

## On the Formality Problem for Symplectic Manifolds

Ivan K. Babenko and Iskander A. Taimanov

ABSTRACT. We discuss the formality problem for symplectic manifolds. In particular, we discuss the first examples of nonformal simply connected symplectic manifolds which we found in 1998.

### 1. Examples and constructions of symplectic manifolds.

A smooth manifold  $M$  endowed with a closed 2-form  $\omega$  is called symplectic if this form is nondegenerate. This form is called symplectic. By the Darboux theorem, in canonical coordinates it is written as  $\omega = \sum_{i=1}^n dx^i \wedge dx^{n+i}$ . Therefore any symplectic manifold is even-dimensional and admits an almost complex structure  $J$  compatible with the symplectic structure. This means that the inner product  $\langle \xi, \eta \rangle = \omega(\xi, J\eta)$  defines a Riemannian metric and  $J$  is skew-symmetric with respect to this metric.

The simplest examples of symplectic manifolds are Kähler manifolds with Kähler forms. For example the complex projective spaces  $\mathbb{C}P^n$  with the forms  $\omega_{\text{FS}} = g_{ij} dz^i \wedge d\bar{z}^j$  where  $g_{ij} dz^i d\bar{z}^j$  is the Fubini–Study metric.

Gromov proved that any open almost complex manifold admits a compatible symplectic structure [14]. For closed manifolds this is not true since  $\omega^n$  is the volume form and therefore  $[\omega]^k \neq 0 \in H^{2k}(M; \mathbb{R})$  for  $k = 1, \dots, n$ .

The Gromov–Tischler theorem [14, 24] reads that if  $M$  is a compact symplectic manifold with an integer symplectic form,  $[\omega] \in H^2(M; \mathbb{Z})$ , then there is a symplectic embedding  $f : (M, \omega) \rightarrow (\mathbb{C}P^{2n+1}, \omega_{\text{FS}})$ , with  $\dim M = n$ , such that  $f^*(\omega_{\text{FS}}) = \omega$ . This implies that any compact symplectic manifold is diffeomorphic to a symplectic submanifold of a complex projective space.

The problem of how symplectic manifolds differ from Kähler manifolds led to the introduction of several geometric methods for constructing symplectic manifolds. These include symplectic fibrations (Thurston [23]), symplectic blow-ups (McDuff [19]), and fiber connected sums (Gompf [13]). The last two constructions were first outlined by Gromov.

There are other constructions by Donaldson [10] and Guan [15] which involve hard global analysis. We recall that a complex manifold  $M$  is called holomorphically symplectic if there is a closed holomorphic  $(2, 0)$ -form  $\omega$  with maximal rank. Notice that  $\omega + \bar{\omega}$  is a symplectic form. Generalizing the construction of K3 surfaces Guan

---

1991 *Mathematics Subject Classification.* 53D05, 55S30.

*Key words and phrases.* symplectic manifolds, rational homotopy type, formality, Massey products, nilmanifolds.

had found the first examples of closed simply connected holomorphically symplectic manifolds which are not Kählerian. <sup>1</sup>

## 2. Symplectic manifolds with no Kähler structure.

The first example of a compact symplectic manifold with no Kähler structure were found by Thurston [23]. This manifold already appeared in the Kodaira classification of complex surfaces. So it is called the Kodaira–Thurston manifold and its construction is as follows.

