Proceedings of the Steklov Institute of Mathematics, Vol. 225, 1999, pp. 322–343.
Translated from Trudy Matematicheskogo Instituta imeni V.A. Steklova, Vol. 225, 1999, pp. 339–361.

The Weierstrass Representation of Spheres in $\mathbb{R}^3$, the Willmore Numbers, and Soliton Spheres

I. A. Taimanov[1]

Accepted December 1998

1. INTRODUCTION

In the present article we consider the Weierstrass representations of spheres in $\mathbb{R}^3$. The existence of a global Weierstrass representation for any compact oriented surface of genus $g \geq 1$ has been established in [17, 18] and this proof, in fact, works for spheres also. Being mostly interested in the relations of these representations to the spectral theory and in possibilities of applying the spectral theory to differential geometry [15, 16], we preferred to consider the sphere case separately because in this case the spectral theory of Dirac operators

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \tag{1}$$

is more developed.

In Section 2 we consider two different representations of the sphere: as a plane completed by adding a point at infinity (a plane representation) and as a cylinder completed by adding a couple of "infinities" (a cylindric representation). For both of them we describe the data of Weierstrass representations which are the potential of a representation and a "wave function" ψ satisfying the equation

$$\mathcal{D}\psi = 0 \tag{2}$$

and some analytic conditions. The data of spectral theory origin are in one-to-one correspondence with the immersed spheres in $\mathbb{R}^3$ (Theorems 1–4). This gives a straightforward procedure for constructing immersions in terms of zero-eigenfunctions of Dirac operators on a plane and on an infinite two-dimensional cylinder.

In Section 3 we consider spheres with one-dimensional potentials, which means that in some cylindric representation the potential U of $\mathcal{D}$ depends on one variable. We prove that a sphere of revolution is uniquely reconstructed from the potential only (Theorem 5), describe all spheres with one-dimensional potentials in terms of the Jost functions (Theorem 6), and prove that

$$\mathcal{W}(\Sigma) \geq 4\pi \left(\dim_{\mathbb{H}} \operatorname{Ker} \mathcal{D}\right)^2, \tag{3}$$

where $\mathcal{W}$ is the Willmore functional and $\mathcal{D}$ is a Dirac operator acting on a spinor bundle over sphere Σ (Theorem 7). We conjecture that this estimate is valid for all spheres.

[1]Institute of Mathematics, 630090 Novosibirsk, Russia. E-mail: taimanov@math.nsc.ru

In Section 4 we consider a special class of spheres with one-dimensional potentials—spheres with soliton (or reflectionless) potentials. This integrable case gives many interesting examples, and, in particular, for each N an equality in (3) is achieved exactly at special soliton spheres, the Dirac spheres [14].

In Appendix we give criterion distinguishing immersions, of universal coverings of compact surfaces of higher genera, converted into immersions of compact surfaces.

2. THE WEIERSTRASS REPRESENTATION OF SPHERES

2.1. The local Weierstrass representation. First, recall the local Weierstrass representation. It is based on the following two facts:

Lemma A (Eisenhart [8]; see also comments in [17]). *Let W be a simply connected domain in $\mathbb{C}$, $z_0 \in W$, and let a vector function $\psi = (\psi_1, \psi_2)\colon W \to \mathbb{C}^2$ satisfy* (2) *where $\mathcal{D}$ is of the form* (1) *and its potential $U(z, \bar z)$ is real-valued. Then the following formulas*

$$X^1(z,\bar z) = \frac{i}{2}\int_{z_0}^{z} \left((\bar\psi_2^2 + \psi_1^2)dz' - (\bar\psi_1^2 + \psi_2^2)d\bar z'\right),$$
$$X^2(z,\bar z) = \frac{1}{2}\int_{z_0}^{z} \left((\bar\psi_2^2 - \psi_1^2)dz' - (\bar\psi_1^2 - \psi_2^2)d\bar z'\right), \tag{4}$$
$$X^3(z,\bar z) = \int_{z_0}^{z} (\psi_1\bar\psi_2 dz' + \bar\psi_1\psi_2 d\bar z')$$

define an immersion of W into $\mathbb{R}^3$ with the induced metric $D(z,\bar z)^2 dz d\bar z$ of the form

$$D(z,\bar z) = |\psi_1(z,\bar z)|^2 + |\psi_2(z,\bar z)|^2$$

and the Gauss curvature and the mean curvature are

$$K(z,\bar z) = -\frac{4}{D^2(z,\bar z)}\partial\bar\partial \ln D(z,\bar z) \qquad \text{and} \qquad H(z,\bar z) = 2\frac{U(z,\bar z)}{D(z,\bar z)}.$$

Lemma B [15]. *Let W be a domain in $\mathbb{C}$ and let $X\colon W \to \mathbb{R}^3$ be a conformal immersion of W into $\mathbb{R}^3$: $z \to X(z,\bar z) = (X^1(z,\bar z), X^2(z,\bar z), X^3(z,\bar z))$. Assume that $\partial X^3/\partial z \neq 0$ near $z_0 \in W$. Then near z_0 the functions*

$$\psi_1(z,\bar z) = \sqrt{-\partial\Phi(z,\bar z)}, \qquad \psi_2(z,\bar z) = \sqrt{\bar\partial\Phi(z,\bar z)}, \tag{5}$$

with

$$\Phi(z,\bar z) = X^2(z,\bar z) + iX^1(z,\bar z),$$

satisfy (2) *with $U(z,\bar z) = H(z,\bar z)D(z,\bar z)/2$, where H is the mean curvature and $D^2 dz d\bar z$ is the metric of the surface $X(W) \subset \mathbb{R}^3$.*

A globalization of this representation requires introducing spinor bundles, generated by ψ, over closed oriented surfaces and considering the operator (1) as acting on them. This had been shown in [15] and the existence of a global Weierstrass representation has been proved for any C^3-regular compact oriented surface of genus $g \geq 1$ (see Theorem 2 in [18]). The proof follows by continuing the sections (5) over the whole spinor bundle and uses the following lemma whose proof is contained in the proof of Theorem 2 from [18]:

Lemma 1. *Let $z_0 \in W$ be a nondegenerate critical point of the function X^3 defined on a domain $W \subset \mathbb{C}$ conformally immersed into $\mathbb{R}^3$. Then near z_0 the branches of (5) are correctly defined as one-valued functions and do not ramify at z_0.*

It is clear that if

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}: \ W \to \mathbb{C}$$

satisfies (2), then

$$\psi^* = \begin{pmatrix} \bar\psi_2 \\ -\bar\psi_1 \end{pmatrix} \tag{6}$$

also satisfies (2). Hence for any $\lambda, \mu \in \mathbb{C}$, such that $|\lambda|^2 + |\mu|^2 \neq 0$, the vector function

$$\Psi_{\lambda,\mu} = \lambda\psi + \mu\psi^* = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \mu \begin{pmatrix} \bar\psi_2 \\ -\bar\psi_1 \end{pmatrix}: \ W \to \mathbb{C}$$

satisfies (2). Consider all immersions

$$X_{\lambda,\mu}: W \to \mathbb{R}^3$$

given via (4) by $\Psi_{\lambda,\mu}$ and normalize them by the condition

$$X_{\lambda,\mu}(z_0) = 0,$$

where a point z_0 is fixed. Denote $\partial X^j_{1,0}/\partial z$ by N^j. It is shown by straightforward computations that

$$\begin{aligned}
\frac{\partial X^1_{\lambda,\mu}}{\partial z} &= \frac{(\lambda^2 + \bar\lambda^2 + \mu^2 + \bar\mu^2)}{2} N^1 + \frac{i(-\lambda^2 + \bar\lambda^2 + \mu^2 - \bar\mu^2)}{2} N^2 + i(\bar\lambda\bar\mu - \lambda\mu)N^3, \\
\frac{\partial X^2_{\lambda,\mu}}{\partial z} &= \frac{i(\lambda^2 - \bar\lambda^2 + \mu^2 - \bar\mu^2)}{2} N^1 + \frac{(\lambda^2 + \bar\lambda^2 - \mu^2 - \bar\mu^2)}{2} N^2 + (\bar\lambda\bar\mu + \lambda\mu)N^3, \\
\frac{\partial X^3_{\lambda,\mu}}{\partial z} &= i(\mu\bar\lambda - \lambda\bar\mu)N^1 + (-\lambda\bar\mu - \mu\bar\lambda)N^2 + (\lambda\bar\lambda - \mu\bar\mu)N^3.
\end{aligned} \tag{7}$$

From (7) we derive

Lemma 2. (1) *The transformation $\Psi_{1,0} \to \Psi_{r,0}$, with $r \in \mathbb{R}$, generates the homothety*

$$(X^1, X^2, X^3) \to (r^2X^1, r^2X^2, r^2X^3)$$

of the immersed surface.

