

Dirac Operators and Conformal Invariants of Tori in 3-Space

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Abstract—It is proved that the multipliers of the Floquet functions that are associated with immersions of tori into \mathbb{R}^3 (or S^3) form a complex curve in \mathbb{C}^2 . The properties of this curve are studied. In addition, it is shown how the curve and its construction are related to the method of finite-gap integration, the Willmore functional, and harmonic mappings of the 2-torus into S^3 .

1. INTRODUCTION

In this paper, we show how to assign to any torus immersed into the three-space \mathbb{R}^3 or the unit three-sphere S^3 a complex curve such that the immersion is described by meromorphic functions defined on this curve (generically, a Riemann surface of infinite genus). We call this curve the spectrum of a torus (with a fixed conformal parameter). This spectrum has many interesting properties and, in particular, relates to the Willmore functional whose value is encoded in it.

Spectra for tori in \mathbb{R}^3 were introduced in [40]. In this text, we do this also for immersed tori in S^3 .

Our conjecture that the spectrum of a torus in \mathbb{R}^3 is invariant under conformal transformations of \mathbb{R}^3 was proved modulo some analytic facts by Grinevich and Schmidt [14]. In fact, their proof is rather physical, as should be expected because the construction of the spectrum originates from soliton theory.

In this paper, we give a complete proof of the conformal invariance of the spectra for isothermic tori. This case already covers many interesting surfaces such as constant mean curvature tori and tori of revolution in \mathbb{R}^3 .

Some spectral curves of finite genus already appeared in the studies of harmonic tori in S^3 by Hitchin [18] and of constant mean curvature (CMC) tori in \mathbb{R}^3 by Pinkall and Sterling [34]. It was shown that such tori are expressed in terms of algebraic functions corresponding to these complex curves [18, 4]. We show that, for minimal tori in S^3 and CMC tori in \mathbb{R}^3 , these spectral curves are particular cases of the general spectrum.

The general construction is based on the global Weierstrass representation of closed surfaces introduced in [38, 40] and a general construction of the Floquet (or Bloch) variety for a periodic differential operator. The existence of this variety is derived from the Keldysh theorem, but an effective construction that gives more information about the analytic behavior of this complex curve was proposed by Krichever in [24], who used perturbation methods. This variety is as follows. Take an immersed torus with the induced metric $e^{2\alpha} dz d\bar{z}$ and consider differential

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operators

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \quad \text{with } U = \frac{1}{2}He^\alpha \quad \text{for a torus in } \mathbb{R}^3,$$

$$\mathcal{D}^S = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} V & 0 \\ 0 & \bar{V} \end{pmatrix} \quad \text{with } V = \frac{1}{2}(H - i)e^\alpha \quad \text{for a torus in } S^3,$$

where H is the mean curvature.

Let Λ be the period lattice of a torus, which means that the torus is an immersion of \mathbb{C}/Λ with a conformal parameter $z \in \mathbb{C}$ on it. Take a basis γ_1, γ_2 for Λ , which is also considered as a basis for $H_1(T^2) \approx \Lambda$. Now, consider all solutions ψ to the equations

$$\mathcal{D}\psi = 0 \quad \text{or} \quad \mathcal{D}^S\psi = 0$$

satisfying the following conditions:

$$\psi(z + \gamma_j) = \mu_j\psi(z), \quad j = 1, 2.$$

These are Floquet(-Bloch) functions, and the pairs (μ_1, μ_2) form a complex curve in \mathbb{C}^2 . This is the Floquet zero-level spectrum of \mathcal{D} and, by definition, this is the spectrum of the immersed torus. The analytic properties of this curve are described by the Pretheorem, which is a modification of its analogues for two-dimensional scalar Schrödinger and heat operators proved in [24]. It is clear that the proof of the Pretheorem can be obtained by a slight modification of the reasonings of [24].

One of the most interesting properties of this construction is its relation to a conformal geometry and to the Willmore functional, which equals

$$4 \int_{\mathbb{C}/\Lambda} U^2 dx \wedge dy \quad \text{or} \quad 4 \int_{\mathbb{C}/\Lambda} |V|^2 dx \wedge dy$$

for tori in \mathbb{R}^3 or S^3 , where $z = x + iy$.

The global Weierstrass representation of closed surfaces represents any closed surface Σ in terms of a solution to the equation $\mathcal{D}\psi = 0$ (a harmonic spinor), where ψ takes values in some bundle over the constant curvature surface Σ_0 that is conformally equivalent to Σ (see Theorems 1–3 in Section 2.3) [38, 40]. The Willmore functional $\int_{\Sigma}(H^2 - K) d\mu = 4 \int_{\Sigma_0} U^2 dx \wedge dy - 2\pi\chi(\Sigma)$ measures the L_2 -norm of the potential U of the surface. For small values of this functional, the equation $\mathcal{D}\psi = 0$ does not admit solutions that describe closed surfaces in \mathbb{R}^3 ; this explains the physical meaning of lower bounds for the Willmore functional proposed by the Willmore conjecture and its generalizations. This also gives a hint that the spectral properties of \mathcal{D} should have a geometric meaning. We will discuss this in detail in Section 4.4.

In [41], it was established that the dimension of the kernel of \mathcal{D} gives quadratic lower estimates for the Willmore functional for spheres. Actually, we proved the following inequality for spheres with one-dimensional potentials U (examples of them are spheres of revolution):

$$4 \int_{\Sigma_0} U^2 dx \wedge dy \geq \pi N^2 \quad \text{with } N = \dim_{\mathbb{C}} \ker \mathcal{D}. \tag{1}$$

The proof of this inequality is based on the inverse scattering problem for the one-dimensional Dirac operator and also works for general Dirac operators with S^1 -symmetry (i.e., with one-dimensional potentials) on special spinor bundles over the 2-sphere. We conjectured that (1) holds for all spheres. Note that it cannot be improved, and the equality is achieved on “soliton spheres” [41].

Another treatment of the global representation belongs to Pedit and Pinkall, who suggested considering spinor \mathbb{C}^2 -bundles introduced in [38] as quaternionic line bundles and harmonic spinors as quaternionic holomorphic sections of such bundles. This enabled them to apply the ideas of algebraic geometry to surface theory and to generalize this representation for surfaces in \mathbb{R}^4 [32]. Very recently, they managed to relate (1) to the quaternionic analogue of the Plücker formula and thereby to prove our conjecture, i.e., to establish inequality (1) for all spheres together with its generalizations for higher genus surfaces. Friedrich has shown how to derive from this inequality lower estimates, for the eigenvalues of the Dirac operator, that are quadratic in the multiplicities of eigenvalues.

This paper is organized as follows.

In Section 2, we recall the notion of the Weierstrass representation.

In Section 3, we prove that the multipliers (μ_1, μ_2) of Floquet functions form a spectral curve in \mathbb{C}^2 and discuss its analytic properties.

In Section 4, we show how to assign such a spectrum to an immersed torus in \mathbb{R}^3 , show how the Willmore functional appears in this picture, and discuss the modern state of the Willmore conjecture.

In Section 5, we show that the spectra of CMC and isothermic tori are particular cases of the spectrum defined in Section 4 and prove our conjecture that the spectra of an isothermic torus and its dual surface coincide [39].

In Section 6, we show how to assign a spectral curve to a torus in S^3 and prove that, for a minimal torus in S^3 , it coincides with a spectral curve defined by Hitchin for harmonic tori in S^3 [18].

In Section 7, we prove that the spectrum of an isothermic torus in \mathbb{R}^3 is invariant with respect to the conformal transformations of $\bar{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\}$ that leave the torus in \mathbb{R}^3 .