Denote by  $\mathcal{H}$  the three-dimensional Heisenberg group formed by the upper-triangular matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with  $x, y, z \in \mathbb{R}$  and the multiplication operation. The matrices with  $x, y, z$  integers form a uniform lattice  $\mathcal{H}_{\mathbb{Z}}$  in  $\mathcal{H}$ . On the circle  $S^1$  take a coordinate  $u$  defined modulo 1 and put

$$M_{\text{KT}} = (\mathcal{H}/\mathcal{H}_{\mathbb{Z}}) \times S^1, \quad \omega = dx \wedge du + dy \wedge dz.$$

We may also construct  $M_{\text{KT}}$  as follows. Let  $T^2$  be a two-torus  $\mathbb{R}^2/\mathbb{Z}^2$  with linear coordinates  $z$  and  $y$  defined modulo integers. Take a product of  $T^2$  with an interval  $[0, 1]$  with a coordinate  $x$ . Glue two components of the boundary by the automorphism  $\begin{pmatrix} z \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix}$ . The resulting nilmanifold is  $\mathcal{H}/\mathcal{H}_{\mathbb{Z}}$ .

Its product with the circle  $S^1$  is  $M_{\text{KT}}$ . It is fibered over the two torus with the coordinates  $x$  and  $u$  defined modulo integers:

$$T^2_{(y,z)} \rightarrow M_{\text{KT}} \rightarrow T^2_{(x,u)}.$$

The symplectic form  $\omega$  splits into the sum of the pullback of the symplectic form  $dx \wedge du$  on the base and the form  $dy \wedge dz$  whose restriction to the fibers defines symplectic structures. This is a simplest example of a symplectic fibration.

The first Betti number of  $M_{\text{KT}}$  is odd (it is actually 3). Therefore it has no Kähler structure.

**THEOREM 1** (Thurston). *The Kodaira–Thurston manifold is a non-simply connected symplectic manifold with no Kähler structure.*

The Kodaira–Thurston manifold admits a complex structure and moreover an indefinite Kähler metric.

Fernandez, Gotay and Gray studied such symplectic fibrations in detail. They had found examples of symplectic manifolds with no complex structure [11] (see also [12]).

**THEOREM 2** (Fernandez–Gotay–Gray). *Let  $M^4$  be a principal  $S^1$  bundle over a three-manifold  $N^3$  which in turn is a principal  $S^1$ -bundle over a two-torus  $T^2$ . Let the first Betti number satisfy the inequalities  $2 \leq b_1(M^4) \leq 4$ . Then*

- 1) if  $b_1 = 2$  then  $M^4$  has symplectic but no complex structures;
- 2) if  $b_1 = 3$  then  $M^4$  has both symplectic and complex structures but no Kähler structure; however it has an indefinite Kähler metric;
- 3) if  $b_1 = 4$  then  $M^4$  is a torus  $T^4$ .

<sup>1</sup>We are grateful to M. Gromov who draw our attention to the papers by Guan after our talk.

Some high-dimensional generalizations of the Kodaira–Thurston manifold were found by Cordero, Fernandez and Gray [8].

Recently we have found other high-dimensional generalizations of these examples. They appear to be interesting for other reasons [4].

Consider the infinite-dimensional algebra  $W$  of formal vector fields on the line. It has the basis

$$e_k = x^{k+1} \frac{d}{dx}, \quad k \geq -1,$$

and the commutation relations are  $[e_i, e_j] = (j-i)e_{i+j}$ . There is a natural filtration  $\cdots \subset \mathcal{L}_1 \subset \mathcal{L}_0 \subset \mathcal{L}_{-1} \subset W$  where  $\mathcal{L}_k$  is a subalgebra spanned by  $e_k, e_{k+1}, \dots$ . Consider the series of finite-dimensional nilpotent Lie algebras

$$\mathcal{V}_n = \mathcal{L}_1 / \mathcal{L}_{n+1}, \quad n = 3, 4, \dots,$$

and denote by  $V_n$  the corresponding Lie groups. The group  $V_n$  is isomorphic to  $(\mathcal{V}_n, \times)$  with the polynomial multiplication law  $\times$  given by the Campbell–Hausdorff formula. The basis  $\{e_1, \dots, e_n\}$  generates a uniform lattice  $\Gamma_n$  and we have a series of nilmanifolds

$$M_n = V_n / \Gamma_n, \quad n = 3, 4, \dots, \quad \dim M_n = n.$$

**THEOREM 3** (Babenko–Taimanov). *The nilmanifolds  $M_{2n}$  have integer symplectic forms  $\omega_{2n} \in H^2(M_{2n}; \mathbb{Z})$ .*

We have  $V_3 = \mathcal{H}$ ,  $M_3 = \mathcal{H} / \mathcal{H}_Z$ , and  $M_3 \times S^1 = M_{KT}$ . The manifold  $M_4$  admits no complex structure.