(2) *For $|\lambda|^2 + |\mu|^2 = 1$ formulas (7) define the isomorphism*

$$\rho: \{|\lambda|^2 + |\mu|^2 = 1, \ \lambda, \mu \in \mathbb{C}\}/\{\pm 1\} \to SO(3)$$

and the immersion $X_{\lambda,\mu}$ is a transformation of $X_{1,0}$ by the rotation $\rho(\lambda, \mu)$.

For instance,

$$\rho(\cos\varphi,\sin\varphi)=\begin{pmatrix}1&0&0\\0&\cos 2\varphi&\sin 2\varphi\\0&-\sin 2\varphi&\cos 2\varphi\end{pmatrix},$$

$$\rho(e^{i\theta},0)=\begin{pmatrix}\cos 2\theta&\sin 2\theta&0\\-\sin 2\theta&\cos 2\theta&0\\0&0&1\end{pmatrix},$$

$$\rho(\cos\tau,i\sin\tau)=\begin{pmatrix}\cos 2\tau&0&\sin 2\tau\\0&1&0\\-\sin 2\tau&0&\cos 2\tau\end{pmatrix}.$$

2.2. A plane representation. For constructing a global Weierstrass representation of a two-sphere immersed into $\mathbb{R}^3$ we may consider a sphere as a plane completed by a point at infinity or as an infinite cylinder completed by two infinities. We analyze both possibilities and start with a *plane representation.*

Let Σ be a 2-sphere immersed into $\mathbb{R}^3$.

Fix a pair of points $\infty_\pm$ on Σ and define a pair of charts with conformal parameters z, on $\mathbb{C}\approx\Sigma\setminus\infty_+$, and

$$u=-\frac{1}{z},$$

on $\mathbb{C}\approx\Sigma\setminus\infty_-$, such that $z(\infty_-)=0$ and $u(\infty_+)=0$. We have

$$du=\frac{dz}{z^2},\qquad \frac{\partial}{\partial u}=z^2\frac{\partial}{\partial z}.$$

Introduce also the following functions

$$\tilde\psi_1(u,\bar u)=\sqrt{-\partial_u(X^2+iX^1)},\qquad \tilde\psi_2(u,\bar u)=\sqrt{\partial_{\bar u}(X^2+iX^1)}.$$

Now we arrive at the definition.

Definition 1. *A sphere Σ, immersed into $\mathbb{R}^3$, possesses a global (plane) Weierstrass representation if there exist real potentials $U(z,\bar z)$ and $\tilde U(u,\bar u)$ and the vector functions $\psi(z,\bar z)$ and $\tilde\psi(u,\bar u)$ defined on a covering of Σ by a pair of charts with parameters z and $u=z^{-1}$ such that*

(1)

$$\begin{cases}\tilde U(u,\bar u)=|z|^2U(z,\bar z),\\ \tilde\psi_1(u,\bar u)=z\psi_1(z,\bar z),\\ \tilde\psi_2(u,\bar u)=\bar z\psi_2(z,\bar z)\end{cases}\tag{8}$$

for $z=-1/u\in\mathbb{C}$;

(2) *the vector functions ψ and $\tilde\psi$ satisfy* (2) *for the Dirac operators with corresponding potentials U and $\tilde U$ and for a suitable choice of coordinates in $\mathbb{R}^3$ define by* (4) *an immersion of Σ.*

Consider the analytic conditions met by the Dirac operator and the spinor sections ψ (8) corresponding to an immersion of a two-sphere into $\mathbb{R}^3$.

First, notice that $\tilde D^2(u,\bar u)du\,d\bar u=D^2(z,\bar z)|z|^4du\,d\bar u$ near ∞_+, which implies that

$$D(z,\bar z)=\frac{C}{|z|^2}+O\left(\frac{1}{|z|^3}\right)\qquad\text{with } C=\text{const}\neq 0,$$

and

$$U(z,\bar{z}) = \frac{U_+}{|z|^2} + O\left(\frac{1}{|z|^3}\right) \qquad \text{with } U_+ = \text{const} \tag{9}$$

as $z \to \infty$. Therefore we conclude

$$|\psi(z,\bar{z})|^2 = |\psi_1(z,\bar{z})|^2 + |\psi_2(z,\bar{z})|^2 = O\left(\frac{1}{|z|^2}\right). \tag{10}$$

In fact, taking into account that the point $\infty_+ \in \Sigma$ is regular, the last equality is refined as follows:

$$|\psi(z,\bar{z})|^2 = \frac{C_+}{|z|^2} + O\left(\frac{1}{|z|^3}\right), \qquad \text{with } C_+ \neq 0, \text{ as } z \to \infty. \tag{11}$$

Assume that Σ is C^3-regularly immersed into $\mathbb{R}^3$. By using the general position argument, we choose coordinates in $\mathbb{R}^3$ such that all critical points of the function X^3 defined on Σ are nondegenerate. Take a point $\infty_+ \in \Sigma$ and introduce a conformal parameter z on $\mathbb{C} \approx \Sigma \setminus \infty_+$. By Lemma 1, the branches of (5) do not ramify anywhere and are correctly defined on $\mathbb{C}$. Now Lemma B and preceding conversations imply

Theorem 1. *Every C^3-regular two-sphere Σ immersed into $\mathbb{R}^3$ possesses a global ("plane") Weierstrass representation and the functions $U(z,\bar{z})$ and $\psi(z,\bar{z})$ defined on $\Sigma \setminus \infty_+$ satisfy* (9) *and* (11).

In fact, conditions (2), (9), and (11) distinguish the data of Weierstrass representations of spheres.

Theorem 2. *Let $U(z,\bar{z})$ be a continuous function and satisfy* (9) *and let ψ be a solution to* (2) *such that $(|\psi_1|^2 + |\psi_2|^2)$ vanishes nowhere on $\mathbb{C}$ and* (11) *holds. Then ψ defines via* (4) *an immersion of $\mathbb{C}$ completed to a C^2-regular immersion of S^2 into $\mathbb{R}^3$. Moreover U and ψ and the functions $\tilde{U}$ and $\tilde{\psi}$ constructed from them by* (8) *form the data of a plane Weierstrass representation of the immersed sphere.*

Proof. By definition ψ defines an immersion of $X\colon \mathbb{C} \to \mathbb{R}^3$ up to translations. Normalize an immersion by $X(i) = 0 \in \mathbb{R}^3$.

Construct the functions $\tilde{U}$ and $\tilde{\psi}$ from U and ψ by (8) and notice that they satisfy the following equation

$$\left[\begin{pmatrix} 0 & \partial_u \\ -\partial_{\bar{u}} & 0 \end{pmatrix} + \begin{pmatrix} \tilde{U} & 0 \\ 0 & \tilde{U} \end{pmatrix}\right]\begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = 0.$$

It follows from (8) and (9) that $\tilde{U}$ is a real-valued continuous function on the whole complex plane parametrized by $u \in \mathbb{C}$. Hence $\tilde{\psi}$ also defines an immersion of $\tilde{X}\colon \mathbb{C} \to \mathbb{R}^3$ up to translations. Normalize an immersion by $\tilde{X}(i) = 0 \in \mathbb{R}^3$.

Since $u = z^{-1}$, we have

$$(\bar{\psi}_2^2 + \psi_1^2)dz - (\bar{\psi}_1^2 + \psi_2^2)d\bar{z} = (\bar{\tilde{\psi}}_2^2 + \tilde{\psi}_1^2)du - (\bar{\tilde{\psi}}_1^2 + \tilde{\psi}_2^2)d\bar{u},$$
$$(\bar{\psi}_2^2 - \psi_1^2)dz - (\bar{\psi}_1^2 - \psi_2^2)d\bar{z} = (\bar{\tilde{\psi}}_2^2 - \tilde{\psi}_1^2)du - (\bar{\tilde{\psi}}_1^2 - \tilde{\psi}_2^2)d\bar{u},$$
$$\psi_1\bar{\psi}_2 dz + \bar{\psi}_1\psi_2 d\bar{z} = \tilde{\psi}_1\bar{\tilde{\psi}}_2 du + \bar{\tilde{\psi}}_1\tilde{\psi}_2 d\bar{u}$$

on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. These formulas imply that

$$X(z) = \tilde{X}\left(-\frac{1}{z}\right) \qquad \text{for } z \in \mathbb{C}^*.$$

Hence, X and $\tilde{X}$ coincide on $\mathbb{C}^*$ and each of them is regularly continued onto the corresponding "infinity point," on $u = 0$ and $z = 0$. By (11), $\tilde{X}$ is regular at $u = 0$.

This proves the theorem.

A nice feature of this theorem is that the closedness problem consisting in distinguishing immersions of planes which are converted into immersions of compact surfaces reduces for spheres to the conditions (9) and (11), which can be easily checked. For surfaces of higher genera this problem is more complicated (see Appendix).

