrm{diag}(z,z,z,z,z^{-4})$, $\mathrm{diag}(z^p,z^p,z^p,z^p,z)$ and $\mathrm{diag}(z,z,z,z,1)$ respectively, where $|z| = 1$ and p is a positive integer.

First we call the known examples (we consider only simply connected manifolds).

2 7-dimensional manifolds

2.1 *7-dimensional Berger space [1]*

This space is isometric to the factor space $Sp(2)/SU(2)$, where $Sp(2)$ is endowed with the standard biinvariant metric and the embedding $SU(2) \subset Sp(2)$ is a nonstandard one. We shall not discuss this space, and consider it as an exceptional one.

2.2 *Aloff–Wallach spaces [2]*

Let $T_{k,l}$ be the subgroup of $SU(3)$ formed by diagonal matrices $(z^k, z^l, z^{-(k+l)})$. We consider the subgroup $G_1 = U(2)$ of $SU(3)$ given by

$$G_1 = \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix}, \quad A \in U(2)$$

and denote by g_1 the Lie algebra of G_1. We denote by $f_{k,l}$ the Lie algebra of $T_{k,l}$. One can see that $f_{k,l}$ is generated by a diagonal matrix $\mathrm{diag}(2\pi\sqrt{-1}k, 2\pi\sqrt{-1}l, -2\pi\sqrt{-1}(k+l))$. Denote by $\langle\ ,\ \rangle_0$ the Killing bi-invariant metric on $SU(3)$. Then one can consider a homogeneous metric on the factor space $N_{k,l,t} = SU(3)/T_{k,l}$, generated by the metric

$$\langle x, y\rangle = (1+t)\langle x_1, y_1\rangle_0 + \langle x_2, y_2\rangle_0, \qquad (1)$$

where $x_i, y_i \in V_i$ and $f^{\perp}_{k,l} = V_1 \oplus V_2$ is an orthogonal decomposition, $V_1 = f^{\perp}_{k,l} \cap g_1$ and $V_2 = g_1^{\perp}$.

Aloff and Wallach showed that if k and l have the same sign and $-1 < t < 0$ then these manifolds $N_{k,l,t}$ are positively curved.

2.3 *Eschenburg spaces [5, 7]*

These spaces are generalizations of the previous ones, and have the form $T_{k,l}\backslash U(3)/T_{p,q}$. For suitable integers k, l, p and q and a metric on $U(3)$, these manifolds have positive curvature. We shall not dwell on them, and only note that most of them are not homeomorphic to homogeneous manifolds and that they were first examples of such a kind.

3 13-dimensional manifolds

3.1 *13-dimensional Berger space [1]*

This manifold B^{13} is the factor space $SU(5)/(Sp(2)\times T^1)$, where $SU(5)$ is endowed with the Killing bi-invariant metric.

3.2 *Bazaikin spaces [6]*

These spaces have the form $S_{\bar p}\backslash U(5)/(Sp(2)\times T_{(0)})$, where $S_{\bar p}$ is formed by diagonal matrices $\mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}, z^{p_4}, z^{p_5})$, with $|z| = 1$. For suitable tuples of integers $\bar p$ and a metric on $U(5)$, these manifolds have positive curvature. We also mention that for these examples, the metric on $U(5)$ is taken to be left-invariant under $U(5)$ action and right-invariant under $U(4)\times U(1)$ action, where the subgroup $U(4)\times U(1)$ is formed by block-diagonal matrices with 4×4 and 1×1 blocks.

4 Main construction

Put

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{-1} \end{pmatrix}$$

and define the mapping

$$\sigma\colon U(5) \to U(5), \quad A \to S\cdot A\cdot S^{-1}.$$

The following two propositions are evident.

Proposition 1 *$\sigma(G) = G$ for $G = SU(5), Sp(2), T^1, T^1_p, T_{(0)}$.*

Proposition 2 *σ^2 is an identity mapping, i.e. σ is an involution.*

Since the metric on $SU(5)$ is bi-invariant for the Berger space and the metric on is $U(5)$-left-invariant and $U(4)\times U(1)$-right-invariant for the Bazaikin spaces $T^1_p\backslash U(5)/(Sp(2)\times T_{(0)})$, the following proposition holds.

Proposition 3 *The involution σ induces isometric involutions on the spaces B^{13} and $T^1_p\backslash U(5)/(Sp(2)\times T_{(0)})$ $(p>0)$.*

First we consider the action of this involution on the Berger space B^{13}.