2. THE WEIERSTRASS REPRESENTATION

2.1. Basic equations of surface theory in \mathbb{R}^3 . Let \mathcal{U} be a domain in \mathbb{R}^2 , with coordinates (x^1, x^2) , regularly immersed into \mathbb{R}^3 :

$$F: \mathcal{U} \rightarrow \mathbb{R}^3.$$

At every point $p \in \mathcal{U}$, the vectors

$$F_1 = \frac{\partial F}{\partial x^1}, \quad F_2 = \frac{\partial F}{\partial x^2}, \quad N = \frac{[F_1 \times F_2]}{|F_1| \cdot |F_2|}$$

form a linear basis $\sigma = (F_1, F_2, N)^\top$ for \mathbb{R}^3 , where F_1 and F_2 are tangent vectors to the surface $\Sigma = F(\mathcal{U})$ and N is a unit normal vector. The variables (x^1, x^2) are local coordinates on Σ , and the induced metric on it is

$$\mathbf{I} = g_{kl} dx^k dx^l \quad (\text{the first fundamental form}),$$

where $g_{kl} = \langle F_k, F_l \rangle$ and $\langle a, b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$.

The derivatives of the basic vectors are expanded in this basis as

$$\frac{\partial^2 F}{\partial x^k \partial x^l} = \Gamma_{kl}^j \frac{\partial F}{\partial x^j} + b_{kl} N, \quad \frac{\partial N}{\partial x^k} = -b_k^j \frac{\partial F}{\partial x^j}, \quad (2)$$

where Γ_{kl}^j are the *Christoffel symbols*, $\mathbf{II} = b_{kl} dx^k dx^l$ is the *second fundamental form*, and $b_k^j = g^{jl} b_{lk}$. Equations (2) are the *Gauss–Weingarten derivation equations* and have the form

$$\frac{\partial \sigma}{\partial x^1} = \mathbf{U} \sigma, \quad \frac{\partial \sigma}{\partial x^2} = \mathbf{V} \sigma, \quad (3)$$

where \mathbf{U} and \mathbf{V} are 3×3 matrices. The compatibility conditions for (3) are given by the *Gauss-Codazzi equations*

$$\frac{\partial^2 \sigma}{\partial x^1 \partial x^2} - \frac{\partial^2 \sigma}{\partial x^2 \partial x^1} = \left(\frac{\partial \mathbf{U}}{\partial x^2} - \frac{\partial \mathbf{V}}{\partial x^1} + [\mathbf{U}, \mathbf{V}] \right) \sigma = 0,$$

which are equivalent to the *zero-curvature equations*

$$\frac{\partial \mathbf{U}}{\partial x^2} - \frac{\partial \mathbf{V}}{\partial x^1} + [\mathbf{U}, \mathbf{V}] = 0 \quad (4)$$

for the connection $(\frac{\partial}{\partial x^1} - \mathbf{U}, \frac{\partial}{\partial x^2} - \mathbf{V})$.

At every point p of the surface, the fundamental forms are diagonalized as

$$\mathbf{I}(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{II}(p) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},$$

and the *principal curvatures* k_1 and k_2 satisfy the equation $\det(b_{kl} - kg_{kl}) = 0$, which, divided by $\det g_{jk}$, takes the form $k^2 - 2Hk + K = 0$, where H is the *mean curvature* and K is the *Gaussian curvature*:

$$H = \frac{k_1 + k_2}{2}, \quad K = k_1 k_2.$$

A point p is called an *umbilic point* if the principal curvatures coincide at p : $k_1 = k_2$, which is equivalent to $H^2 - K = 0$.

Let $z = x^1 + ix^2$ be a conformal parameter on the surface, i.e., the first fundamental form is

$$\mathbf{I} = e^{2\alpha(z, \bar{z})} dz d\bar{z},$$

which means that

$$\langle F_z, F_z \rangle = \langle F_{\bar{z}}, F_{\bar{z}} \rangle = 0, \quad \langle F_z, F_{\bar{z}} \rangle = \frac{1}{2} e^{2\alpha}.$$

The family $\tilde{\sigma} = (F_z, F_{\bar{z}}, N)^\top$ is a basis for \mathbb{C}^3 , and the Gauss-Weingarten equations are written as

$$\tilde{\sigma}_z = \tilde{\mathbf{U}}\sigma, \quad \tilde{\sigma}_{\bar{z}} = \tilde{\mathbf{V}}\sigma,$$

with

$$\tilde{\mathbf{U}} = \begin{pmatrix} 2\alpha_z & 0 & A \\ 0 & 0 & B \\ -2e^{-2\alpha}B & -2e^{-2\alpha}A & 0 \end{pmatrix}, \quad \tilde{\mathbf{V}} = \begin{pmatrix} 0 & 0 & B \\ 0 & 2\alpha_{\bar{z}} & \bar{A} \\ -2e^{-2\alpha}\bar{A} & -2e^{-2\alpha}B & 0 \end{pmatrix},$$

$A = \langle F_{zz}, N \rangle$, and $B = \langle F_{z\bar{z}}, N \rangle$. The second fundamental form equals

$$\mathbf{II} = (2B + (A + \bar{A}))(dx^1)^2 + 2i(A - \bar{A}) dx^1 dx^2 + (2B - (A + \bar{A}))(dx^2)^2,$$

and we have

$$H = 2Be^{-2\alpha}, \quad K = 4(B^2 - A\bar{A})e^{-4\alpha}.$$

Now, the Gauss-Codazzi equations $\tilde{\mathbf{U}}_{\bar{z}} - \tilde{\mathbf{V}}_z + [\tilde{\mathbf{U}}, \tilde{\mathbf{V}}] = 0$ take the form

$$\alpha_{z\bar{z}} + e^{-2\alpha}(B^2 - A\bar{A}) = 0, \quad A_{\bar{z}} - B_z + 2\alpha_z B = 0. \quad (5)$$

The first of them is the *Gauss egregium theorem*, and the other equation

$$A_{\bar{z}} = \frac{1}{2} H_z e^{2\alpha}$$

splits into two real-valued equations.

A quadratic differential $\omega = A dz^2$ is called the *Hopf differential* and has important geometrical properties. For instance, ω vanishes at a point if and only if this is an umbilic point.

It is said that the *Gauss map* of a surface $\Sigma = F(\mathcal{U})$,

$$G: \Sigma \rightarrow S^2, \quad G(p) = N(p),$$

is *harmonic* if $\Delta G(p) = \lambda(p)N(p)$, where Δ is the *Laplace–Beltrami operator*, $\Delta = 4e^{-2\alpha}\partial\bar{\partial}$. We have

$$\Delta F = 2HN.$$

Since $N_{z\bar{z}} = -2e^{-2\alpha}(\bar{A}_z F_z + A_{\bar{z}} F_{\bar{z}} + (A\bar{A} + B^2)N)$, we conclude that

- the Gauss map G is harmonic if and only if the Hopf differential ω is holomorphic, which, by (5), is equivalent to $H_z = H_{\bar{z}} = 0$, i.e., $H = \text{const}$ and Σ is a constant mean curvature (CMC) surface [36].

There are two other important classes of surfaces:

- (1) a surface is called *minimal* if $H = 0$,
- (2) a surface is called *isothermic* if there is a conformal parameter on it such that $\text{Im } A = 0$.

2.2. The local representation of a surface. Here, we follow [22, 38].

Denote by \mathcal{Q} a quadric in \mathbb{C}^3 defined by the equation

$$Z_1^2 + Z_2^2 + Z_3^2 = 0, \quad Z = (Z_1, Z_2, Z_3) \in \mathbb{C}^3.$$

For a conformal parameter z on a surface Σ , there is a mapping

$$f: \mathcal{U} \xrightarrow{F} \Sigma \rightarrow \mathcal{Q}, \quad \text{where } f(p) = F_z(p), \tag{6}$$

satisfying the conditions

$$\text{Im } \frac{\partial f}{\partial \bar{z}} = 0. \tag{7}$$

It is clear that any mapping $f: \mathcal{U} \rightarrow \mathcal{Q}$ satisfying (7) has the form $f = \partial_z \Phi$ for some real-valued function Φ and therefore has the form (6) for some surface.