If we have a solution ψ to (2) such that $|\psi|$ decays slower than $|z|^{-1}$ as $z \to \infty$ then nevertheless we may construct by (4) an immersion of a sphere into $\mathbb{R}^3$ with a peak singularity at the "infinity." If $|\psi|$ decays faster than $|z|^{-1}$ than we have a branch point at "infinity." This also occurs when $|\psi(z, \bar{z})| = 0$ at $z \in \mathbb{C}$.

Let us now admit branch points and consider more general situation.

Denote by $\mathcal{E}$ a $\mathbb{C}^2$-bundle

$$\mathbb{C}^2 \to \mathcal{E} \to S^2$$

whose sections ψ satisfy (8).

Theorem 3. *Let U satisfy* (8) *and* (9). *Then*

(1) *$\mathcal{D}$ is defined on sections of $\mathcal{E}$;*

(2) *solutions to* (2) *satisfying* (10) *are in one-to-one correspondence with zero-eigenfunctions of $\mathcal{D}$;*

(3) *the kernel of $\mathcal{D}$ is finite-dimensional and moreover it is even-dimensional.*

The first and second statements are evident. Since $\mathcal{D}$ is elliptic, its kernel is finite-dimensional. We know that there exists an automorphism $*$, of the kernel, given by (6). Since $(\psi^*)^* = -\psi$, the kernel splits into two-dimensional subspaces invariant under $*$ and therefore $\dim_{\mathbb{C}} \operatorname{Ker} \mathcal{D} = 2n$ with n integer. Each section $\psi \in \operatorname{Ker} \mathcal{D}$ generates via (4) an immersed sphere which may have branch points.[2]

2.3. A cylindric representation. Here we introduce a representation of an immersed sphere as an immersed cylinder completed by two points.

We set $z' = \ln z = x' + iy'$ and take a cylinder $\mathcal{Z} = \mathbb{C}/i\mathbb{Z}$ with a conformal parameter z' defined modulo $2\pi i$.

Let (U, ψ) be the data of a plane representation. We have

$$dz' = \frac{1}{z} dz, \qquad \frac{\partial}{\partial z'} = z \frac{\partial}{\partial z}.$$

Consider the following functions on $\mathcal{Z}$:

$$\hat{D}(z', \bar{z}') = |z| D(z, \bar{z}), \qquad \hat{U}(z', \bar{z}') = |z| U(z, \bar{z}) \tag{12}$$

and

$$\hat{\psi}_1(z', \bar{z}') = \sqrt{z} \psi_1(z, \bar{z}), \qquad \hat{\psi}_2(z', \bar{z}') = \sqrt{\bar{z}} \psi_2(z, \bar{z}). \tag{13}$$

By straightforward computations we obtain

[2] It was proposed in [9] to treat $\mathcal{E}$ as a quaternion vector bundle and to treat $\operatorname{Ker} \mathcal{D}$ as a quaternion vector space identifying $*$ with a multiplication by $\mathbf{j} \in \mathbb{H}$. In this event $\dim_{\mathbb{H}} \operatorname{Ker} \mathcal{D} = n$.

Lemma 3. (1) *The functions* $\hat{\psi}$ *satisfy the equation*

$$\left[\begin{pmatrix} 0 & \partial_{z'} \\ -\partial_{\bar{z}'} & 0 \end{pmatrix} + \begin{pmatrix} \hat{U} & 0 \\ 0 & \hat{U} \end{pmatrix}\right]\begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{pmatrix} = 0. \tag{14}$$

(2) *There are the asymptotics*

$$\hat{U} = \frac{U_{\pm}}{e^{|x'|}} + O\left(\frac{1}{e^{2|x'|}}\right), \qquad |\hat{\psi}_1|^2 + |\hat{\psi}_2|^2 = \frac{C_{\pm}}{e^{|x'|}} + O\left(\frac{1}{e^{2|x'|}}\right), \tag{15}$$

with $U_{\pm}$ *and* $C_{\pm}$ *constants such that* $C_{\pm} \neq 0$, *as* $x' \to \pm\infty$.

Now it is clear how to derive from Theorems 1 and 2 the following result:

Theorem 4. (1) *Any vector function* $\hat{\psi}$ *satisfying* (14) *and* (15) *defines via* (4) *an immersion of* $\mathcal{Z}$ *into* $\mathbb{R}^3$, *which is completed to a regular immersion of a two-sphere.*

(2) *For any two-sphere* Σ *which is* C^3*-immersed into* $\mathbb{R}^3$ *and any pair of distinct points* $\infty_{\pm} \in \Sigma$ *there exists an immersion of a cylinder* $\mathcal{Z} = \Sigma \setminus \{\infty_{\pm}\}$ *such that*

(a) *this immersion is defined, for a suitable choice of coordinates in* $\mathbb{R}^3$, *via* (4) *by functions* $\hat{\psi}$ *and* $\hat{U}$ *satisfying* (14) *and* (15);

(b) *by adding a point to each end of the cylinder this immersion is completed to a regular immersion of* Σ.

Theorems 1 and 4 imply the following:

Corollary 1. *Every* C^3*-regular two-sphere* Σ *immersed into* $\mathbb{R}^3$ *possesses a cylindric Weierstrass representation.*

We mention above that $\psi(z, \bar{z})$ are sections of $\mathcal{E}$. Formulas (12) and (13) show that vector functions $\hat{\psi}$ on $\mathcal{Z}$ meeting conditions (15) and

$$\hat{\psi}(x', y' + 2\pi) = -\hat{\psi}(x', y') \tag{16}$$

are sections of $\mathcal{E}$ and these formulas just establish an equivalence between two different representations of $\mathcal{E}$. Moreover, these formulas also establish the equivalence between Dirac operators and, therefore, we have

Corollary 2. *Solutions* $\hat{\psi}$ *to* (14) *satisfying* (16) *and*

$$|\hat{\psi}|^2 = O\left(\frac{1}{e^{|x'|}}\right) \qquad as\ \ x' \to \pm\infty$$

form the kernel of $\mathcal{D}$. *The dimension of* $\operatorname{Ker}\mathcal{D}$ *is finite and even.*

We also mention that each section from this kernel generates, via (4), an immersed sphere which may have branch points.

3. SPHERES WITH ONE-DIMENSIONAL POTENTIALS

3.1. The spectral data for one-dimensional potentials. In this section we consider spheres admitting cylinder representations with one-dimensional real-valued potentials $U(x)$. This means that being defined on an infinite cylinder $\mathcal{Z} = \{(x, y): -\infty < x < \infty,\ 0 \le y \le 2\pi\}$ the potential $U(z)$ depends on x only. We assume that $U(x)$ decays exponentially:

$$U(x) = O\left(\frac{1}{e^{|x|}}\right) \qquad \text{as } x \to \pm\infty.$$

For $U = 0$ solutions to (2) are linear combinations of

$$\begin{pmatrix} 0 \\ \exp(ik\bar{z}) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \exp(ilz) \\ 0 \end{pmatrix},$$

with $z = x + iy$ and $k, l \in \mathbb{C}$. These functions are defined on $\mathcal{Z}$ if and only if $k, l \in 2\pi\mathbb{Z}$. Hence we look for solutions to (2) of the form

$$\psi(x, y) = \exp(ky)\varphi(x, y), \qquad \varphi(x, y + 2\pi) = \varphi(x, y), \tag{17}$$

i.e., we consider solutions which are defined on the universal covering $\mathbb{R}^2$ of $\mathcal{Z}$ and satisfy the periodicity condition

$$\psi(x, y + 2\pi) = \mu\psi(x, y),$$

with μ a constant. For this ansatz equation (2) reduces to

$$\left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & ik \\ ik & 0 \end{pmatrix}\right]\varphi = 0 \tag{18}$$

and, decomposing its solutions into Fourier series in y

$$\varphi(x, y) = \sum_{m \in \mathbb{Z}} \varphi_m(x) e^{imy},$$

we conclude that each $\varphi_m(x)$ satisfies (18) with $k + im$ substituted for k. Hence for studying all solutions to (2) of the form (17) it is enough to study solutions to

$$\left[\begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} + \begin{pmatrix} 2U & 0 \\ 0 & 2U \end{pmatrix} - \begin{pmatrix} 0 & ik \\ ik & 0 \end{pmatrix}\right]\varphi = 0, \tag{19}$$

depending on x only.

This problem called the Zakharov–Shabat problem was studied in its relation to soliton equations (see, for instance, [1, 5, 11, 19]) and we give the brief summary of results which we need in the sequel.