Theorem 1 *Let W^7 be a submanifold of B^{13} that contains the point $E = 1\cdot(Sp(2)\times T^1)\in B^{13}$, where 1 is the unit of $SU(5)$, and that is formed by fixed points of involution $\sigma\colon B^{13}\to B^{13}$. Then the manifold W^7 is a totally geodesic submanifold that is isometric to the Aloff–Wallach space $N_{1,1,-1/2}$, and the minimum and maximum values of the sectional curvature of B^{13} are attained on 2-planes tangent to W^7.*

Proof of Theorem 1 First note that, since W^7 is a component of the set formed by fixed points of involution, this embedding $W^7\to B^{13}$ is a totally geodesic one.

Now let us compute the dimension of W^7 and find the generators of the tangent space of W^7 at the point E.

We use the notation of Heintze [3], who denoted by H_i $(1\le i\le 11)$ the set of orthonormal vectors that form a basis of the tangent space to $Sp(2)\times T^1$ and denoted by M_j $(1\le j\le 13)$ the basic orthonormal vectors of its orthogonal complement.

Let E_{kl} be the 5×5-matrix $(\delta_{ak}\delta_{bl})_{(1\le a,b\le 5)}$. Then introduce $Q_{kl} = E_{kl} - E_{lk}$, $R_{kl} = \sqrt{-1}(E_{kl}+E_{lk})$ and $P_k = \sqrt{-1}(E_{kk} - E_{55})$.

Heintze used the following basis :

$$M_j = \sqrt{2}Q_{j5}, \quad M_{j+4} = \sqrt{2}R_{j5}, \quad j = 1,2,3,4,$$

$$M_9 = Q_{12} - Q_{34}, \quad M_{10} = Q_{14} - Q_{23},$$

$$M_{11} = R_{12} + R_{34}, \quad M_{12} = R_{14} - R_{23}, \quad M_{13} = P_1 - P_2 + P_3 - P_4.$$

One can find by direct computation that the space V generated by these vectors splits into two pairwise orthogonal subspaces V^+ and V^- such that $\sigma|_{V^\pm} = \pm 1$.

The orthonormal bases of these subspaces are:
for V^+,

$$\frac{M_1+M_7}{\sqrt{2}},\ \frac{M_2-M_8}{\sqrt{2}},\ \frac{M_3-M_5}{\sqrt{2}},\ \frac{M_4-M_6}{\sqrt{2}},\ M_{11},\ M_{12},\ M_{13};$$

for V^-,

$$\frac{M_1-M_7}{\sqrt{2}},\ \frac{M_2-M_8}{\sqrt{2}},\ \frac{M_3+M_5}{\sqrt{2}},\ \frac{M_4+M_6}{\sqrt{2}},\ M_9,\ M_{10}.$$

Since $\dim V^+ = 7$ and the submanifold W^7 is homogeneous,

$$\dim W^7 = 7.$$

Let us show that this submanifold is isometric to the space $N_{1,1,-1/2}$.

We introduce another action given by

$$\rho\colon SU(5) \to SU(5), \quad A \to R\cdot A\cdot R^{-1},$$

where

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 & \sqrt{-1} & 0 \\ \sqrt{-1} & 0 & 1 & 0 & 0 \\ 0 & \sqrt{-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

Since $R \in Sp(2) \subset SU(5)$, this action is an isometry.

The action ρ generates an action on the Lie algebra $su(5)$, which we also denote by ρ. We compute the action of ρ only on V^+:

$$\begin{aligned} \rho(\frac{M_1+M_7}{\sqrt{2}}) &= M_7, \\ \rho(\frac{M_2+M_8}{\sqrt{2}}) &= M_8, \\ \rho(\frac{M_3-M_5}{\sqrt{2}}) &= M_3, \\ \rho(\frac{M_4-M_6}{\sqrt{2}}) &= M_4, \\ \rho(M_{11}) &= M_{11}, \\ \rho(M_{12}) &= M_9, \\ \rho(M_{13}) &= M_{13}. \end{aligned} \qquad (2)$$

Moreover, we have

$$\begin{aligned} M_9 &= -2Q_{34}+\rho(H_{10}), \\ M_{11} &= 2R_{34}+\rho(H_8), \\ M_{13} &= 2(P_3-P_4)+\rho(H_5), \end{aligned} \qquad (3)$$

where H_j are unit basic orthonormal vectors from the tangent space to $Sp(2)\times T^1$ (see [3]).