Interesting properties of these manifolds were observed by Buchstaber [7] who described them by a sequence  $M_{n+1} \rightarrow M_n$  of principal  $S^1$ -bundles and showed that  $H^*(\varprojlim M_n) = H^*(\mathcal{S})$  where  $\mathcal{S}$  is the Landweber–Novikov algebra from the complex cobordism theory. These ideas were recently developed by Alaniya [1], Artel'nykh [2], and Millionshchikov [20].

Before speaking about simply connected symplectic manifolds we would like to mention the following theorem by Gompf [13] which solved the Kotschick problem.

**THEOREM 4** (Gompf). *For any even  $n \geq 4$  and any finitely-presented group  $G$  there is a compact symplectic  $n$ -manifold  $M$  such that  $\pi_1(M) = G$  and this manifold does not satisfy the Hard Lefschetz Condition. Therefore it is not homotopically equivalent to any Kähler manifold.*

The proof of Gompf uses the fiber connected sums construction. The Hard Lefschetz Condition is valid for Kähler manifolds and says that for any  $k$  the multiplication by the  $k$ -th power of the cohomology class  $[\omega]$  of the Kähler form defines an isomorphism  $H^{n-k}(M; \mathbb{C}) \xrightarrow{\times[\omega]^k} H^{n+k}(M; \mathbb{C})$  where  $\dim M = 2n$ . Gompf had shown that for his manifolds this condition is not satisfied for any 2-dimensional cohomology class.

### 3. Simply connected symplectic manifolds with no Kähler structure.

A symplectic blow-up was used by McDuff for constructing a simply connected compact symplectic manifold with no Kähler structure.

This method is as follows. Let  $X$  be a symplectic manifold of dimension  $2n$  and  $Y$  be a  $2k$ -dimensional symplectic submanifold of  $X$ . There is a symplectic normal bundle  $\nu$  to  $Y$  in  $X$  which is a vector bundle over  $Y$  with  $2(n-k)$  dimensional fibers. On this bundle there is a fiberwise almost complex structure induced by a symplectic structure and thus we identify the fibers with  $\mathbb{C}^{n-k}$ . The blow up of

$\mathbb{C}^{n-k}$  at the origin results in replacing any nonzero vector  $\xi$  by a pair  $(\xi, l)$  where  $l$  is a line passing through the origin in the direction of  $\xi$ . Such a replacement is smoothly extended up to replacing  $\mathbb{C}^{n-k}$  by the canonical line bundle  $E$  over  $\mathbb{C}P^{n-k-1}$ . There is a mapping

$$\pi : E \rightarrow \mathbb{C}$$

which maps the zero section of  $E$  to the origin  $0 \in \mathbb{C}$  and maps diffeomorphically the complement of the zero section of  $E$  onto  $\mathbb{C} \setminus \{0\}$ .

Take a small normal neighborhood of  $Y$  in  $X$ , identify it with a neighborhood of the zero section of  $\nu$ , and blow up the fibers of  $\nu$  at the origins. The resulting manifold  $\tilde{X}$  is called the blow-up of  $X$  along  $Y$ . There is a natural mapping

$$\pi : \tilde{X} \rightarrow X$$

which is a diffeomorphism outside  $\pi^{-1}(Y)$ . This construction was proposed by Gromov and explained in detail by McDuff [19].