Since $U(x)$ decays exponentially as $|x| \to \infty$, we have

Summary. *The potential $U(x)$ of an operator*

$$L = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} + \begin{pmatrix} 2U & 0 \\ 0 & 2U \end{pmatrix}$$

is uniquely reconstructed from the spectral data which are

(i) *the reflection coefficient* $R(k) = b(k)/a(k)$, *with* $k \in \mathbb{R} \setminus \{0\}$,

(ii) *the poles* $\varkappa_1, \dots, \varkappa_N$ *of the transmission coefficient* $T(k)$ *with* $\operatorname{Im} k > 0$,

(iii) *some additional quantities* $\lambda_1, \dots, \lambda_N \in \mathbb{C}$, *attached to* $\varkappa_j$.

The poles of $T(k)$ *are simple and correspond to exponentially decaying solutions to* (19). *For each pole* $\varkappa_j$ *every such solution is a multiple of* $\varphi_1^+(x, \varkappa_j)$ *which is a unique solution to the equation*

$$\varphi(x) = \begin{pmatrix} 0 \\ e^{i\varkappa_j x} \end{pmatrix} + \int_x^{+\infty} \begin{pmatrix} 0 & -e^{-i\varkappa_j(x-x')} \\ e^{i\varkappa_j(x-x')} & 0 \end{pmatrix} \cdot 2U(x') \cdot \varphi(x')\, dx'.$$

Since $U(x)$ *is real-valued,*

(a) $\varkappa_j$ *are symmetric with respect to the imaginary axis; if* $\varkappa_j$ *and* $\varkappa_l = -\overline{\varkappa_j}$ *are different poles of* $T(k)$ *then* $\lambda_j = \bar{\lambda}_l$, *and if* $\operatorname{Re} \varkappa_m = 0$ *then* $\lambda_m \in \mathbb{R}$;

(b) $R(k) = \overline{R(-k)}$.

The function $\varphi_1^+(x, k)$ *is represented in the form*

$$\varphi_1^+(x, k) = \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix} + \int_x^{+\infty} dx' \begin{pmatrix} B_1(x, x') \\ B_2(x, x') \end{pmatrix} e^{ikx'}. \tag{20}$$

Introduce the function

$$\Omega(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R(k) e^{-ikz}\, dk - \sum_{j=1}^{N} \lambda_j e^{i\varkappa_j z},$$

which is uniquely constructed from the spectral data. Then the kernel $B_1(x, y)$ *and* $B(x, y)$ *from* (20) *satisfy the Gelfand–Levitan–Marchenko* (*GLM*) *equations* (*for the Zakharov–Shabat problem*)

$$\begin{cases} B_2(x, y) + \displaystyle\int_x^{+\infty} B_1(x, x')\Omega(x' + y)\, dx' = 0, \\ \Omega(x + y) - B_1(x, y) + \displaystyle\int_x^{+\infty} B_2(x, x')\Omega(x' + y)\, dx' = 0, \end{cases} \tag{21}$$

where $y > x$. *These equations are of the Volterra type and uniquely solvable in the domain* $y > x$. *Moreover,* $\lim_{y \to +\infty} B_1(x, y) = \lim_{y \to +\infty} B_2(x, y) = 0$ *and the limits of* $B_j(x, y)$ *as* $y \to x$ *are defined and we denote them by* $B_j(x, x)$. *The reconstruction of* $U(x)$ *is given by the formula*

$$U(x) = -B_1(x, x). \tag{22}$$

Moreover

$$\frac{dB_2(x, x)}{dx} = 2U^2(x). \tag{23}$$

We recall the definition of the Kruskal integrals:

$$I_n(U) = \int_{-\infty}^{+\infty} U(x) q_n(x)\, dx,$$

where

$$q_1(x) = U(x)$$

and the other quantities $q_j(x)$ are defined by the recursion relation

$$q_{j+1}(x) = -i\frac{dq_j(x)}{dx} - 4U(x)\sum_{m=1}^{j-1} q_m(x)q_{j-m}(x).$$

The first of them is obviously the squared L_2-norm of $U(x)$:

$$I_1(U) = \int_{-\infty}^{+\infty} U^2(x)\,dx.$$

These quantities are related to the spectral data via the trace formulas (see [5, formulas (7.20) and (7.21) in Chapter 1 of Part I][3]):

$$I_n(U) = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} \ln(1-|b(k)|^2)(-2k)^{n-1}\,dk + \frac{i2^{n-2}}{n}\sum_{j=1}^{N}(\bar{\varkappa}_j^n - \varkappa_j^n), \tag{24}$$

where $b(k)$ is the ratio of $R(k)$ and $T(k)$ and satisfies the inequality

$$0 \le |b(k)| < 1.$$

For $n = 1$ we have

$$\int_{-\infty}^{+\infty} U^2(x)\,dx = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} \ln(1-|b(k)|^2)\,dk + \sum_{j=1}^{N}\operatorname{Im}\varkappa_j \tag{25}$$

and we conclude that

$$\int_{-\infty}^{+\infty} U^2(x)\,dx \ge \sum_{j=1}^{N}\operatorname{Im}\varkappa_j \tag{26}$$

and an equality in (26) is achieved exactly at *reflectionless* potentials, i.e., $b(k) \equiv 0$, which is equivalent to $R(k) \equiv 0$.

3.2. Construction of spheres with one-dimensional potentials. A reconstruction of a sphere of revolution from its potential. By Corollary 1, every sphere Σ regularly immersed into $\mathbb{R}^3$ possesses a cylindric Weierstrass representation. In this subsection we describe spheres which admits cylindric representations with potentials depending on x only. The simplest and most important examples are spheres of revolution [16].

Let $U(x)$ be a potential of an immersed sphere Σ. By Lemma 3, it decays exponentially and we may apply the spectral theory of L exposed in 3.1. In particular, all exponentially decaying solutions to (18) are linear combinations of $\varphi_1^+(x,\varkappa_j)$ and their $*$-transforms (6).

For any sphere of revolution Σ there exists a cylindric representation with a one-dimensional potential, and Σ is immersed into $\mathbb{R}^3$ via (4) where ψ has the form

$$\psi(x,y) = \varphi(x)e^{iy/2}$$

[3]In [5] these formulas are written in terms of $\varkappa$, $\psi(x)$, and λ which, in our notation, are (-4), $-iU(x)$, and $-2k$, respectively.

(see [16]). The function ψ takes the form

$$\psi_{\lambda,\mu}(x,y) = \lambda\left(\varphi_1^+(x,i/2)e^{iy/2}\right) + \mu\left(\varphi_1^+(x,i/2)e^{iy/2}\right)^*,$$

where $\lambda,\mu \in \mathbb{C}$. By Lemma 2, for different λ and μ such immersions are transformed one into another by homotheties of spheres and rigid motions in $\mathbb{R}^3$. Since the potentials of different Weierstrass representations are reconstructed one from another by the formulas (8) and (12), we conclude that the following theorem is valid.

Theorem 5. *Any sphere of revolution without branch points is uniquely defined (up to homotheties of the sphere and rigid motions in $\mathbb{R}^3$) by the potential of any of its Weierstrass representations.*

The condition on absence of branch points is added just for the following reason. Notice that we may consider the linear combination

$$\psi_{\lambda,\mu,\varkappa}(x,y) = \lambda\left(\varphi_1^+(x,\varkappa)e^{\varkappa y}\right) + \mu\left(\varphi_1^+(x,\varkappa)e^{\varkappa y}\right)^*$$

with $\varkappa = in/2$ and $n > 1$. Then the sphere constructed from $\psi_{\lambda,\mu,\varkappa}$ via (4) would be an n-sheeted covering of a sphere of revolution with branch points at the infinities.

For general spheres with one-dimensional potentials the statement of Theorem 5 is not valid.

Let $\varkappa_1,\dots,\varkappa_N$ be the poles of the transmission coefficient $T(k)$ coming into the spectral data of $U(x)$; we divide them into three groups:

$\varkappa_j = \dfrac{in_j}{2}$ with n_j an odd positive integer for $1 \le j \le L$,

$\varkappa_j = \dfrac{in_j}{2}$ with n_j an even positive integer for $L+1 \le j \le M$, and

$\varkappa_j$ is not of the form $\dfrac{in}{2}$ with n integer for $j \ge M+1$.

Set

$$\psi_j(x,y) = \varphi_1^+(x,\varkappa_j)e^{\varkappa_j y}.$$

It is clear that for $j \ge M+1$ the functions ψ_j and ψ_j^* are neither periodic nor antiperiodic in y. Therefore squares of linear combinations of such functions are not defined on $\mathcal{Z}$ and do not generate via (4) immersions of cylinders.