The set \mathcal{Q} is parameterized by $(\varphi_1, \varphi_2) \in \mathbb{C}^2$ as follows:

$$Z_1 = \varphi_1^2 - \varphi_2^2, \quad Z_2 = i(\varphi_1^2 + \varphi_2^2), \quad Z_3 = 2\varphi_1\varphi_2.$$

For simplicity, renormalize φ as follows:

$$\psi_1 = \sqrt{\frac{2}{i}} \varphi_1, \quad \psi_2 = \sqrt{\frac{2}{i}} \bar{\varphi}_2.$$

In terms of ψ , equation (7) is expressed as

$$\mathcal{D}\psi = 0, \tag{8}$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \tag{9}$$

is a Dirac operator with a real-valued potential $U(z, \bar{z})$.

Now, if $F(p_0) = (x_0^1, x_0^2, x_0^3) \in \mathbb{R}^3$, then the surface is described by the *Weierstrass formulas*

$$\begin{aligned} x^1(p) &= x^1(p_0) + \int_{p_0}^p \left(\frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2) dz - \frac{i}{2}(\psi_2^2 + \bar{\psi}_1^2) d\bar{z} \right), \\ x^2(p) &= x^2(p_0) + \int_{p_0}^p \left(\frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2) dz + \frac{1}{2}(\psi_2^2 - \bar{\psi}_1^2) d\bar{z} \right), \\ x^3(p) &= x^3(p_0) + \int_{p_0}^p (\psi_1 \bar{\psi}_2 dz + \bar{\psi}_1 \psi_2 d\bar{z}). \end{aligned} \tag{10}$$

Different versions of these local formulas were known before (see, for instance, [21] and comments in [38, 40]); in the form (10), they were introduced by Konopelchenko [22], who, when working with the formulas of Eisenhart [11], elaborated them into a form most convenient for applications. He considered them as applied to the construction of some surfaces via solutions to (8) and to the definition of soliton deformations of “induced surfaces”; however, as we can see, this gives a general local construction of surfaces.

Straightforward computations yield

$$U = \frac{He^\alpha}{2}, \quad e^\alpha = |\psi_1|^2 + |\psi_2|^2, \tag{11}$$

and we see that, for $H = 0$, formulas (10) reduce to the classical formulas for minimal surfaces.

Compute $N = e^{-\alpha}(-i(\bar{\psi}_1 \bar{\psi}_2 - \psi_1 \psi_2), -(\bar{\psi}_1 \bar{\psi}_2 + \psi_1 \psi_2), |\psi_2|^2 - |\psi_1|^2)$ and, taking (8) into account, derive

$$A = \langle F_{zz}, N \rangle = \psi_{1z} \bar{\psi}_2 - \bar{\psi}_{2z} \psi_1, \quad B = \langle F_{z\bar{z}}, N \rangle = Ue^\alpha. \tag{12}$$

The Gauss–Weingarten equations written in terms of ψ describe the deformations of ψ , and the first half of them is just the Dirac equation (8). To obtain the other half of equations, differentiate e^α with respect to z ,

$$\alpha_z e^\alpha = \bar{\psi}_1 \psi_{1z} + \psi_2 \bar{\psi}_{2z},$$

and, taking (12) into account, arrive at

$$\psi_{1z} = \alpha_z \psi_1 + Ae^{-\alpha} \psi_2, \quad \psi_{2\bar{z}} = -\bar{A}e^{-\alpha} \psi_1 + \alpha_{\bar{z}} \psi_2.$$

Now, the Gauss–Weingarten equations are written as

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z & Ae^{-\alpha} \\ -U & 0 \end{pmatrix} \right] \psi = \left[\frac{\partial}{\partial \bar{z}} - \begin{pmatrix} 0 & U \\ -\bar{A}e^{-\alpha} & \alpha_{\bar{z}} \end{pmatrix} \right] \psi = 0, \tag{13}$$

and the compatibility conditions for them, the Gauss–Codazzi equations, are

$$A_{\bar{z}} = (U_z - \alpha_z U)e^\alpha, \quad \alpha_{z\bar{z}} + U^2 - A\bar{A}e^{-2\alpha} = 0. \tag{14}$$

In fact, equation (8) already provides a compatibility condition for the existence of a surface with the Gauss map given by ψ . The other half of equations (13) follows from it.

In [13], Friedrich explained this representation within the classical theory of Dirac operators. We also would like to mention a paper by Matsutani [30], who considered this representation from the physical point of view.

2.3. The global Weierstrass representation. Here, we follow our paper [38], where the global representation was introduced.

Any closed oriented surface Σ in \mathbb{R}^3 is conformally equivalent to a constant curvature surface Σ_0 , and a choice of a conformal parameter z on Σ means that a conformal equivalence $\Sigma_0 \rightarrow \Sigma$ is fixed.

To define a compact oriented surface globally via formulas (10), we have to introduce bundles over surfaces and Dirac operators on them.

Consider two cases:

1. *Tori.* Let Σ be a torus immersed into \mathbb{R}^3 . Then, it is conformally equivalent to a flat torus $\Sigma_0 = \mathbb{C}/\Lambda$, and z is a conformal parameter. The vector function ψ is expanded to a section of a \mathbb{C}^2 -fiber bundle over Σ defined by the monodromy rules

$$\psi(z + \gamma) = \varepsilon(\gamma)\psi(z), \tag{15}$$

where $\gamma \in \Lambda$ and $\varepsilon: \Lambda \rightarrow \{\pm 1\}$ is a character of Λ , i.e., a homomorphism to $\{\pm 1\}$. The Dirac operator \mathcal{D} is defined on smooth sections of this bundle, and

$$U(z + \gamma) = U(z). \tag{16}$$

Theorem 1 [38]. *Formulas (15) and (16) define a \mathbb{C}^2 -bundle over a flat torus Σ_0 . To any section ψ of this bundle satisfying the Dirac equation (8), there corresponds a surface in \mathbb{R}^3 defined by formulas (10) up to translations in \mathbb{R}^3 . This surface covers Σ_0 , and its Gauss map descends through Σ_0 .*

2. *Surfaces of genus $g > 1$.* Let Σ_0 be a hyperbolic surface conformally equivalent to a surface Σ immersed into \mathbb{R}^3 and z be a conformal parameter. The surface Σ_0 is isometric to \mathcal{H}/Λ , where \mathcal{H} is the Lobachevsky upper-half plane and Λ is a discrete subgroup of $PSL(2, \mathbb{R})$.

Any element $\gamma \in \Lambda \subset PSL(2, \mathbb{R})$ is represented by a matrix

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1.$$

The action on \mathcal{H} is

$$z \rightarrow \gamma(z) = \frac{az + b}{cz + d}.$$

Define a \mathbb{C}^2 -bundle over Σ_0 by the monodromy rules

$$\psi_1(\gamma(z)) = (cz + d)\psi_1(z), \quad \psi_2(\gamma(z)) = (\bar{c}z + d)\psi_2(z). \tag{17}$$

The Dirac operator is defined on smooth sections of this bundle, and

$$U(\gamma(z)) = |cz + d|^2 U(z). \tag{18}$$

Theorem 2 [38]. *Formulas (17) and (18) define a \mathbb{C}^2 -bundle over a hyperbolic surface Σ_0 . To any section ψ of this bundle satisfying the Dirac equation (8), there corresponds a surface in \mathbb{R}^3 defined by formulas (10) up to translations in \mathbb{R}^3 . This surface covers Σ_0 , and its Gauss map descends through Σ_0 .*

Notice that $\psi_1\sqrt{dz}$ and $\bar{\psi}_2\sqrt{dz}$ are defined modulo ± 1 . In fact, they are spinors, and, therefore, we will also call ψ_1 and ψ_2 spinors. There are 2^{2g} such nonequivalent spinor bundles over Σ_0 , where g is the genus of Σ_0 .