**THEOREM 5 (Gromov–McDuff).** *Given a symplectic form  $\omega$  on  $X$ , there is a symplectic form  $\tilde{\omega}$  on  $\tilde{X}$  such that outside a small neighborhood of  $\pi^{-1}(Y)$  we have  $\tilde{\omega} = \pi^*\omega$ .*

There is a symplectic embedding of  $M_{KT}$  into  $(\mathbb{C}P^5, \omega_{FS})$ . It was used by McDuff for constructing the first example of a simply connected compact symplectic manifold with no Kähler structure [19]:

**THEOREM 6 (McDuff).** *The third Betti number of the symplectic blow-up  $\widetilde{\mathbb{C}P}^5$  of  $\mathbb{C}P^5$  along  $M_{KT}$  equals  $b_1(M_{KT})$  (which is 3). Therefore  $\widetilde{\mathbb{C}P}^5$  has no Kähler structure.*

Many exciting examples coming from the Seiberg–Witten theory are now known in the four-dimensional case however we do not discuss them here.

#### 4. Minimal models and formality.

Minimal models of simply connected and nilpotent spaces were invented by Sullivan [22].

Let  $\mathcal{M}$  be a free graded commutative algebra over  $\mathbb{Q}$  with homogeneous generators  $x_1, \dots$  such that  $1 \leq \deg x_i \leq \deg x_j$  for  $i \leq j$  and for any dimension  $l \geq 1$  there are finitely many generators  $x_j$  of dimension  $l$ . Such an algebra with a differential  $d$  is called minimal if  $dx_i \in \bigwedge(x_1, \dots, x_{i-1})$  for  $i \geq 2$ .

In the sequel we consider only connected differential graded (d.g.) algebras  $\mathcal{A}$  which means that  $\mathcal{A}^0 = \mathbb{Q}$  and the multiplication by the elements from  $\mathcal{A}^0$  is identified with the multiplication by rationals.

It is said that a minimal algebra  $\mathcal{M}$  is the minimal model for  $\mathcal{A}$  if there is a homomorphism of d. g. algebras  $f : \mathcal{M} \rightarrow \mathcal{A}$  inducing an isomorphism of cohomology:  $f^* : H^*(\mathcal{M}) \xrightarrow{\cong} H^*(\mathcal{A})$ . For the algebra of  $\mathbb{Q}$ -polynomial forms on  $X$  its minimal model  $\mathcal{M}_X$  is called the minimal model for  $X$ .

**THEOREM 7 (Sullivan).** *Any compact simply connected polyhedron or nilmanifold has a minimal model which is unique up to isomorphism. The algebra  $\mathcal{M}_X$  describes up to isomorphism the rational homotopy type of  $X$  and in particular we have*

$$\text{Hom}(\pi_*(X), \mathbb{Q}) = \mathcal{M}_X / \mathcal{M}_X \wedge \mathcal{M}_X.$$

A minimal algebra  $\mathcal{M}$  is called formal if there is a homomorphism of d.g. algebras  $f : (\mathcal{M}, d) \rightarrow (H^*(\mathcal{M}), 0)$  inducing an isomorphism of cohomology. The space  $X$  is called formal if its minimal model is formal. This implies that  $\mathcal{M}$  is the minimal model for the cohomology ring  $H^*(\mathcal{M})$  with the zero differential.

A d.g. algebra  $\mathcal{A}$  is called formal if its minimal model is formal. Notice that if  $\mathcal{A}$  is one-connected d.g. algebra, then its minimal model is uniquely reconstructed from  $\mathcal{A}$ . Therefore the formality of a simply connected space means that its rational homotopy type is reconstructed from its rational cohomology ring. Examples of formal spaces are; global symmetric spaces, classifying spaces of Lie groups, and closed simply connected manifolds of dimension  $\leq 6$  (Neisendorfer–Miller).

A nilmanifold is formal if and only if it is a torus.