By Corollary 2, since ψ_j and ψ_j^* satisfy (16) for $j \le L$, they are sections of $\mathcal{E}$ and we conclude

Lemma 4. $\operatorname{Ker}\mathcal{D}$ *is spanned by $\psi_j(x,y)$ and $\psi_j^*(x,y)$ where $j \le L$.*

Any linear combination $\psi(x,y)$ of ψ_j and ψ_j^* for $L+1 \le j \le M$ also generate via (4) an immersion of a sphere. It is easy to see that, if for all ψ_j and ψ_k^* coming into this combination the frequencies n_j are represented in the form

$$n_j = 2^k l_j,$$

with l_j odd integers, then the immersion would be a 2^k-sheeted covering over its image with branch points of order 2^k at infinities. The potential of the representation of a covered sphere given by the function $\psi'(x,y) = \psi(x/2^k, y/2^k)$ would be $U'(x) = U(x/2^k)$. Otherwise the immersion would have branch points of odd order at the infinities.

Since by the definition, a vector function ψ coming into a cylindric representation belongs to $\operatorname{Ker}\mathcal{D}$, we summarize these conversations as follows:

Theorem 6. *Let* $\mathbf{a} = (a_1, \ldots, a_{2L}) \in \mathbb{C}^{2L} \setminus \{0\}$. *Then the function*

$$\psi_{\mathbf{a}}(x,y) = a_1\psi_1(x,y) + \ldots + a_L\psi_L(x,y) + a_{L+1}\psi_1^*(x,y) + \ldots + a_{2M}\psi_L^*(x,y) \tag{27}$$

defines via (4) *an immersed sphere* $\Sigma_{\mathbf{a}}$ *in* $\mathbb{R}^3$.

If there exist nonzero coefficients a_j *and* a_k *such that* $(j-k) \neq \pm L$, *then* $\Sigma_{\mathbf{a}}$ *is not a sphere of revolution.*

These spheres are exactly the spheres which have $U(x)$ *as a potential of some of their cylindric Weierstrass representations.*

3.3. The Willmore functional via the trace formula and the Willmore numbers.

Consider the following problem:

Problem 1. *How to estimate the dimension of* $\operatorname{Ker}\mathcal{D}$?

For a Dirac operator with a one-dimensional potentials (in some cylindric representation of a sphere) the trace formula (25) enables us to give a precise estimate.

Indeed, by Lemma 4, $\operatorname{Ker}\mathcal{D}$ is spanned by $\varphi_1^+(x)e^{\varkappa_j y}$ and $(\varphi_1^+(x)e^{\varkappa_j y})^*$ with $\varkappa_j$ of the form

$$\varkappa_j = \frac{i(2n_j+1)}{2} \tag{28}$$

with n_j nonnegative integers. By (25) and (26), we have

$$\int_{\infty}^{+\infty} U^2(x)\,dx \geq \frac{1}{2}\sum_{j=1}^{L}(2n_j+1) \tag{29}$$

and an equality in (29) is achieved exactly in the case then $U(x)$ is a reflectionless potential and the whole discrete spectrum with $\operatorname{Im}\varkappa > 0$ consists of eigenvalues of the form (28). We obtain

Theorem 7. *Let* $\mathcal{D}$ *be a Dirac operator* (1) *on* $\mathcal{E}$ *with a one-dimensional potential* $U(x)$ *in some cylindric representation* (*for instance, a Dirac operator generating an immersion of a sphere of revolution*). *If*

$$\frac{\dim_{\mathbb{C}}\operatorname{Ker}\mathcal{D}}{2} = \dim_{\mathbb{H}}\operatorname{Ker}\mathcal{D} \geq N,$$

then

$$\int_{-\infty}^{+\infty} U^2(x)\,dx \geq \frac{N^2}{2}. \tag{30}$$

Proof. Indeed, any level $\varkappa = i(2n+1)/2$ may be filled just by one eigenfunction of the form $\varphi_1^+(x,\varkappa_j)e^{\varkappa_j y}$ and, given $N = \dim_{\mathbb{H}}\operatorname{Ker}\mathcal{D}$, the left-hand side in (30) achieves its minimal possible value if just the first N levels are filled. This means that

$$\frac{1}{2}\sum_{j=1}^{N}(2n_j+1) = \frac{1}{2}(1+3+\ldots+(2N-1)) = \frac{N^2}{2}.$$

This proves the theorem.

An example of the Dirac spheres, constructed by U. Pinkall and J. Richter [14], shows that for each N an equality in (30) is achieved at the potential of such a sphere and, therefore, we have

Corollary 3. *For any N estimate* (30) *is precise and an equality is achieved at*

$$U_N(x) = \frac{N}{2\cosh x}.$$

We discuss the spectral data of such potentials in Section 4.

To rewrite (30) for general spheres it needs to integrate the left-hand side over y and obtain an integral over S^2 (by (8) and (12) this integral is correctly defined for any representation):

$$\int_\Sigma U^2(z,\bar z)\,dx\wedge dy \geq \pi N^2. \tag{31}$$

We would like to conjecture the following.

Conjecture 1. *Estimate* (31) *holds for any Dirac operator on $\mathcal{E}$.*

In fact, by Theorem 4, the dimension of $\operatorname{Ker}\mathcal{D}$ measures the dimension of a family of isopotential spheres in $\mathbb{R}^3$. If there exists $\psi\in\operatorname{Ker}\mathcal{D}$ such that ψ vanishes nowhere then it generates an immersion of a sphere without branch points. It is also known that for any compact surface Σ immersed via (4) the value of the Willmore functional $\mathcal{W}$, an integral of a squared mean curvature, is given by

$$\mathcal{W}(\Sigma) = \int H^2\,d\mu = 4\int_\Sigma U^2(z,\bar z)\,dx\wedge dy \tag{32}$$

(see [15]), i.e., a multiple of the squared L_2-norm of the potential U. We have

$$\int_\Sigma (H^2-K)\,d\mu = \int_\Sigma \left(\frac{k_1-k_2}{2}\right)^2 d\mu \geq 0,$$

where k_j are the principal curvatures and K is the Gauss curvature of Σ. By the Gauss–Bonnet theorem, for spheres

$$\int_\Sigma K\,d\mu = 4\pi$$

and this implies

$$\int_\Sigma H^2\,d\mu \geq 4\pi.$$

Therefore we conclude that

$$\int_\Sigma U^2(z,\bar z)\,dx\wedge dy \geq \pi.$$

This is just the inequality (31) for $N=1$ and an equality is achieved at $U_1(x)$, the potential of the unit sphere [16]. Recalling for $N=2$ the recent result of F. Pedit and U. Pinkall [13] obtained by methods of so-called quaternion algebraic geometry, we conclude the following:

Proposition 1. *Assume that there exists a zero-eigenfunction ψ of $\mathcal{D}$ such that ψ defines via* (4) *a regular immersion of S^2 into $\mathbb{R}^3$. Then Conjecture* 1 *is valid*

(1) *for $N=1$ (Gauss–Bonnet);*

(2) *for $N=2$ (Pedit–Pinkall).*

For such operators estimate (31) in terms of $\mathcal{W}$ takes the form

$$\mathcal{W}(\Sigma) \geq 4\pi N^2. \tag{33}$$

We show in Section 4 that the dimension of $\operatorname{Ker}\mathcal{D}$ cannot be estimated from below in terms of the Willmore functional.

This treatment of (31) fits into a general approach to estimates for the Willmore functional based on the Weierstrass representation (see [18] where the spectral approach to the Willmore conjecture for tori is introduced). The right-hand sides of (33), *the Willmore numbers*, measure not only the existence of a sphere with given value of the Willmore functional but the dimension of a family of isopotential spheres. It looks natural that, for given dimension of $\operatorname{Ker}\mathcal{D}$, the Willmore functional has to attain its minimal possible value on a very symmetric operator which has to have a one-dimensional potential, i.e., to be of the form covered by Theorem 7. The analogous idea is discussed in [18] for the Willmore conjecture for tori.

4. SOLITON SPHERES

4.1. Solving the Gelfand–Levitan–Marchenko equations for reflectionless potentials. We call a sphere reflectionless if it admits a Weierstrass representation with a one-dimensional reflectionless potential $U(x)$. This means that the reflection coefficient $R(k)$ of $U(x)$ vanishes identically, i.e., $R(k) \equiv 0$, and the spectral data of $U(x)$ are just

(1) a half, of a discrete spectrum of L with $U(x)$ its potential, lying in the upper-half plane: $\varkappa_1, \dots, \varkappa_N$; this spectrum is symmetric with respect to the imaginary axis;

(2) some quantities λ_j corresponding to $\varkappa_j$ such that if $\operatorname{Re}\varkappa_j = 0$ then $\lambda_j \in \mathbb{R}$ and if $\varkappa_k$ and $\varkappa_l = -\bar{\varkappa}_k$ do not coincide then $\lambda_k = \bar{\lambda}_l$.

The potential $U(x)$ is reconstructed from the spectral data via the GLM equations. For reflectionless potentials solutions to these equations can be found explicitly. We explain this procedure following [11].