This representation of compact oriented surfaces in \mathbb{R}^3 in terms of solutions of the Dirac equation (i.e., harmonic spinors) in spinor bundles over constant curvature surfaces is called the *Weierstrass representation* of surfaces.

Theorem 3 ([38] for real-analytic surfaces and [40] for C^3 -regular surfaces). *Every smooth closed oriented surface in \mathbb{R}^3 has a Weierstrass representation.*

We see from the direct construction in Section 2.2 that, for a surface with a fixed conformal parameter, such a representation is unique. This gives rise to the following definition.

Definition 1. Let (Σ, z) be an immersed surface with a fixed conformal parameter. Then, the potential U of its Weierstrass representation is called the *potential of a surface*.

The criterion of closedness of surfaces constructed by formulas (10) is as follows.

Proposition 1. *A surface represented by a harmonic spinor ψ over a compact surface Σ_0 is closed if and only if*

$$\int_{\Sigma_0} \bar{\psi}_1^2 d\bar{z} \wedge \omega = \int_{\Sigma_0} \psi_2^2 d\bar{z} \wedge \omega = \int_{\Sigma_0} \bar{\psi}_1 \psi_2 d\bar{z} \wedge \omega = 0$$

for any holomorphic differential on Σ_0 .

This proposition was proved by M. Schmidt for tori (in this case, $\omega = \text{const} \cdot dz$) and by the author for higher genera surfaces [41].

One of the most important properties of this representation is the equality

$$4 \int_{\Sigma_0} U^2 dx \wedge dy = \int_{\Sigma} H^2 d\mu,$$

where $d\mu$ is the measure given by the induced metric on Σ [38]. The functional

$$\mathcal{W}(\Sigma) = \int_{\Sigma} (H^2 - K) d\mu = \int_{\Sigma} \left(\frac{k_1 - k_2}{2} \right)^2 d\mu$$

is called the *Willmore functional*. By the Gauss–Bonnet theorem, for closed oriented surfaces, it equals

$$\mathcal{W}(\Sigma) = \int_{\Sigma} H^2 d\mu - 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface and, therefore, for tori $\mathcal{W} = \int_{\Sigma} H^2 d\mu$.

The famous Willmore conjecture says that, for tori, this functional is greater than or equal to $2\pi^2$. We will discuss it in Section 4.4.

We see that the main advantage of the global Weierstrass representation lies in using the spectral properties of \mathcal{D} for studying the conformal geometry of surfaces. The present paper is devoted to developing this idea for tori.

3. THE FLOQUET SPECTRUM OF A PERIODIC OPERATOR

3.1. Floquet functions and the spectral curve. Let L be a differential operator acting on functions or vector functions on \mathbb{R}^n , and let the coefficients of L be periodic with respect to translations by vectors from a lattice Λ isomorphic to \mathbb{Z}^n .

To any vector $\gamma \in \Lambda$, there corresponds a translation operator T_γ ,

$$T_\gamma f(x) \rightarrow f(x + \gamma).$$

Since L is Λ -periodic, if $Lf = \lambda f$, then $LT_\gamma f = \lambda T_\gamma f$. Moreover,

$$[T_\gamma, L] = 0,$$

and there are common eigenfunctions of these commuting operators. Such functions are called *Floquet* (or *Bloch*) functions. A rigorous definition is as follows.

Definition 2. A function $f: \mathbb{R}^n \rightarrow \mathbb{C}$ is called a Floquet function of a Λ -periodic operator L if

$$Lf = Ef \quad \text{and} \quad f(x + \gamma) = e^{2\pi i \langle k, \gamma \rangle} f(x)$$

for all $\gamma \in \Lambda$ and some $E \in \mathbb{C}$. The quantities k_1, \dots, k_n are called the quasimomenta of f .

Any Floquet function defines a multiplier homomorphism $\mu: \Lambda \rightarrow \mathbb{C}$:

$$f(x + \gamma) = \mu(\gamma)f(x).$$

Consider the case when $L = \mathcal{D}$ is the two-dimensional Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

with a double-periodic continuous potential $U(z)$, where $z = x^1 + ix^2 \in \mathbb{C}$.

Theorem 4. There is an analytic set $Q(\mathcal{D}) \subset \mathbb{C}^3$ of positive codimension such that there exists a Floquet function of \mathcal{D} with quasimomenta $k = (k_1, k_2)$ and eigenvalue E if and only if $(k_1, k_2, E) \in Q(\mathcal{D})$.

The intersection of this set with the plane $\lambda = 0$ is an analytic set $Q_0(\mathcal{D})$ of complex dimension one.

Proof. Take a constant C such that the operator $\mathcal{D} + C$ is invertible on $L_2(T^2) = L_2(\mathbb{C}/\Lambda)$. Then, consider a polynomial operator pencil

$$A_{k,E} = 1 + \begin{pmatrix} U - (C + E) & \pi(k_2 + ik_1) \\ \pi(k_2 - ik_1) & U - (C + E) \end{pmatrix} \begin{pmatrix} C & \partial \\ -\bar{\partial} & C \end{pmatrix}^{-1}.$$

Since, for any function g , we have

$$e^{2\pi i \langle k, x \rangle} [A_{k,E}(\mathcal{D} + C)]g = [\mathcal{D} - E]e^{2\pi i \langle k, x \rangle} g,$$

there is a Floquet function $f(x)$ with quasimomenta (k_1, k_2) and eigenvalue E if and only if there is a Λ -periodic function $g(x)$ satisfying the equation

$$A_{k,E}[\mathcal{D} + C]g = 0.$$

Such a solution exists if and only if there is a solution $\varphi \in L_2(T^2)$ to the equation

$$A_{k,E}\varphi = 0. \tag{19}$$

If such a solution φ exists, then $f = e^{2\pi i \langle k, x \rangle} [\mathcal{D} + C]^{-1}\varphi$ is the desired Floquet function. The operator pencil

$$1 - A_{k,E}$$

is polynomial in k_1, k_2 , and E . Since U is bounded, the multiplication operator

$$\times U: L_2(T^2) \rightarrow L_2(T^2)$$

is bounded and the pencil $1 - A_{k,E}$ consists of compact operators on $L_2(T^2)$.

Now, we apply the Keldysh theorem (or the polynomial Fredholm alternative) to it. This theorem reads that there is a regularized determinant $\widetilde{\det} A_{k,E}$ of this pencil that is analytic in k_1, k_2 , and E and such that equation (19) is solvable if and only if $\widetilde{\det} A_{k,E} = 0$ [19, 20]. Now, it remains to put

$$Q(\mathcal{D}) = \{\widetilde{\det} A_{k,E} = 0\}.$$

In the same manner, for the pencil $1 - A_{k,0}$, we obtain a complex curve

$$Q_0(\mathcal{D}) = \{\widetilde{\det} A_{k,0} = 0\} \subset \mathbb{C}^2.$$

As is shown by perturbation methods, the codimensions of these sets are positive (see [40]). The nontriviality of such determinants follows from their construction by Keldysh. This proves the theorem.

We applied this method to the operators $\Delta + u$ and $\partial_t - \Delta$ in 1985. Later, it became known to us that Kuchment also proposed the same approach in [25]; therefore, we did not publish our paper, which is cited in [24] as an unpublished paper. The theory of such determinants is developed in [26], and one can show that they are entire functions of k and E .

We will discuss another, more effective but technically difficult, approach of Krichever in Section 3.3.

Recall that the dual lattice $\Lambda^* \subset \mathbb{C}^2$ consists of vectors γ^* such that $\langle \gamma, \gamma^* \rangle = 0$ for any $\gamma \in \Lambda$. The following proposition is evident.