One can see that $(M_3, M_4, M_7, M_8, \sqrt{2}Q_{34}, \sqrt{2}R_{34}, \sqrt{2}(P_3-P_4))$ form an orthonormal (up to multiplication by a constant) basis in a subalgebra V_1 (see (1)) for $k=l=1$ and the group $SU(3)$ given by

$$\tilde{G} = \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix}, \quad A \in SU(3). \qquad (4)$$

But, since (2) and (3), we see that $(M_3, M_4, M_7, M_8, 2Q_{34}, 2R_{34}, 2(P_3-P_4))$ form an orthonormal basis in the tangent space of $\rho(W)$ at the point E. This coincides with (1) for $t=-1/2$.

Since W^7 is a homogeneous manifold and the homogeneous manifold $N_{1,1,-1/2}$ is simply connected, there exists a finite isometric covering

$$N_{1,1,-1/2} \to W^7.$$

Let us prove that this covering is a diffeomorphism.

From (2) and (3), one can see that W^7 is formed by $(Sp(2)\times T^1)$-orbits of elements g such that $\rho(g) \in \tilde{G}$ (see (4)). But, by direct computation, one can find that

i if $\rho(h) \in \tilde{G}$ and $h \in Sp(2)$ then $h = 1 \in Sp(2)$;

ii if

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2)$$

and $\rho(h)$ is a diagonal matrix then $a=d, b=-c, a^2+b^2 = I_2$, and $\rho(h)$ has the form $\mathrm{diag}(\lambda, \mu, \lambda^{-1}, \mu^{-1}, 1)$.

It follows now that the orbits of two elements $g_1, g_2 \in \tilde{G}$ coincide if and only if $g_1 g_2^{-1} \in T_{1,1} \subset SU(3)$, and we conclude that

$$W^7 = N_{1,1,-1/2}.$$

We are left to prove that the curvature of B^{13} attains its minimum and maximum values on 2-planes that are tangent to W^7.

Using the formula from Lemma 2 of [3], one computes the sectional curvature of the 2-plane generated by vectors X and Y :

$$K(X,Y) = \frac{29}{4}$$

for $X = (M_1 + M_7)/\sqrt{2}$ and $Y = (M_3 - M_5)/\sqrt{2}$, and it was proved in [3] that this value is the maximum curvature of the space B^{13}.

Let take the matrix $Q = \mathrm{diag}(1, -\sqrt{-1}, 1, \sqrt{-1}, 1) \in Sp(2) \subset SU(5)$. Since $Q \in Sp(2)$, the action ξ: $X \to Q \cdot X \cdot Q^{-1}$ generates an isometry of B^{13} (see [3]). Let us take

$$X = \sqrt{\frac{12}{37}}\left(\frac{M_1+M_7}{\sqrt{2}} + \frac{M_2+M_8}{\sqrt{2}}\right) + \sqrt{\frac{13}{37}}M_{11},$$
$$Y = -\sqrt{\frac{12}{37}}\left(\frac{M_3-M_5}{\sqrt{2}} - \frac{M_4-M_6}{\sqrt{2}}\right) - \sqrt{\frac{13}{37}}M_{12}.$$

One can compute

$$\xi^{-1}(X) = \sqrt{\frac{12}{37}}\left(\frac{M_1+M_7}{\sqrt{2}} + \frac{M_4+M_6}{\sqrt{2}}\right) + \sqrt{\frac{13}{37}}M_9,$$
$$\xi^{-1}(Y) = -\sqrt{\frac{12}{37}}\left(\frac{M_3-M_5}{\sqrt{2}} - \frac{M_2-M_8}{\sqrt{2}}\right) + \sqrt{\frac{13}{37}}M_{10},$$

apply to these vectors Lemma 2 of [3], and derive that

$$K(X,Y) = K(\xi^{-1}(X), \xi^{-1}(Y)) = \frac{4}{37}.$$

It was proved in [3] that this value is the minimum curvature of B^{13}.

We can conclude that the pinching constants $K_{\min}/K_{\max}$ for B^{13} and $W^7 = N_{1,1,-1/2}$ coincide and are equal to $16/(29 \times 37)$.