Given the spectral data for a reflectionless potential, consider the following ansatz:

$$B_j(x, y) = \langle B_j(x) | T(y) \rangle,$$

where

$$T(z) = (e^{i\varkappa_1 z}, \dots, e^{i\varkappa_N z})$$

and

$$\langle u | v \rangle = u_1 v_1 + \ldots + u_N v_N$$

is a standard inner product. Then Ω takes the form

$$\Omega(x + y) = \langle \Psi(x) | T(y) \rangle,$$

where

$$\Psi(z) = (-\lambda_1 e^{i\varkappa_1 z}, \dots, -\lambda_N e^{i\varkappa_N z}).$$

In terms of these functions the GLM equations (21) are written as

$$\langle B_2(x) | T(y) \rangle + \int_x^{+\infty} \langle B_1(x) | T(x') \rangle \langle \Psi(x') | T(y) \rangle \, dx' = 0$$

and

$$\langle \Psi(x) | T(y) \rangle - \langle B_1(x) | T(y) \rangle + \int_x^{+\infty} \langle B_2(x) | T(x') \rangle \langle \Psi(x') | T(y) \rangle \, dx' = 0.$$

Introduce the matrix

$$M(x) = \int_x^{+\infty} |T(x')\rangle\langle\Psi(x')|\,dx'$$

(here we use Dirac's notation treating the inner product as a product of a bra vector $\langle u|$ and a ket vector $|v\rangle$, [4]), rewrite the GLM equations as

$$\langle B_2(x) + B_1(x)M(x)|T(y)\rangle = \langle\Psi(x) - B_1(x) + B_2(x)M(x)|T(y)\rangle = 0$$

and finally arrive at the following form of them:

$$B_2(x) + B_1(x)M(x) = \Psi(x) - B_1(x) + B_2(x)M(x) = 0. \qquad (34)$$

The entries of $M(x)$ are simply computed

$$M_{jk}(x) = \int_x^{+\infty} T_j(x')\Psi_k(x')\,dx' = \frac{\lambda_k}{i(\varkappa_j + \varkappa_k)} e^{i(\varkappa_j+\varkappa_k)x},$$

and (34) implies that

$$B_1(x) = \Psi(x)\cdot(1 + M^2(x))^{-1}$$

and

$$B_2(x) = -B_1(x)\cdot M(x).$$

By (22), we derive

$$U(x) = -\langle\Psi(x)\cdot(1 + M^2(x))^{-1}|T(x)\rangle.$$

Now represent $B_1(x,x)$ and $B_2(x,x)$ as follows:

$$B_1(x,x) = -\mathrm{Tr}\left[\frac{dM(x)}{dx}(1 + M^2(x))^{-1}\right],$$

$$B_2(x,x) = \mathrm{Tr}\left[\frac{dM(x)}{dx}M(x)(1 + M^2(x))^{-1}\right].$$

It follows from (22) and (23) that

$$2U^2(x) + i\frac{dU(x)}{dx} = \frac{d}{dx}\mathrm{Tr}\left[\frac{dM(x)}{dx}(M(x) + i)(1 + M^2(x))^{-1}\right],$$

and, since $1 + M^2(x) = (1 + iM(x))(1 - iM(x))$, we have

$$2U^2(x) + i\frac{dU(x)}{dx} = \frac{d}{dx}\mathrm{Tr}\left[\frac{d(1 + iM(x))}{dx}(1 + iM(x))^{-1}\right].$$

Using the well-known identity

$$\frac{d}{dx}\ln\det A(x) = \mathrm{Tr}\left(\frac{dA(x)}{dx}A^{-1}(x)\right),$$

we obtain

$$2U^2(x) + i\frac{dU(x)}{dx} = \frac{d^2}{dx^2}\ln\det(1 + iM(x)).$$

Since $U(x)$ is real-valued and fast decaying, we have

$$U(x) = \frac{d}{dx}\mathrm{Im}\,\ln\det(1+iM(x)) = \frac{d}{dx}\arctan\frac{\mathrm{Im}\,\det(1+iM(x))}{\mathrm{Re}\,\det(1+iM(x))}.$$

For reflectionless potentials $\varphi_1^+(x,k)$ is simply written. Set

$$W(x,k) = \int_x^{+\infty} T(x')e^{ikx'}\,dx' = \left(\frac{i}{\varkappa_1+k}e^{i(\varkappa_1+k)x},\dots,\frac{i}{\varkappa_N+k}e^{i(\varkappa_N+k)x}\right),$$

and, by (20), obtain

$$\varphi_1^+(x,k) = \begin{pmatrix} \langle\Psi(x)\cdot(1+M^2(x))^{-1}|W(x,k)\rangle \\ e^{ikx} - \langle\Psi(x)\cdot(1+M^2(x))^{-1}M(x)|W(x,k)\rangle \end{pmatrix}. \tag{35}$$

4.2. Construction and properties of reflectionless spheres. We consider some explicit examples of reflectionless spheres which are spheres with N-soliton potentials.

4.2.1. $N=1$, $\varkappa_1 = \frac{i}{2}$. In this case

$$M(x) = -\lambda e^{-x},$$

where $\lambda = \lambda_1 \in \mathbb{R}\setminus\{0\}$, and we obtain

$$U(x,\lambda) = \frac{\lambda e^{-x}}{1+\lambda^2 e^{-2x}}.$$

Since for an immersed surface $U(x)$ is defined up to a sign, we assume that $\lambda = e^{-a} > 0$ and derive $U(x,\lambda) = U_1(x+a)$ with

$$U_1(x) = \frac{1}{2\cosh x}.$$

This is the potential of a round sphere and it is easily checked that $\varphi_1^+(x,i/2)e^{iy/2}$ defines it via (4).

4.2.2. $N=2$, $\varkappa_1 = \frac{i}{2}$, $\varkappa_2 = \frac{3i}{2}$. A general potential corresponding to this data is

$$U(x,\lambda_1,\lambda_2) = \frac{144\lambda_1 e^{-x} + 144\lambda_2 e^{-3x} + 36\lambda_1^2\lambda_2 e^{-5x} + 4\lambda_1\lambda_2^2 e^{-7x}}{144 + 144\lambda_1^2 e^{-2x} + 72\lambda_1\lambda_2 e^{-4x} + 16\lambda_2^2 e^{-6x} + \lambda_1^2\lambda_2^2 e^{-8x}},$$

which for $\lambda_1 = 2$, $\lambda_2 = 6$ takes the form

$$U_2(x) = \frac{1}{\cosh x}.$$

4.2.3. *The potentials of the Dirac spheres.* U. Pinkall and J. Richter constructed the Dirac spheres using the representation theory [14]. We mentioned above that, for these spheres, estimate (31) is precise. Their potentials are

$$U_N(x) = \frac{N}{2\cosh x}.$$

We show above how $U_2(x)$ is obtained via the inverse scattering method, and, from (25) and Theorem 7, follows

Proposition 2. *The discrete spectrum of $U_N(x)$ consists of $\pm\frac{(2j-1)i}{2}$ with $j \le N$.*

For completing a description of these potentials it needs to find the coefficients $\lambda_1, \dots, \lambda_N$. Recall that there exists an infinite family of soliton equations, *the modified Korteweg–de Vries hierarchy* of nonlinear equations, such that

(1) the mth mKdV equation has the form

$$\frac{\partial}{\partial t_m} U = \frac{\partial^{2m-1}}{\partial x^{2m}} U + \dots,$$

preserves the spectrum of L, and transforms λ_j and $R(k)$ as follows:

$$\lambda_j \to \lambda_j \exp(i 2^{2m-1} \varkappa_j t_m), \qquad R(k) \to R(k) \exp(i 2^{2m-1} k t_m);$$

(2) all flows generated by these equation pairwise commute.

Stationary solutions to linear combinations of these flows satisfy *the Novikov equations* [12]:

$$\left(a_1 \frac{\partial}{\partial t_1} + \dots + a_m \frac{\partial}{\partial t_m} \right) U(x, t_1, \dots) = 0.$$

These facts are exposed in [1, 5, 11, 19]. It follows from the formulas that reflectionless potentials are fast decaying solutions to the Novikov equations (for the mKdV hierarchy) and, given $\varkappa_1, \dots, \varkappa_N$, the mKdV-orbit of a reflectionless potential consists of solutions to these equations with given $a_1, \dots, a_m$.

Proposition 3. *Every reflectionless potential with $\varkappa_j = \frac{(2j-1)i}{2}$, where $1 \leq j \leq N$, is obtained from $U_N(x)$ by the mKdV-deformations.*

4.2.4. *The Dirac spheres as rational spheres.* A nice property of the Dirac spheres is that they are described in terms of rational functions. Indeed, return back to a plane representation of spheres. It means that we represent S^2 as a complex plane $\mathbb{C}$ completed by a point at infinity. A conformal parameter Z on $\mathbb{C}$ is related with x and y, coming in (27) and (35), as $Z = e^{x+iy}$ and, by (13), a spinor field $\Psi(Z, \bar{Z})$ defining via (4) a plane representation of a sphere is

$$\Psi_1(Z, \bar{Z}) = \frac{1}{e^{(x+iy)/2}} \psi_1(x, y), \quad \Psi_2(Z, \bar{Z}) = \frac{1}{e^{(x-iy)/2}} \psi_2(x, y).$$

Now it is easy to see that Ψ is a rational function of Z and $\bar{Z}$.