**THEOREM 8** (Deligne–Griffiths–Morgan–Sullivan [9]). *Kähler manifolds are formal.*

### 5. On simply connected nonformal symplectic manifolds.

We see that symplectic nilmanifolds different from tori are nonformal. However Lupton and Oprea conjectured that on simply connected compact manifolds the existence of a symplectic structure implies formality [17] (see also [25]). They also found some additional conditions on the minimal model which together with a symplectic structure imply formality.

The Lupton–Oprea conjecture was disproved in [3, 4] by the following result.

**THEOREM 9** (Babenko–Taimanov). *For  $N \geq 5$  the symplectic blow up  $\widetilde{\mathbb{C}P}^N$  of the complex projective space along an embedded Kodaira–Thurston manifold is a simply connected nonformal symplectic manifold.*

The proof is as follows. Nontrivial Massey products obstruct formality and it appeared that if the codimension of a symplectic submanifold  $Y \subset X$  is rather large then nontrivial Massey products on  $Y$  induce nontrivial Massey products on the symplectic blow-up of  $X$  along  $Y$ . Generically for such an inheritance of a triple product it needs to have  $\text{codim } Y \geq 8$ . However in [4], the extended version of [3], we showed that for the special pairs  $(\mathbb{C}P^N, M_{\text{KT}})$  this estimate is improved to  $\text{codim } M_{\text{KT}} \geq 6$  (that is  $N \geq 5$ ).

### 6. The Massey products.

Let  $\mathcal{A}$  be a differential graded module. That is a graded module  $\mathcal{A} = \bigoplus_{k \geq 0} \mathcal{A}^k$  with the associative multiplication  $\wedge$  and with a differential  $d$  meeting the conditions  $d^2 = 0$  and

$$d(x \wedge y) = dx \wedge y + \bar{x} \wedge dy.$$

The conjugation on  $\mathcal{A}$  is defined on homogeneous elements as  $x \rightarrow \bar{x} = (-1)^p x$  for  $x \in \mathcal{A}^p$  and is linearly extended onto the whole ring.

Denote by  $M_n(\mathcal{A})$  the algebra of  $(n \times n)$ -matrices with coefficients from  $\mathcal{A}$ . For a matrix  $A = (a_{ij}) \in M_n(\mathcal{A})$  we mean by  $\bar{A}$  the matrix whose elements are conjugates of elements of  $A$ :  $\bar{A} = (\bar{a}_{ij})$ . We have the Leibniz rule:

$$d(A \wedge B) = dA \wedge B + \bar{A} \wedge dB.$$

For any matrix  $A \in M_n(\mathcal{A})$  denote by  $\text{Ker } A$  the  $\mathcal{A}$ -submodule of  $M_n(\mathcal{A})$  generated by all matrices  $(1)_{ij}$  such that  $(1)_{ij} A = A(1)_{ij} = 0$ . Here  $(1)_{ij}$  stands for the matrix such that all its entries but the entry on the intersection of the  $i$ -th

line and the  $j$ -th column vanish and this nonzero entry is 1. In particular, we have  $AB = BA = 0$  for any  $B \in \text{Ker } A$ .

We say that a matrix  $A \in M_n(\mathcal{A})$  satisfies the (generalized) Maurer–Cartan equation if

$$dA - \bar{A} \wedge A = 0 \pmod{\text{Ker } A}.$$

In this case the matrix  $\mu(A) = dA - \bar{A} \wedge A$  is called the curvature matrix.

The following Bianchi identity holds for any matrix  $A \in M_n(\mathcal{A})$ :

$$d\mu(A) = \overline{\mu(A)} \wedge A - A \wedge \mu(A).$$

If  $\mu(A) \in \text{Ker } A$ , then  $d\mu(A) = 0$  and the elements of  $\mu(A)$  define a (matrix) cohomology class with coefficients from  $H^*(\mathcal{A})$  which we call a generalized Massey product  $[\mu(A)]$  [5].

We introduce the Massey products (but not in the most general situation). We may assume that the elements of  $\mathcal{A}$  are also matrices such that the associative multiplication  $A \wedge B$  is correctly defined and the Leibniz rule holds.