Proposition 2. *The sets $Q(\mathcal{D})$ and $Q_0(\mathcal{D})$ are invariant with respect to translations by the vectors from Λ^* :*

$$k_1 \rightarrow k_1 + \operatorname{Re} \gamma^*, \quad k_2 \rightarrow k_2 + \operatorname{Im} \gamma^*. \tag{20}$$

Definition 3. $Q_0(\mathcal{D})$ is called the (zero) Floquet spectral data of \mathcal{D} . The genus of the normalization of $Q_0(\mathcal{D})/\Lambda^*$ is called the spectral genus of \mathcal{D} .

Another two properties of the Floquet spectrum are easily derived in a manner conventional in soliton theory.

Proposition 3. *If U is real-valued, then $Q_0(\mathcal{D})$ is invariant under an antiholomorphic involution $k \rightarrow -\bar{k}$.*

Proof. If $(\psi_1, \psi_2)^\top$ is a Floquet function with quasimomenta (k_1, k_2) , then $(\bar{\psi}_2, -\bar{\psi}_1)^\top$ is a Floquet function with quasimomenta $(-\bar{k}_1, -\bar{k}_2)$. This proves the proposition.

Proposition 4. $Q_0(\mathcal{D})$ is invariant under a holomorphic involution $k \rightarrow -k$.

Proof. Consider the pencil $L_k = A_{k,0}(\mathcal{D} + C)$. We have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{L_k^*} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = L_{-k}. \tag{21}$$

The index of a Fredholm operator A is $\operatorname{ind} A = \dim \ker A - \dim \ker A^*$. Since \mathcal{D} is self-adjoint, its index vanishes. The operators L_k have the same principal terms as \mathcal{D} and, by the index theorem, their indices also vanish. Hence, if $k \in Q_0(\mathcal{D})$, then $\dim L_k = \dim L_k^* > 0$ and identity (21) implies that $\dim L_{-k}^* > 0$. Therefore, $(-k) \in Q_0(\mathcal{D})$, and this proves the proposition.

Given a basis (γ_1, γ_2) for Λ , we have a mapping

$$\mathcal{M}: Q_0(\mathcal{D})/\Lambda^* \rightarrow \mathbb{C}^2, \quad \mathcal{M}(k) = (e^{2\pi i \langle k, \gamma_1 \rangle}, e^{2\pi i \langle k, \gamma_2 \rangle}),$$

which maps quasimomenta into multipliers.

The submanifold $\mathcal{M}(Q_0(\mathcal{D})/\Lambda^*) \subset \mathbb{C}^2$ is generically singular and its normalization is the normalization of $Q_0(\mathcal{D})/\Lambda^*$.

Definition 4. A complex curve Γ that is the normalization of $Q_0(\mathcal{D})/\Lambda^*$ is called the spectral curve of \mathcal{D} .

A normalization generically consists in unsticking double points: a pair of points of Γ corresponding to a double point of $Q_0(\mathcal{D})/\Lambda^*$ is called a resonance pair.

The definition of \mathcal{M} depends on the choice of a basis for Λ . If $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ is another basis for Λ such that

$$\begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

then the multipliers $(\mu_1, \mu_2) = (\mu(\gamma_1), \mu(\gamma_2))$ and $(\tilde{\mu}_1, \tilde{\mu}_2) = (\mu(\tilde{\gamma}_1), \mu(\tilde{\gamma}_2))$ are related as follows:

$$\tilde{\mu}_1 = \mu_1^a \mu_2^b, \quad \tilde{\mu}_2 = \mu_1^c \mu_2^d, \tag{22}$$

and the sets of multipliers $\{(\mu_1, \mu_2)\}$ and $\{(\tilde{\mu}_1, \tilde{\mu}_2)\}$ with respect to different bases are biholomorphically equivalent.

We call the image of \mathcal{M} the (Floquet) spectrum of \mathcal{D} (on the zero energy level $E = 0$). Given a basis of Λ , this image is uniquely defined. In general, the spectral data are defined modulo the $SL(2, \mathbb{Z})$ -action (22), and we say that the spectral data of two operators coincide if the $SL(2, \mathbb{Z})$ -orbits of their spectral data coincide.

3.2. Examples of spectra. 1. $U = 0$. Let $\Lambda = \mathbb{Z} + i\mathbb{Z}$. The Floquet functions are parameterized by two planes

$$\psi^+ = (e^{\lambda+z}, 0) \quad \text{and} \quad \psi^- = (0, e^{\lambda-\bar{z}});$$

Γ is the union of these planes, which we compactify by two points at infinity where ψ has exponential singularities.

The quasimomenta of ψ^+ are

$$k_1 = \frac{\lambda_+}{2\pi i} + n_1, \quad k_2 = \frac{\lambda_+}{2\pi} + n_2, \quad n_j \in \mathbb{Z}, \tag{23}$$

and the quasimomenta of ψ^- are

$$k_1 = \frac{\lambda_-}{2\pi i} + m_1, \quad k_2 = -\frac{\lambda_-}{2\pi} + m_2, \quad m_j \in \mathbb{Z}. \tag{24}$$

Hence,

$$Q_0 = \left(\bigcup_{n_1, n_2 \in \mathbb{Z}} A_{n_1, n_2} \right) \cup \left(\bigcup_{m_1, m_2 \in \mathbb{Z}} B_{m_1, m_2} \right),$$

where A_{n_1, n_2} and B_{m_1, m_2} are the planes described by (23) and (24).

The functions ψ^+ and ψ^- have the same multipliers at the points

$$\lambda_+^{m, n} = \pi(n + im), \quad \lambda_-^{m, n} = \pi(n - im), \quad m, n \in \mathbb{Z}.$$

These are resonance pairs for this potential with $\Lambda = \mathbb{Z} + i\mathbb{Z}$.

Consider the zero potential to be Λ -periodic with respect to a general lattice $\gamma_1\mathbb{Z} + \gamma_2\mathbb{Z}$. The Floquet functions are the same, but the resonance pairs are

$$\lambda_+^{m, n} = \frac{2\pi i}{\bar{\gamma}_1\gamma_2 - \gamma_1\bar{\gamma}_2}(\bar{\gamma}_1 n - \bar{\gamma}_2 m), \quad \lambda_-^{m, n} = \frac{2\pi i}{\bar{\gamma}_1\gamma_2 - \gamma_1\bar{\gamma}_2}(\gamma_1 n - \gamma_2 m). \tag{25}$$

2. $U = C = \text{const} \neq 0$. Assume, for simplicity, that $\Lambda = \mathbb{Z} + i\mathbb{Z}$.

The Floquet functions are

$$\psi(z, \bar{z}, \lambda) = \left(\exp\left(\lambda z - \frac{C^2}{\lambda} \bar{z}\right), -\frac{C}{\lambda} \exp\left(\lambda z - \frac{C^2}{\lambda} \bar{z}\right) \right),$$

where $\lambda \in \Gamma = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Compactify Γ by the points 0 and ∞ and define the Floquet function on Γ globally as

$$\psi(z, \bar{z}, \lambda) = \frac{\lambda}{\lambda - C} \left(\exp\left(\lambda z - \frac{C^2}{\lambda} \bar{z}\right), -\frac{C}{\lambda} \exp\left(\lambda z - \frac{C^2}{\lambda} \bar{z}\right) \right).$$

It has the following asymptotics:

$$\psi \approx \begin{pmatrix} \exp(k_+ z) \\ 0 \end{pmatrix} \quad \text{as } \lambda \rightarrow \infty \quad \text{and} \quad \psi \approx \begin{pmatrix} 0 \\ \exp(k_- \bar{z}) \end{pmatrix} \quad \text{as } \lambda \rightarrow 0 \tag{26}$$

with $k_+ = \lambda$ and $k_- = -C^2/\lambda$. In this case, ψ has a pole at $\lambda = C$.