Theorem 1 is established. ■

Of course, this explanation can be simplified by replacing the involution σ by another one:

$$A \to \Sigma \cdot A \cdot \Sigma^{-1}, \quad \Sigma = R \cdot S \cdot R^{-1}.$$

Nevertheless, we have prefered to follow the proof as it was originally obtained.

By using analogous arguments one can prove the following theorem.

Theorem 2 *Let M_p^{13} be the Bazaikin space of the form $T_p^1 \backslash U(5)/(Sp(2) \times T_{(0)})$. Then the fixed-point set, of involution σ: $M_p^{13} \to M_p^{13}$, that contains the point $E = T_p^1 \cdot 1 \cdot Sp(2) \times T_{(0)}$, is diffeomorphic to the space*

$$W_p^7 = \begin{pmatrix} z^p & 0 & 0 \\ 0 & z^p & 0 \\ 0 & 0 & z \end{pmatrix} \backslash U(3) \Big/ \begin{pmatrix} \bar{w} & 0 & 0 \\ 0 & \bar{w} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad |z| = |w| = 1.$$

One can easy see that W_p^7 is a totally geodesic submanifold of M_p^{13}. The spaces M_p^{13} are nonhomogeneous for $p \geq 2$, and thus the problem of comparing pinchings of W_p^7 and M_p^{13} is not reduced to local computations, as was done in the proof of Theorem 1.

Remark These spaces W_p^7 were not presented directly in this form in [5, 7], and probably some of these examples were not known before. We note that they are also of the biquotient form that was introduced by Gromoll and Meyer [8]. Since these spaces are totally geodesic submanifolds of positively curved spaces, they have positive sectional curvature.

The following question is of interest.

Question *Does there exist a correspondence of 7-dimensional Aloff-Wallach and Eschenburg spaces to 13-dimensional Berger and Bazaikin spaces that is realized by totally geodesic embeddings ?*

If such correspondence exists only for some subfamilies, what are these subfamilies ?

If such correspondence exists is it realized by pinching-essential embeddings (i.e. embeddings with the same pinching constants of manifolds and submanifolds), as in the case of Theorem 1 ?

Let consider the topological properties of the manifolds W^7 and B^{13}.

Put

$$\hat{G}(\approx SU(2)) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \subset SU(3), \quad A \in SU(2).$$

One can see that $SU(3)/T_{1,1} = W^7$ and $SU(3)/\hat{G} = S^5$. The group $T_{1,1}$ acts on $S^5 = \{z_1^2 + z_2^2 + z_3^2 = 1 \mid z_i \in \mathbb{C}\}$ by multiplications:

$$(z_1, z_2, z_3) \to (\lambda^{-2}z_1, \lambda^{-2}z_2, \lambda^{-2}z_3)$$

where $\mathrm{diag}(\lambda, \lambda, \lambda^{-2}) \in T_{1,1}$. Moreover, the actions of $T_{1,1}$ and $\hat{G}$ on $SU(3)$ commute.

Let us consider the fiber bundle

$$SU(3) \to \mathbb{C}P^2. \qquad (5)$$

Its fiber is diffeomorphic to $U(2)$ which one can represent in the following manner. Put

$$\hat{Q} = SU(2) \times (\mathbb{R}/2\pi\mathbb{Z}).$$

We denote by $\hat{Q}_1$ the factor space of $\hat{Q}$ under the following $\mathbb{Z}_2$ action:

$$(X,t) \to (-X, t+\pi), \quad X \in SU(2).$$

This factor space is diffeomorphic to the fiber of the bundle (5). It is fibered over S^1 in the usual manner:

$$(X,t) \to t \in S^1 = \mathbb{R}/\pi\mathbb{Z}.$$

In these terms, the action of $T_{1,1}$ on the fiber bundle (5) has the form

$$(X,t) \to (\exp(\sqrt{-1}\pi\phi)\cdot X, t+\phi),$$

$$\mathrm{diag}(\exp(\sqrt{-1}\pi\phi), \exp(\sqrt{-1}\pi\phi), \exp(-2\sqrt{-1}\pi\phi)) \in T_{1,1}.$$