*Example. The ordinary Massey products.* Let  $\alpha_1, \dots, \alpha_n$  be homogeneous elements of the cohomology ring  $H^*(\mathcal{A})$ . Let these classes be represented by elements  $a_1, \dots, a_n$ . For  $n \geq 3$  the  $n$ -tuple Massey product  $\langle \alpha_1, \dots, \alpha_n \rangle$  is correctly defined if there is a matrix  $A \in M_{n+1}(\mathcal{A})$  of the form

$$A = \begin{pmatrix} 0 & a_1 & * & \dots & * & * \\ 0 & 0 & a_2 & \dots & * & * \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & a_n \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

satisfying the Maurer–Cartan equation. This means that

$$dA - \bar{A} \wedge A = \begin{pmatrix} 0 & \dots & 0 & \tau \\ 0 & \dots & 0 & 0 \\ & & \dots & \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

To define the classical product  $\langle \alpha_1, \dots, \alpha_n \rangle$  we have to consider all solutions  $[\tau]$  to these equations for all initial data  $a_1, \dots, a_n$ . Therefore it is actually the set of cohomology classes  $[\tau]$ . For instance for  $n = 3$  the triple product of  $\alpha_1 \in H^p(\mathcal{A})$ ,  $\alpha_2 \in H^q(\mathcal{A})$ ,  $\alpha_3 \in H^r(\mathcal{A})$  is defined modulo  $\alpha_1 \wedge H^{q+r-1}(\mathcal{A}) + \alpha_3 \wedge H^{p+q-1}(\mathcal{A})$ . Moreover it is not defined for all tuples of cohomology classes  $\alpha_1, \dots, \alpha_n$  because for some tuples the Maurer–Cartan equation may have no solutions. Hence

*the classical Massey product  $\langle \alpha_1, \dots, \alpha_n \rangle$  is the set of all generalized Massey products  $[\mu(A)]$  for all matrices  $A$  with  $[a_{i,i+1}] = \alpha_i$  and  $a_{ij} = 0$  for  $i \geq j$ ,  $i, j = 1, \dots, (n+1)$ .*

Such a definition of the ordinary Massey products can be extracted from the paper by Kraines [16] and that was actually done by May [18] who used it for defining matrix Massey products. The matrix products are defined as ordinary products with replacing elements from  $\mathcal{A}$  by multipliable matrices with coefficients from  $\mathcal{A}$ . Notice, that the given definition of Massey products is rather general. The classical (ordinary and matrix) Massey products correspond to nilpotent upper triangular matrices only.

A generalized Massey product  $[\mu(A)]$  is called completely reducible if all coefficients of  $[\mu(A)]$  belong to  $H^+(\mathcal{A}) \wedge H^+(\mathcal{A})$ . A classical product  $\langle \alpha_1, \dots, \alpha_n \rangle$  is called trivial if it contains the zero.

**THEOREM 10.** *If a differential graded algebra  $\mathcal{A}$  is formal then*

- 1) *all generalized Massey products  $[\mu(A)]$  are completely reducible;*
- 2) *all classical (ordinary and matrix) Massey products are trivial.*

### 7. On Massey products in symplectic manifolds.

In [5] we developed the observation from [3, 4] and described some new classes of nonformal blow-ups of symplectic manifolds.

**THEOREM 11** (Babenko–Taimanov). *a) Let a simply connected symplectic manifold  $X$  have an irreducible generalized Massey product of dimension  $k$ . Then for any symplectic submanifold  $Y \subset X$  with  $\text{codim } Y > k$  the corresponding symplectic blow-up  $\tilde{X}$  also has an irreducible generalized Massey product of dimension  $k$ .*

*b) Let a symplectic manifold  $Y$  have a nontrivial ordinary or matrix triple Massey product. Then for any symplectic embedding  $Y \subset X$  of codimension greater than or equal to 8 the corresponding symplectic blow-up  $\tilde{X}$  also has a nontrivial triple matrix Massey product.*