The resonance pairs (λ, λ') are

$$\lambda = \frac{q\bar{q} \pm \sqrt{(q\bar{q})^2 - 4C^2q\bar{q}}}{2\bar{q}}, \quad \lambda' = \lambda - q, \quad q = \pi(n + im), \quad m, n \in \mathbb{Z},$$

and they are parameterized by $q \in \pi\mathbb{Z}^2 \setminus \{0\}$.

3. $U = U(x)$ is a function of one variable. Let $U(x + T) = U(x)$, where T is the minimal period. Then, equation (8) for Floquet functions $\psi(x, y) = \varphi(x)e^{\lambda y}$ is the *Zakharov–Shabat* system

$$L\varphi = \begin{pmatrix} U & \frac{1}{2}\partial_x \\ -\frac{1}{2}\partial_x & U \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{2}\lambda \\ \frac{i}{2}\lambda & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{27}$$

and, in terms of $\eta_1 = \varphi_1 + i\varphi_2$ and $\eta_2 = \varphi_1 - i\varphi_2$, it is given by

$$(\partial_x + 2iU)\eta_1 = -i\lambda\eta_2, \quad (\partial_x - 2iU)\eta_2 = -i\lambda\eta_1.$$

We see that $f = \eta_2$ satisfies the equation

$$[-\partial_x^2 + (2iU_x - 4U^2)]f = \nu f,$$

where $\nu = \lambda^2$. The transformation $L \rightarrow [-\partial_x^2 + (2iU_x - 4U^2)]$ is called the *Miura transformation*. The same name is used for the transformation

$$[-\partial_x^2 + (2iU_x - 4U^2)] \leftrightarrow [-\partial_x^2 + (-2iU_x - 4U^2)]. \tag{28}$$

For any $\lambda \in \mathbb{C}$, take a two-dimensional space \mathcal{V}_λ of solutions to (27) and consider the monodromy operator

$$\widehat{T}: \mathcal{V}_\lambda \rightarrow \mathcal{V}_\lambda, \quad \widehat{T}(\varphi)(x) = \varphi(x + T).$$

For any pair $(\varphi(x, \lambda), \vartheta(x, \lambda))$ of solutions to (27), their Wronskian $W(\vartheta, \varphi) = \vartheta_1(x)\varphi_2(x) - \vartheta_2(x)\varphi_1(x)$ is constant. Take a basis $(c(x, \lambda), s(x, \lambda))$ for \mathcal{V}_λ normalized as

$$c(0, \lambda) = s_x(x, \lambda) = 1, \quad c_x(0, \lambda) = s(x, \lambda) = 0.$$

One can show that, in this basis, the entries of the matrix \widehat{T} are entire functions of λ . Since $W(c, s)$ is constant, $\det \widehat{T}(\lambda) = 1$ and the characteristic equation for $\widehat{T}(\lambda)$ takes the form

$$k^2 - \text{Tr} \widehat{T}(\lambda)k + 1 = 0. \tag{29}$$

If $\widehat{T}(\lambda)$ is diagonalized, then its eigenvectors are the Floquet functions. The operator $\widehat{T}(\lambda)$ is not diagonalized if and only if λ is a simple root of the equation

$$\text{Tr}^2 \widehat{T}(\lambda) - 4 = 0, \tag{30}$$

which can have only simple and double roots. In this case, the Jordan form for $\widehat{T}(\lambda)$ is a nondiagonal upper triangular matrix, and there is only one (up to a constant factor) eigenvector.

The Floquet function is globally defined on the two-sheeted covering of \mathbb{C} branched at points λ_1, \dots that are simple roots of (30), and this complex curve is exactly Γ . Resonance pairs of the spectrum are pairs of points that are projected into double roots of (30).

If there are finitely many simple roots of (30), L is called a *finite-gap* operator. In this case, Γ is compactified by two infinities and the Floquet functions are pasted in a meromorphic function on Γ with punctured infinities where it has asymptotics (26).

These analytic properties of the Floquet function for a one-dimensional Schrödinger operator are explained in [9]. For the Dirac operator, similar results are obtained by using the Miura transformation or derived straightforwardly by the same reasoning.

3.3. Spectra via perturbation theory. Generically, Floquet functions and the Floquet spectrum cannot be found by solving ordinary differential equations as in examples in Section 3.2. For proving the existence of the Floquet spectrum, we used the Keldysh theorem in Section 3.1. Another approach for finding this spectrum and describing it in a rather efficient manner was proposed by Krichever, who realized it for a two-dimensional Schrödinger operator and for the operator $\partial_y - \partial_x^2 + U(x, y)$ [24]. This approach is based on perturbation theory.

The examples discussed above demonstrate how the spectrum is deformed under the deformation of U . The main picture is as follows: deforming the potential, we deform double points on $Q_0(\mathcal{D})/\Lambda^*$ into handles and remove the singularities. The norm of a deformation measures the “sizes” of such handles.

For the two-dimensional Dirac operator (9), this has not yet been done; however, since it is clear that the method of [24] works for this operator after a slight modification, we explain what is the expected picture.

Pretheorem. *For a smooth potential U , the spectral curve Γ consists of two parts, M_0 and M_∞ , where*

- (1) M_0 is a complex curve of finite genus whose boundary consists in a pair of circles;
- (2) M_∞ is diffeomorphic to the union of the domains $|\lambda_\pm| \geq R$ in the λ_\pm -planes for a certain R with some resonance pairs $(\lambda_+^{m,n}, \lambda_-^{m,n})$ (25) “joined by handles” of decreasing sizes as $m^2 + n^2 \rightarrow \infty$;
- (3) M_0 and M_∞ are pasted along their boundaries.

This “joining by a handle” means that some small disks $|\lambda_+ - \lambda_+^{m,n}| < \varepsilon_+^{m,n}$ and $|\lambda_- - \lambda_-^{m,n}| < \varepsilon_-^{m,n}$ are removed and replaced by a cylinder pasted to their boundaries, and $\varepsilon_\pm^{m,n} \rightarrow 0$ as $m^2 + n^2 \rightarrow \infty$.

To any point of the subsets $M_+, M_- \subset M_\infty$, where

$$M_+ = \mathbb{C} \setminus \left\{ \{|\lambda_+| > R\} \cup \left(\bigcup_{m,n} \{|\lambda_+ - \lambda_+^{m,n}| \leq \varepsilon_+^{m,n}\} \right) \right\},$$

$$M_- = \mathbb{C} \setminus \left\{ \{|\lambda_-| > R\} \cup \left(\bigcup_{m,n} \{|\lambda_- - \lambda_-^{m,n}| \leq \varepsilon_-^{m,n}\} \right) \right\},$$

there corresponds a unique (up to a constant factor) Floquet function. These functions and their multipliers $\mu(\gamma_j, \lambda_\pm)$ asymptotically behave as in the case $U = 0$; in particular,

$$\mu(\gamma_j, \lambda_+) = e^{\lambda_+ \gamma_j} \left(1 + O\left(\frac{1}{\lambda_+}\right) \right), \quad \mu(\gamma_j, \lambda_-) = e^{\lambda_- \gamma_j} \left(1 + O\left(\frac{1}{\lambda_-}\right) \right)$$

as $\lambda_\pm \rightarrow \infty$.

If U does not vanish identically, then Γ is irreducible.

A potential U is *finite-gap* if there is such a representation with no handles joining resonance points in M_∞ . In this case, Γ is compactified by a pair of infinities ∞_\pm to a Riemann surface of finite genus.

In fact, this is a general description of the Floquet spectra of operators with periodic coefficients. The physical picture from [24] was chosen by Feldman, Knörrer, and Trubowitz as a most convenient and reasonable definition of general (nonhyperelliptic) Riemann surfaces of infinite genus, and they developed a nice theory of such surfaces for which analogues of many classical theorems on algebraic curves are valid [12].