4.2.5. *Soliton deformations of reflectionless spheres.* Each mKdV-flow $U(x, t)$ deforms the eigenfunctions $\varphi_1^+(x, k)$ via linear differential equations. This gives a deformation of a sphere defined by any linear combination (27). These deformations are called the mKdV-deformations of spheres and they are reductions (for one-dimensional potentials) of the more general modified Novikov–Veselov deformations introduced in [10] as deformations of surfaces locally presented via (4).

Here we only mention some interesting facts:

(1) *If $N \geq 2$, then generically a function of the form* (27) *defines not a sphere of revolution and moreover a sphere admitting no S^1-isometries. Nevertheless, soliton deformations of this sphere are described by $1+1$-dimensional soliton equations;*

(2) *The Kruskal integrals are first integrals for all mKdV-flows. For reflectionless spheres these integrals are presented in terms of $\varkappa_1, \dots, \varkappa_N$ by the trace formulas* (24);

(3) *The mKdV-deformations preserve closedness of spheres (for tori this has been proved in* [16]*).*

Generically the values of these integrals, as well as the modified Novikov–Veselov deformations, depend on a choice of a conformal parameter on a sphere. For a surface of revolution there exists a distinguished parameter (see [16]).

4.2.6. *Reflectionless spheres with* $\dim_{\mathbb{H}} \operatorname{Ker} \mathcal{D} = 1$ *and with large values of the Willmore functional.* The construction of such spheres is simple: take $\varkappa_1 = i/2, \varkappa_2 = -a + it$, and $\varkappa_3 = -a + it$, and fix admissible λ_j. Then for each $t > 0$ take a sphere of revolution Σ_t given by $\varphi_1^+(x, i/2)$. By (25), we have

$$\mathcal{W}(\Sigma_t) = 4\pi + 8\pi t.$$

Therefore we obtain

Proposition 4. *For any $C > 0$ there exists a regular sphere Σ in $\mathbb{R}^3$ such that $\dim_{\mathbb{H}} \operatorname{Ker} \mathcal{D} = 1$, Σ is defined via (4) by a zero-eigenfunction of $\mathcal{D}$, and $\mathcal{W}(\Sigma) > C$.*

We do not check the regularity condition but it is easy using (35).

4.2.7. *Deformations of reflectionless spheres via deformations of the spectrum.* We consider a simple example which relates to the previous one (see 4.2.6).

Take a reflectionless potential $U(x,t)$ whose spectral data are $\varkappa_1, \dots, \varkappa_{N+2}$ and $\lambda_1, \dots, \lambda_{N+2}$ with $\varkappa_{N+1} = -a + it, \varkappa_{N+2} = a + it$ and $\lambda_{N+1} = \alpha t, \lambda_{N+2} = \bar{\alpha} t$ with $a, t \in \mathbb{R}$. Take a linear combination (27) of eigenfunctions of L with eigenvalues of the form $(2m+1)i/2$. Since $\varphi_1^+(x, \varkappa_{N+1})$ and $\varphi_1^+(x, \varkappa_{N+2})$ do not come into this combination, then for each t this combination defines a sphere Σ_t immersed into $\mathbb{R}^3$.

Consider the limit $t \to 0$. Then it is easy to see that

$U(x,t) \to U(x,0)$ with the spectral data $\varkappa_1, \dots, \varkappa_N$ and $\lambda_1, \dots, \lambda_N$, and Σ_t tends to a sphere Σ_0 immersed into $\mathbb{R}^3$.

It is quite sure from the construction that the spectral data of L depends on $U(x)$ continuously but we do not know any estimates for stability of the inverse scattering problem for Dirac operators. Hence we treat the Dirac spheres as "constrained" Willmore only on a physical rigor level.

Let Σ be a Dirac sphere corresponding to an N-soliton potential and assume that the eigenfunctions corresponding to all levels of discrete spectrum come into a linear combination (27) defining Σ. Consider small perturbations of the sphere preserving the class of spheres with one-dimensional potentials. Such perturbations reflect in small perturbations of the potential and therewith in small perturbations of the discrete spectrum. These perturbations may result in appearance of new eigenvalues and perturbations of $R(k)$ which transform $U(x)$ into non-reflectionless potential. But the discrete spectrum $\varkappa_j = (2j-1)i/2$ has to be preserved because it is all coming in the representation of the sphere. Now it follows from (25) and (32) that all such perturbations have to increase the value of the Willmore functional. By the completeness argument, we conclude that also for all Dirac spheres and moreover for all spheres such that each eigenvalue of L takes the form $(2m+1)i/2$. Therefore we have:

Soliton spheres, such that each eigenvalue of L is of the form $(2m+1)i/2$, are critical points of the Willmore functional restricted onto the class of spheres with one-dimensional potentials.

5. FINAL REMARKS

(1) In the present paper we mostly use a cylindric representation which enables us to apply well-developed inverse spectral theory for one-dimensional Dirac operators.

In a plane representation we may treat the two-dimensional Zakharov–Shabat problem by the $\bar{\partial}$ method and the nonlocal Riemann problem. In this case we consider the spectral data related to one level of energy E:

$$\mathcal{D}\psi = E\psi,$$

and in [3] the reconstruction problem has been solved for positive E and assuming that the L_2-norm of U is sufficiently small. For geometric reasons we have to consider this problem for $E = 0$ and for potentials with sufficiently large L_2-norms, $\|U\|^2_{L_2} \geq \pi$ (the Gauss–Bonnet theorem, (32)), and the vanishing of E itself leads to appearance of logarithmic singularities of the Green–Faddeev functions coming into the kernels of these integral equations.

(2) The spinor field ψ on an immersed surface may be obtained also as the restriction of a parallel spinor field on $\mathbb{R}^3$ onto the surface. This recently has been exposed in [7] and also had been conjectured to the author by Pinkall just after author's talk in Amherst (November 1995) about the paper [15].

Being geometrically invariant, this treating of Lemma B does not enable us to extract spectral properties of the Weierstrass representation which are of a global origin and to develop construction of surfaces from spectral data as we do that in Sections 3 and 4.

But this fits into the more general consideration of Dirac operators on hypersurfaces in $\mathbb{R}^{n+1}$ with induced spin structures. Recently it has been shown by C. Bär ([2]) that for such hypersurface M there exists at least $2^{[n/2]}$ distinct eigenvalues λ of the induced operator $\mathcal{D}$ such that

$$\lambda^2 \leq \frac{n^2}{4\,\mathrm{Vol}(M)}\mathcal{W}(M).$$

APPENDIX

PERIOD PROBLEM FOR IMMERSIONS OF SURFACES OF GENUS ≥ 2

Recall the definition of a global Weierstrass representation of a surface Σ of genus $g \geq 2$ which is conformally equivalent to $\Sigma_0 = \mathcal{H}/\Lambda$ with $\mathcal{H}$ the Lobachevskii upper half-plane and Λ a lattice in $PSL(2,\mathbb{R})$ [15, 18]:

Definition. *A sphere Σ with g (> 1) handles, immersed into $\mathbb{R}^3$, possesses a global Weierstrass representation if there exist a real potential U and functions ψ_1 and ψ_2 defined on the universal covering of Σ, i.e., on $\mathcal{H}$, such that*

(1)

$$\begin{cases} U(\gamma(z)) = |cz+d|^2 U(z), \\ \psi_1(\gamma(z)) = (cz+d)\psi_1(z), \\ \psi_2(\gamma(z)) = (c\bar{z}+d)\psi_2(z) \end{cases} \tag{A.1}$$

for $z \in \mathcal{H}$ and $\gamma \in \Lambda$ represented by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a,b,c,d \in \mathbb{R}, \quad ad - bc = 1;$$

(2) *the vector function ψ satisfies* (2) *and for a suitable choice of coordinates in $\mathbb{R}^3$ defines by* (4) *an immersion of Σ.*

We have

Theorem. *Let Σ_0 be a compact oriented surface of genus $g \geq 2$ and let U and ψ satisfy (A.1) and define via (4) an immersion of the universal covering of Σ_0 into $\mathbb{R}^3$. Then this immersion converts into an immersion of Σ_0 if and only if*

$$\int_{\Sigma_0} \bar{\psi}_1^2 \, d\bar{z} \wedge \omega = \int_{\Sigma_0} \psi_2^2 \, d\bar{z} \wedge \omega = \int_{\Sigma_0} \bar{\psi}_1 \psi_2 \, d\bar{z} \wedge \omega = 0 \tag{A.2}$$

for any holomorphic differential ω on Σ_0.