*c) Let a symplectic manifold  $Y$  have a strictly irreducible quadruple matrix Massey product  $\langle S_1, S_2, S_3, S_4 \rangle$ . Then for any symplectic embedding  $Y \subset X$  such that  $\text{codim } Y > 2 \text{sdeg } \langle S_1, S_2, S_3, S_4 \rangle$  the corresponding symplectic blow-up  $\tilde{X}$  has a nontrivial quadruple matrix Massey product.*

Inheritance of triple ordinary Massey products observed in [3] was later studied by Rudyak and Tralle in the general situation. They proved statement b) of the theorem for ordinary triple Massey products [21].

To explain part c) we recall that a classical product  $\langle \alpha_1, \dots, \alpha_n \rangle$  is called strictly irreducible if it does not contain a matrix with coefficients from  $H^+(\mathcal{A}) \wedge H^+(\mathcal{A})$  and  $\text{sdeg } \alpha$  equal to the maximal degree of elements of a matrix cohomology class  $\alpha$ .

Parts b) and c) do not generalize for  $n$ -tuple products with  $n \geq 5$ . Due to a high indeterminacy of higher products they contain many more elements than lower-order products. Typically the indeterminacy substantially grows with the order and reasonable conditions do not guarantee that the product would not contain the zero.

### 8. Some problems and conjectures.

Now it is clear that a symplectic structure does not strongly restrict topology of symplectic manifolds. The only known obstruction is still the existence of a two-class  $[\omega] \in H^*(X)$  such that its powers  $[\omega]^k$  are nontrivial for  $2k \leq \dim X$ .

In [5] we propose the following

**CONJECTURE 1.** *For any finite polyhedron  $P$  and any  $N \geq 1$  there is a symplectic manifold  $X$  and an embedding  $f : P \rightarrow X$  such that*

$$f_* : \pi_k(P) \rightarrow \pi_k(X) \text{ is a monomorphism for } k \leq N.$$

For  $N = 1$  and  $\dim P = 2$  the conjecture follows from Gompf's results [13].

There is another interesting

**PROBLEM 1.** *Does there exist a symplectic simply connected closed manifold satisfying the Hard Lefschetz Condition and with no Kähler structure? Or, more strongly, nonformal?*

Another conjecture proposed by Benson and Gordon [6] is very close to the subject of our talk.

**CONJECTURE 2.** *If a compact quotient of a simply connected completely solvable Lie group by a lattice has a Kähler structure, then it is a complex torus.*

Recall that a solvable group  $G$  is called completely solvable if all eigenvalues of the operator  $\text{ad} : \mathcal{G} \rightarrow \mathcal{G}$  on its Lie algebra are real.

During their research the authors were partially supported by the Russian Foundation for Basic Research (grants 99-01-01202a (I.K.B.), 00-01-00915 and 00-15-99252 (I.A.T.)) and the Switzerland National Foundation for Basic Research (I.K.B.)