4. THE SPECTRUM OF THE WEIERSTRASS REPRESENTATION

4.1. The spectral curve of an immersed torus. Let $F: \mathbb{C} \rightarrow \mathbb{R}^3$ be a conformal immersion of a plane whose Gauss map descends through a torus \mathbb{C}/Λ , i.e., is doubly periodic. We will consider tori as a particular case of such planes when the immersion is also doubly periodic. The potential $U(z)$ of a Weierstrass representation of such a surface is Λ -periodic.

It is said that two surfaces (with fixed conformal parameters) $F_1: \mathbb{C} \rightarrow \mathbb{R}^3$ and $F_2: \mathbb{C} \rightarrow \mathbb{R}^3$ are *isopotential* if their potentials coincide.

Definition 5. The spectral curve Γ of the operator \mathcal{D} with the potential U is called the *spectral curve of a surface*, and the genus of this curve is called the *spectral genus of a surface*. If a basis γ_1, γ_2 for Λ is fixed, the image of the multiplier map

$$\mathcal{M}: Q_0(\mathcal{D})/\Lambda^* \rightarrow \mathbb{C}^2, \quad \mathcal{M}(k) = (e^{2\pi i \langle k, \gamma_1 \rangle}, e^{2\pi i \langle k, \gamma_2 \rangle})$$

is called the *spectrum of a surface*.

Proposition 5. *The spectral curve and the spectral genus of a surface do not depend on a choice of a conformal parameter.*

The spectrum of a surface depends on a choice of a basis for $H_1(T^2) = H_1(\mathbb{C}/\Lambda) = \Lambda$, and, for different bases $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ and (γ_1, γ_2) related by an $SL(2, \mathbb{Z})$ -transformation

$$\begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},$$

the spectra $\{(\tilde{\mu}_1, \tilde{\mu}_2)\}$ and $\{(\mu_1, \mu_2)\}$ are related as

$$\tilde{\mu}_1 = \mu_1^a \mu_2^b, \quad \tilde{\mu}_2 = \mu_1^c \mu_2^d.$$

Proof. One needs to verify that all these data are preserved under a passage from z to $w = t^2 z$ with $t \in \mathbb{C} \setminus \{0\}$. For these conformal parameters, the surface is defined by the functions $\psi(z) = (\psi_1(z), \psi_2(z))^T$ and $\tilde{\psi}(w) = (\tilde{\psi}_1(w), \tilde{\psi}_2(w))^T$, which are solutions to the equations

$$\mathcal{D}\psi = \begin{pmatrix} U & \partial_z \\ -\partial_{\bar{z}} & U \end{pmatrix} \psi = 0, \quad \tilde{\mathcal{D}}\tilde{\psi} = \begin{pmatrix} \tilde{U} & \partial_w \\ -\partial_{\bar{w}} & \tilde{U} \end{pmatrix} \tilde{\psi} = 0.$$

We have $dw = t^2 dz$, $\partial_z = \frac{1}{t^2} \partial_w$, and, since $e^{2\tilde{\alpha}(w)} dw d\bar{w} = e^{2\alpha(z)} dz d\bar{z}$ and $\tilde{H}(w) = H(z)$, formulas (11) imply that $\tilde{U}(w) = t\bar{t}U(z)$ for $w = t^2 z$. Therefore,

$$\begin{pmatrix} U & \partial_z \\ -\partial_{\bar{z}} & U \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0 \quad \text{if and only if} \quad \begin{pmatrix} \tilde{U} & \partial_w \\ -\partial_{\bar{w}} & \tilde{U} \end{pmatrix} \begin{pmatrix} t\varphi_1 \\ \bar{t}\varphi_2 \end{pmatrix} = 0,$$

and $\varphi = (\varphi_1, \varphi_2)^T$ and $\tilde{\varphi} = (t\varphi_1, \bar{t}\varphi_2)^T$ have the same multipliers.

The transformation of the spectra was already discussed in Section 3.1, and we just repeat here these formulas because now they appear in another situation.

The proposition is proved.

We say that two planes, which may be converted to tori by immersions, are *isospectral* if

- (1) there are conformal parameters on them such that both of them are represented by mappings $F_1: \mathbb{C} \rightarrow \mathbb{R}^3$ and $F_2: \mathbb{C} \rightarrow \mathbb{R}^3$;
- (2) the corresponding potentials of the Dirac operators are Λ -periodic with the same lattice Λ ;
- (3) the spectra of these operators with respect to a fixed basis for Λ coincide.

4.2. On the Willmore functional. Given a Weierstrass representation of a torus Σ , we have

$$\mathcal{W}(\Sigma) = 4 \int_{\Sigma} U^2 dx \wedge dy = 2i \int_{\Sigma} U^2 dz \wedge d\bar{z} \tag{31}$$

(see [38]), which shows that the Willmore functional \mathcal{W} measures the deviation of a connection in a spinor bundle defined by \mathcal{D} from the trivial connection. The relation

$$\mathcal{W}(\Sigma) = \int_{\Sigma} H^2 d\mu$$

between the spectrum and the Willmore functional was first established in [39].

Let U be a Λ -periodic potential with $\Lambda = \gamma_1\mathbb{Z} + \gamma_2\mathbb{Z}$, and let Γ be of finite genus and be compactified by two infinities ∞_{\pm} . As is shown in the Pretheorem, these infinities are inherited during the perturbation of U from the compactification of the spectrum of the zero potential (see example 1 in Section 3.2).

Take the Floquet function $\psi(z, P)$ meromorphic outside the infinities and with the asymptotics

$$\psi \approx \begin{pmatrix} \exp(\lambda_+ z) \\ 0 \end{pmatrix} \text{ as } P \rightarrow \infty_+ \quad \text{and} \quad \psi \approx \begin{pmatrix} 0 \\ \exp(\lambda_- \bar{z}) \end{pmatrix} \text{ as } P \rightarrow \infty_-, \tag{32}$$

where λ_{\pm}^{-1} are local parameters near ∞_{\pm} .

The theory of finite-gap integration gives a recipe for reconstructing U from such asymptotic expansions. Let

$$\psi(z, \lambda_+) = \exp(\lambda_+ z) \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \frac{1}{\lambda_+} + O\left(\frac{1}{\lambda_+^2}\right) \right) \text{ as } \lambda_+ \rightarrow \infty.$$

Substitute this expansion into the Dirac equation and expand $\mathcal{D}\psi = 0$ in powers of λ_+ . Every coefficient in this expansion equals zero. Take the first two of them:

$$U + \zeta_2 = 0, \quad U\zeta_2 - \bar{\partial}\zeta_1 = 0. \tag{33}$$

This gives a reconstruction formula for U and the identity

$$-U^2 = \bar{\partial}\zeta_1.$$

Let us show how the Willmore functional appears in this picture. By perturbation theory, we expect that the spectrum asymptotically behaves as the spectrum of the zero potential and this leads to the following conclusion: there is a function $W(\lambda_+)$ defined near ∞_+ such that

- (1) $W(\lambda_+) = C_1\lambda_+^{-1} + O(\lambda_+^{-2})$ and
- (2) $\psi(z, \lambda_+) = e^{\lambda_+ z + W(\lambda_+) \bar{z}} \varphi(z, \lambda_+)$, where $\varphi(z, \lambda_+)$ is Λ -periodic.

The function W measures the deviation of the Floquet spectrum from the spectrum of the zero potential. Indeed, the multipliers of ψ are

$$(\mu(\gamma_1), \mu(\gamma_2)) = (e^{\lambda_+ \gamma_1 + W(\lambda_+) \bar{\gamma}_1}, e^{\lambda_+ \gamma_2 + W(\lambda_+) \bar{\gamma}_2}),$$

and the multipliers of the Floquet function of the zero potential (which is considered to be Λ -periodic) are

$$(\mu_0(\gamma_1), \mu_0(\gamma_2)) = (e^{\lambda+\gamma_1}, e^{\lambda+\gamma_2}).$$

It is easy to notice that

$$\zeta_1 = C_1 \bar{z} + (\text{a } \Lambda\text{-periodic function});$$

hence,

$$\int_{\Pi} U^2 dx \wedge dy = -C_1 \text{Vol } \Pi,$$

where Π is a parallelogram spanned by γ_1 and γ_2 . Now, using (31), we conclude that

$$\mathcal{W}(\Sigma) = -4C_1 \text{Vol } \Pi, \tag{34}$$

where $\text{Vol } \Pi$ is the area of the fundamental domain of Λ . This derivation of formula (34) was exposed in [14]. An analogous derivation of a formula for the area of minimal tori is given in [18].