Now one can show that

i $S^5/T_{1,1} = SU(3)/(\hat{G}\times T_{1,1}) = \mathbb{C}P^2$;

ii these actions generate a fiber bundle

$$W^7 = SU(3)/T_{1,1} \overset{\mathbb{R}P^3}{\longrightarrow} \mathbb{C}P^2; \qquad (6)$$

iii it follows from computations of the cohomology groups of W^7 (see [5]) that the transgression d_4 in the spectral sequence of the fiber bundle (6) is given by

$$d_4\colon E_4^{0,3} = \mathbb{Z} \overset{\times 3}{\longrightarrow} E_4^{4,0} = \mathbb{Z}, \qquad (7)$$

and

$$H^4(W^7) = \mathbb{Z}_3. \qquad (8)$$

Put

$$\bar{G}\,(\approx SU(4)) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \subset SU(5), \quad A \in SU(4).$$

One can see that $SU(5)/(Sp(2)\times T^1) = B^{13}$ and $SU(5)/\bar{G} = S^9$. The group T^1 acts on $S^9 = \{z_1^2 + \dots + z_5^2 = 1 \mid z_i \in \mathbb{C}\}$ by multiplications :

$$(z_1, \dots, z_5) \to (\lambda^{-4} z_1, \dots, \lambda^{-4} z_5),$$

where $\mathrm{diag}(\lambda,\lambda,\lambda,\lambda,\lambda^{-4}) \in T^1$. Moreover, the actions of T^1 and $\bar{G}$ on $SU(5)$ commute.

Let us consider the fiber bundle

$$SU(5)/Sp(2) \to \mathbb{C}P^4. \qquad (9)$$

Put

$$\bar{Q} = SU(4)/Sp(2) \times (\mathbb{R}/\pi\mathbb{Z}).$$

We denote by $\bar{Q}_1$ the factor space of $\bar{Q}$ under the following $\mathbb{Z}_2$ action:

$$(X,t) \to \left(\sqrt{-1}X, t + \frac{1}{2}\pi\right), \quad X \in SU(4)/Sp(2).$$

This factor space is diffeomorphic to the fiber of the bundle (9). It is fibered over S^1 in the usual manner:

$$(X,t) \to t \in S^1 = \mathbb{R}/\frac{1}{2}\pi\mathbb{Z}.$$

In these terms, the action of T^1 on the fiber bundle (9) has the form

$$(X,t) \to (\exp(\sqrt{-1}\pi\phi)\cdot X, t+\phi),$$

$$\mathrm{diag}(\exp(\sqrt{-1}\pi\phi), \dots, \exp(\sqrt{-1}\pi\phi), \exp(-4\sqrt{-1}\pi\phi)) \in T^1.$$

Now one can show that

i $S^5/T^1 = SU(5)/(\bar{G}\times T^1) = \mathbb{C}P^4$;

ii these actions generate a fiber bundle

$$B^{13} \overset{\mathbb{R}P^5}{\longrightarrow} \mathbb{C}P^4; \qquad (10)$$

iii it follows from computations of the cohomology groups of B^{13} (see [6]) that the transgression d_6 in the spectral sequence of the fiber bundle (10) is given by

$$d_6\colon E_6^{0,5} = \mathbb{Z} \overset{\times 5}{\longrightarrow} E_6^{6,0} = \mathbb{Z}, \qquad (11)$$

and

$$H^6(B^{13}) = \mathbb{Z}_5. \qquad (12)$$

The similarity of the formulas (5)–(8) for W^7 and (9)–(12) for B^{13} and Theorem 1 give us a reason to pose the following question.

Question *Is it true that for every positive integer k there exist a space Γ_k such that*

i there exists a fiber bundle

$$\Gamma_k \overset{\mathbb{R}P^{2k+1}}{\longrightarrow} \mathbb{C}P^{2k}; \qquad (13)$$

ii the transgression d_{2k+2} in the spectral sequence of (13) is given by

$$d_{2k+2}\colon H^{2k+1}(S^{2k+1}) \overset{\times(2k+1)}{\longrightarrow} H^{2k+2}(\mathbb{C}P^{2k}), \qquad (14)$$

and

$$H^{2k+2}(\Gamma_k) = \mathbb{Z}_{2k+1}; \qquad (15)$$

iii the manifold Γ_k has positive sectional curvature;

iv $\Gamma_1 = W^7$ and $\Gamma_2 = B^{13}$?