### References

- [1] Alaniya, L. A. Cohomologies with local coefficients of some nilmanifolds. *Russian Math. Surveys* **54:5** (1999), 1019–1020.
- [2] Artel'nykh, I. V. Massey products and the Buchstaber spectral sequence. *Russian Math. Surveys* **55:3** (2000).
- [3] Babenko, I. K., and Taimanov, I. A. On existence of nonformal simply connected symplectic manifolds, *Russian Math. Surveys* **53:5** (1998), 1082–1083; also: [math.SG/9810065](#).
- [4] Babenko, I. K., and Taimanov, I. A. On nonformal simply connected symplectic manifolds. *Siberian Math. Journal* **41:2** (2000), 204–217; also: [math.SG/9811167](#).
- [5] Babenko, I. K., and Taimanov, I. A. Massey products in symplectic manifolds. *Math. Sbornik* **191:8** (2000), 3–44 (in Russian); also: [math.SG/9911132](#).
- [6] Benson, C., and Gordon, C. Kähler and symplectic structures on nilmanifolds, *Topology* **27** (1988), 513–518.
- [7] Buchstaber, V. M. Groups of polynomial transformations of a line, nonformal symplectic manifolds, and the Landweber-Novikov algebra. *Russian Math. Surveys* **54:4** (1999), 837–838.
- [8] Cordero, L. A., Fernandez, M., and Gray, A. Symplectic manifolds with no Kähler structure, *Topology* **25** (1986), 375–380.
- [9] Deligne, P., Griffiths, P., Morgan, J., and Sullivan, D. Real homotopy theory of Kähler manifolds, *Invent. Math.* **19** (1975), 245–274.
- [10] Donaldson, S., Symplectic submanifolds and almost-complex geometry, *J. Diff. Geom.* **44** (1996), 666–705.
- [11] Fernandez, M., Gotay, M., and Gray, A. Compact parallelizable four dimensional symplectic and complex manifolds, *Proc. Amer. Math. Soc.* **103** (1988), 1209–1212.
- [12] Fernandez, M., Gray A., and Morgan J.W. Compact symplectic manifolds with free circle actions, and Massey products, *Michigan Math. J.* **38** (1991), 271–283.
- [13] Gompf, R. E. A new construction of symplectic manifolds, *Ann. of Math.* **142** (1995), 527–595.
- [14] Gromov, M. L. A topological technique for the construction of solutions of differential equations and inequalities, *Actes Congrès Intern. Math.* (Nice, 1970), Gauthier-Villars, Paris, No. 2, 1971, 221–225.
- [15] Guan, D., Examples of compact holomorphic symplectic manifolds which are not Kählerian. II, *Invent. Math.* **121** (1995), 135–145.
- [16] Kraines, D., Massey higher products, *Trans. of Amer. Math. Soc.* **124** (1966), 431–449.
- [17] Lupton, G., and Oprea, J. Symplectic manifolds and formality, *J. Pure Appl. Algebra* **91** (1994), 193–207.
- [18] May, J. P., Matric Massey products, *J. Algebra.* **12** (1969), 533–568.
- [19] McDuff, D. Examples of symplectic simply connected manifolds with no Kähler structure, *J. Diff. Geom.* **20** (1984), 267–277.
- [20] Millionshchikov, D. V., Cohomology of nilmanifolds and Gontcharova's theorem. *Russian Math. Surveys*, to appear.
- [21] Rudyak, Yu., and Tralle, A. On Thom spaces, Massey products, and non-formal symplectic manifolds. *Internat. Math. Res. Notices* **10** (2000), 495–513; also: [math.DG/9907035](#).
- [22] Sullivan, J. Infinitesimal computations in topology, *Publ. IHES* **47** (1978), 269–331.



- [23] Thurston, W. Some simple examples of compact symplectic manifolds, Proc. Amer. Math. Soc. **55** (1976), 467–468.
- [24] Tischler, D. Closed 2-forms and an embedding theorem for symplectic manifolds, J. Diff. Geom. **12** (1977), 229–235.
- [25] Tralle, A., and Oprea, J. Symplectic Manifolds with no Kähler Structure, Lect. Notes in Math. **1661**, Springer, Berlin Heidelberg, 1997.

DÉPARTEMENT DE MATHÉMATIQUES,, UNIVERSITÉ MONTPELLIER II,, PLACE EUGÈNE BATAILLON,, 34095 MONTPELLIER CEDEX 5, FRANCE. ON LEAVE OF ABSENCE FROM, DEPARTMENT OF MATHEMATICS AND MECHANICS, OF THE MOSCOW STATE UNIVERSITY

*E-mail address:* babenko@math.univ-montp2.fr

INSTITUTE OF MATHEMATICS,, 630090 NOVOSIBIRSK, RUSSIA

*E-mail address:* taimanov@math.nsc.ru