Proof. Take a canonical basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ for $H_1(\Sigma_0)$. This means that its intersection form is

$$\alpha_j \circ \beta_k = \delta_{jk}, \quad \alpha_j \circ \alpha_k = \beta_j \circ \beta_k = 0$$

and there are loops representing these cycles such that after cutting along them we obtain a domain M bounded by a polygon

$$\partial M = \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \dots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}.$$

To this basis corresponds a unique basis of holomorphic differentials $\omega_1, \dots, \omega_g$ normalized by the condition

$$\int_{\alpha_k} \omega_j = \delta_{jk}.$$

Define the period matrix Ω by

$$\Omega_{jk} = \int_{\beta_k} \omega_j.$$

This matrix is symmetric and its imaginary part is positive definite [6].

It follows from (4) and (A.1) that $\bar{\psi}_1^2 d\bar{z}, \psi_2^2 d\bar{z}$, and $\bar{\psi}_1 \psi_2 d\bar{z}$ are correctly defined 1-forms on Σ_0 and

$$\bar{\psi}_1^2 = \bar{\partial}(iX^1 - X^2), \qquad \psi_2^2 = \bar{\partial}(iX^1 + X^2), \qquad \bar{\psi}_1 \psi_2 = \bar{\partial} X^3. \tag{A.3}$$

Introduce the following vectors of translation periods:

$$V_j = \int_{\alpha_j} (\bar{\psi}_1^2 \, d\bar{z} - \bar{\psi}_2^2 dz), \qquad V_{g+j} = \int_{\beta_j} (\bar{\psi}_1^2 \, d\bar{z} - \bar{\psi}_2^2 dz),$$

$$W_j = \int_{\alpha_j} (\psi_1 \bar{\psi}_2 \, dz + \bar{\psi}_1 \psi_2 \, d\bar{z}), \qquad W_{g+j} = \int_{\beta_j} (\psi_1 \bar{\psi}_2 \, dz + \bar{\psi}_1 \psi_2 \, d\bar{z}).$$

Denote by Y and Z the vectors

$$Y_j = \int_{\Sigma_0} \bar{\psi}_1^2 \, d\bar{z} \wedge \omega_j, \qquad Y_{j+g} = -\overline{\int_{\Sigma_0} \psi_2^2 \, d\bar{z} \wedge \omega_j},$$

$$Z_j = \int_{\Sigma_0} \bar{\psi}_1 \psi_2 \, d\bar{z} \wedge \omega_j, \qquad Z_{j+g} = \overline{\int_{\Sigma_0} \bar{\psi}_1 \psi_2 \, d\bar{z} \wedge \omega_j}.$$

It is evident that an immersion of $\mathcal{H}$ is converted into an immersion of Σ_0 if and only if $V = W = 0$.

Now, by the Stokes theorem and (A.3), we have

$$\int_{\Sigma_0} \bar{\psi}_1^2\, d\bar{z} \wedge \omega_j = \int_{\partial M} (iX^1 - X^2)\omega_j = \sum_{k=1}^{g} \left(V_k \int_{\beta_k} \omega_j - V_{k+g} \int_{\alpha_k} \omega_j \right),$$

$$\int_{\Sigma_0} \psi_2^2\, d\bar{z} \wedge \omega_j = \int_{\partial M} (iX^1 + X^2)\omega_j = \sum_{k=1}^{g} \left(-\bar{V}_k \int_{\beta_k} \omega_j + \bar{V}_{k+g} \int_{\alpha_k} \omega_j \right), \tag{A.4}$$

$$\int_{\Sigma_0} \bar{\psi}_1 \psi_2\, d\bar{z} \wedge \omega_j = \int_{\partial M} X^3 \omega_j = \sum_{k=1}^{g} \left(W_k \int_{\beta_k} \omega_j - W_{k+g} \int_{\alpha_k} \omega_j \right).$$

Consider the $2g\times 2g$-matrix

$$\tilde{\Omega} = \begin{pmatrix} \Omega & -1 \\ \bar{\Omega} & -1 \end{pmatrix}.$$

Since $\operatorname{Im}\Omega$ is positive definite, $\tilde{\Omega}$ is nondegenerate. Rewrite (A.4) as follows:

$$\tilde{\Omega} V = Y, \qquad \tilde{\Omega} W = Z,$$

and conclude that $V = W = 0$ if and only if

$$Y = Z = 0. \tag{A.5}$$

Since ω_j form a basis for holomorphic differentials, (A.5) is equivalent to vanishing integrals (A.2) for any holomorphic differential on Σ_0.

This proves the theorem.

Proposition 4 from [18] which settles the period problem for tori may be reformulated as (A.2).

ACKNOWLEDGMENTS

This work has been done when the author was a guest of SFB 288 in Technische-Universität in Berlin and was also supported by INTAS–RFBR (the Russian Foundation for Basic Research), joint grant no. 95-0418.

The author thanks U. Pinkall for helpful discussions.

REFERENCES

1. Ablowitz, M.J. and Segur, H., *Solitons and the Inverse Scattering Method*, Philadelphia: SIAM, 1981.
2. Bär, C., Extrinsic Bounds for Eigenvalues of the Dirac Operator, *Preprint, Freiburg Univ.*, 1997.
3. Bogdanov, L.V., On the Two-Dimensional Zakharov–Shabat Problem, *Teor. Mat. Fiz.*, 1987, vol. 72, no. 1, pp. 155–159 [*Theor. Math. Phys.* (Engl. Transl.), 1987, vol. 72, pp. 790–793].
4. Dirac, P.A.M., *The Principles of Quantum Mechanics*, 4th ed., Oxford: Oxford Univ. Press, 1958.
5. Takhtadzhyan, L.A. and Faddeev, L.D., *Gamil'tonov podkhod v teorii solitonov* (Hamiltonian Methods in the Theory of Solitons), Moscow: Nauka, 1986. Berlin: Springer, 1987 (Engl. Transl.).
6. Fay, J., *Theta Functions on Riemann Surfaces*, Berlin: Springer, 1973 (*Lect. Notes Math.*, vol. 352).
7. Friedrich T. On the Spinor Representation of Surfaces in Euclidean 3-Space, *Preprint* no. 295, SFB 288, Berlin: Techn. Univ., 1997.
8. Eisenhart, L.P., *A Treatise on the Differential Geometry of Curves and Surfaces*, New York: Dover, 1909.

9. Kamberov, G., Pedit, F., and Pinkall, U., Bonnet Pairs and Isothermic Surfaces, *Duke Math. J.*, 1998, vol. 92, pp. 637–644.

10. Konopelchenko, B.G., Induced Surfaces and Their Integrable Dynamics, *Stud. Appl. Math.*, 1996, vol. 96, pp. 9–52.

11. Lamb, G.L., Jr., *Elements of Soliton Theory*, New York: J. Wiley & Sons, 1980.

12. Novikov, S.P., The Periodic Problem for the Korteweg–De Vries Equation, *Funk. Anal. Ego Pril.*, 1974, vol. 8, no. 3, pp. 54–66 [*Funct. Anal. Appl.* (Engl. Transl.), 1974, vol. 8, pp. 236–246].

13. Pedit, F. and Pinkall, U., Quaternionic Analysis on Riemann Surfaces and Differential Geometry, *Proc. Int. Congr. Math. Berlin, 1998*, Doc. Math., 1998. Extra vol. 2, pp. 389–400 (electronic).

14. Richter, J., Conformal Maps of a Riemann Surface into the Space of Quaternions, *PhD Thesis*, Berlin: Techn. Univ., 1997.

15. Taimanov, I.A., Modified Novikov–Veselov Equation and Differential Geometry of Surfaces, in *Solitons, Geometry and Topology: On the Crossroad*, Buchstaber, V.M. and Novikov, S.P., Eds., Providence (R.I.): Amer. Math. Soc., 1997, pp. 133–151 (*AMS Transl., Ser. 2*, vol. 179).

16. Taimanov, I.A., Surfaces of Revolution in Terms of Solitons, *Ann. Global Anal. and Geom.*, 1997, vol. 15, pp. 419–435.

17. Taimanov, I.A., Global Weierstrass Representation and Its Spectrum, *Usp. Mat. Nauk*, 1997, vol. 52, no. 6, pp. 187–188 [*Russian Math. Surveys* (Engl. Transl.)].

18. Taimanov, I.A., The Weierstrass Representation of Closed Surfaces in $\mathbb{R}^3$, *Funk. Anal. Ego Pril.*, 1998, vol. 32, no. 4, pp. 49–62 [*Funct. Anal. and Appl.* (Engl. Transl.)].

19. *Teoriya solitonov* (Theory of Solitons), Novikov, S.P., Ed., Moscow: Nauka, 1980.

Translated by the author