One can pose more rigorous conjecture by adding

v the spaces Γ_k form a tower

$$\Gamma_1 \to \Gamma_2 \to \dots \to \Gamma_n \to \Gamma_{n+1} \to \dots$$

of pinching-essential totally geodesic embeddings, and thus the pinching constants of Γ_k are equal to $16/(29 \times 37)$.

If such a tower exists, one can expect that its properties are similar to the properties of $\mathbb{C}P^n$ or $\mathbb{H}P^n$ towers.

5 Final remarks

i Let consider the topological properties of the spaces W_p^7 and M_p^{13}.

Since a left multiplication transforms orbits under right action into orbits under left action, the group $\tilde{T}_p = \mathrm{diag}(z^p, z^p, z)$ acts on the space $U(3)/(SU(2) \times \mathrm{diag}(\bar{w}, \bar{w}, 1)) = S^5$ (where $|z| = |w| = 1$) and the group T_p^1 acts on the space $U(5)/(SU(4) \times T_{(0)}) = S^9$. These actions are not free.

One can see that the elements $\mathrm{diag}(z^p, z^p, z)$ for $z^p = 1$ have nontrivial fixed point sets and other elements of $\tilde{T}_p$ act freely. These fixed-point sets are the same for all pth roots of unity, and they are diffeomorphic to the 3-dimensional equator sphere in S^5. Let us consider the $\mathbb{Z}_p$ action on S^5 given by the subgroup of $\tilde{T}_p$ formed by elements with $z^p = 1$. The factor space of $U(3)/(SU(2) \times \mathrm{diag}(\bar{w}, \bar{w}, 1))$ under this $\mathbb{Z}_p$ action is diffeomorphic to S^5 (one can consider S^5 as the cyclic p-covering of S^5 ramified at the 3-dimensional equator sphere). The factor group $\tilde{T}_p/\mathbb{Z}_p$ acts freely on this factor space, and for every p we obtain the mapping

$$W_p^7 \longrightarrow \begin{pmatrix} z^p & 0 & 0 \\ 0 & z^p & 0 \\ 0 & 0 & z \end{pmatrix} \backslash U(3) \Big/ \left(SU(2) \times \begin{pmatrix} \bar{w} & 0 & 0 \\ 0 & \bar{w} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \mathbb{C}P^2.$$

It is almost evident that this mapping is an $\mathbb{R}P^3$ bundle, but the full proof needs additional work.

Recently Ya.V. Bazaikin, answering our question, computed that the order of $H^4(W_p^7)$ is $r_p = 4p - 1$.

The 13-dimensional case is completely analogous to the 7-dimensional case, and for every p we obtain the mapping

$$M_p^{13} \longrightarrow T_p^1 \backslash U(5)/(SU(4) \times T_{(0)}) = \mathbb{C}P^4.$$

This mapping also ought to be an $\mathbb{R}P^5$ bundle.

It was computed in [6] that the order of $H^6(M_p^{13})$ is $s_p = (8p^2 - 4p + 1)$.

One can note that the very interesting formula

$$s_p = \frac{1}{2}(r_p^2 + 1)$$

holds. For $p = 1$, it coincides with (8) and (12).

One can also ask how to extend these embeddings $W_p^7 \to M_p^{13}$ to towers.

ii When we discussed the results of this paper with K. Grove, he asked about the existence of such embeddings for the even-dimensional Wallach spaces [9]. The existence of topological embeddings of these flag spaces is evident. We can give the simplest example of an involution

$$A \to \hat{S} \cdot A \cdot \hat{S}^{-1},$$

$$A = \begin{pmatrix} \sqrt{-1}I_3 & 0 \\ 0 & -\sqrt{-1}I_3 \end{pmatrix}.$$

This involution generates an involution on the space $Sp(3)/(Sp(1) \times Sp(1) \times Sp(1))$. The component of the fixed-point set that contains the orbit of the unit is diffeomorphic to the space $SU(3)/T^2$, where T^2 is the maximal torus. Thus we obtain the totally geodesic embedding $SU(3)/T^2 \to Sp(3)/(Sp(1) \times Sp(1) \times Sp(1))$.

We call that it was proved in [10] that homogeneous metrics on the even-dimensional Wallach spaces have the same maximal pinching, which is equal to 1/64. Thus one can expect that this embedding is pinching-essential and that such an embedding also exists for the other pair of Wallach spaces $(F_4/Spin(8), Sp(3)/(Sp(1) \times Sp(1) \times Sp(1)))$.

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