Consider the whole series

$$W(\lambda) = C_1 \lambda^{-1} + C_2 \lambda^{-2} + \dots$$

Since the involution $k \rightarrow -k$ inverts λ and preserves the spectrum, we have $C_{2k} = 0$ for $k = 1, 2, \dots$. The quantities C_{2k-1} for $k \geq 2$ depend on the choice of a conformal parameter z and a parameter λ on the spectral curve. If a parameter λ is fixed, then the holomorphic differentials

$$\mathcal{W}_k = -4(C_{2k-1} \text{Vol } \Pi) dz^{2k-2}$$

are geometric invariants. The first of them is the Willmore functional $\mathcal{W} = \mathcal{W}_1$.

If the conformal parameter is fixed, then C_{2k-1} are first integrals of the modified Novikov–Veselov equation. The question of what are their geometrical meanings was posed in [38].

4.3. Surfaces in terms of theta functions. Assume that the spectrum Γ of a torus Σ has a finite genus equal to g and take two different points ∞_{\pm} on Γ with local parameters λ_{\pm}^{-1} near them such that $\lambda_{\pm}^{-1}(\infty_{\pm}) = 0$. Then, the theory of Baker–Akhieser functions [23] states that, for a generic effective divisor D , i.e., for a formal sum of points on Γ , of degree $g + 1$ ($D = P_1 + \dots + P_{g+1}$), there is a unique function $\psi(z, \bar{z}, P)$ such that

- (1) ψ is meromorphic in $P \in \Gamma \setminus \{\infty_{\pm}\}$ and has asymptotics (32) as $P \rightarrow \infty_{\pm}$;
- (2) ψ has poles only in D .

This function is constructed in terms of theta functions of a Riemann surface Γ . Using this function, one can reconstruct a Dirac operator \mathcal{D} by (33). Therefore, to each point $P \in \Gamma \setminus \{\infty_{\pm}, P_1, \dots, P_{g+1}\}$, there corresponds a surface in \mathbb{R}^3 constructed from $\psi(P)$ by (10).

In this case, the Riemann surface Γ parameterizes isopotential surfaces, and each of these surfaces is described in terms of the theta functions of Γ . The detailed formulas are given in [40, 42]. For CMC tori in \mathbb{R}^3 , analogous formulas were derived by another method in [4].

For tori of infinite spectral genera, one can apply the theory of theta functions on Riemann surfaces of infinite genera developed by Feldman, Knörrer, and Trubowitz [12].

4.4. On Willmore surfaces and the Willmore conjecture. The Willmore functional \mathcal{W} appeared for the first time in the 1920s in the papers by Blaschke [3] and Thomsen [43]. It was

called there a *conformal area*, and its extrema were called *conformally minimal surfaces*. Blaschke and Thomsen also established the main properties of this functional, which are as follows:

1. The Willmore functional is invariant with respect to conformal transformations of the ambient space: let $\Sigma \subset \mathbb{R}^3$ be a compact immersed oriented surface, $z = x + iy$ be a conformal parameter on it, and $G: \overline{\mathbb{R}^3} \rightarrow \overline{\mathbb{R}^3}$ be a conformal transformation that maps Σ into \mathbb{R}^3 ; then, $\mathcal{W}(\Sigma) = \mathcal{W}(G(\Sigma))$, which follows from the conformal invariance of the quantity

$$(k_1 - k_2)^2 d\mu = 4(H^2 - K) d\mu = 16|A|^2 e^{-2\alpha} dx \wedge dy,$$

where $e^{2\alpha} dz d\bar{z}$ is the first fundamental form and $A dz^2$ is the Hopf differential of the surface.

2. If Σ is a minimal surface in S^3 and $\pi: S^3 \rightarrow \overline{\mathbb{R}^3}$ is a stereographic projection that maps Σ into \mathbb{R}^3 , then $\pi(\Sigma)$ is a conformally minimal (Willmore) surface.

Hence, in contrast to minimal surfaces, there exist compact Willmore surfaces in \mathbb{R}^3 .

All Willmore spheres were described by Bryant [6]. The classification of Willmore tori has not yet been completed, and we only mention the papers [1, 2], where the finite-gap integration was applied to this problem. The theory of integrable systems was applied for constructing general Willmore surfaces in [16].

The simplest example of a Willmore torus is given by the stereographic projection of the Clifford torus $\{(x^1)^2 + (x^2)^2 = 1/2, (x^3)^2 + (x^4)^2 = 1/2\} \subset S^3 \subset \mathbb{R}^4$ into \mathbb{R}^3 . In another way, it may be obtained as a circular torus of revolution with the ratio of the distance from the center of the circle to the axis of revolution to the radius of the circle equal to $\sqrt{2}$. This torus in \mathbb{R}^3 is also called the Clifford torus.

Willmore [45] conjectured that

- *the functional \mathcal{W} for tori attains its minimum on the Clifford torus and its conformal transformations, and this minimum equals $2\pi^2$ (the Willmore conjecture).*

This conjecture implies the Hsiang–Lawson conjecture that the area of a minimal torus in S^3 is no less than $2\pi^2$, the area of the Clifford torus in S^3 , but not equivalent to it since there are Willmore tori in \mathbb{R}^3 that are not images of minimal tori under the stereographic projection [33].

The Willmore conjecture is still open, and there are some particular cases for which it was proved:

1. Langer and Singer proved it for tori of revolution [28], and Hertrich-Jeromin and Pinkall generalized their result for channel tori that are the enveloping tori for one-parameter families of spheres [17].

2. Li and Yau proved this conjecture for tori that are conformally equivalent to flat tori $\mathbb{C}/\{\mathbb{Z} + \tau\mathbb{Z}\}$, where $\tau = a + ib$, $0 \leq a \leq 1/2$, $b > 0$, and $\sqrt{1 - a^2} \leq b \leq 1$ [29]. Later, Montiel and Ros improved the latter inequality to $(a - 1/2)^2 + (b - 1)^2 \leq 1/4$ [31].

Simon proved that the minimum of the Willmore functional is attained on a real-analytic torus [37].

For higher genera surfaces, a generalization of the Willmore conjecture was proposed by Kusner [27].

In [40] (see also [42]), we conjectured that

- *for a fixed conformal class of tori, the minimum of \mathcal{W} is attained on tori with the minimal spectral genus.*

If this conjecture is valid, then we may reduce the proof of the Willmore conjecture to estimating \mathcal{W} for Willmore tori of small spectral genera and then to checking the conjecture by the methods of soliton theory.

The global Weierstrass representation gives a physical explanation for lower bounds for \mathcal{W} : for small perturbations of the zero potential $U = 0$, the surfaces constructed from solutions to (8) via (10) do not convert into tori, and the lower bound for \mathcal{W} , the squared L_2 -norm of the perturbation, shows how large a perturbation has to be to convert planes into tori.

5. THE SPECTRA OF INTEGRABLE TORI

5.1. Constant mean curvature tori. Let Σ be a CMC torus in \mathbb{R}^3 , i.e., $H = \text{const}$, and let it be conformally equivalent to a flat torus \mathbb{C}/Λ with $\Lambda = \gamma_1\mathbb{Z} + \gamma_2\mathbb{Z}$.

As is shown in Section 2.3, for CMC surfaces, the Hopf differential $\omega = Adz^2$ is holomorphic, and therefore $Adz^2 = \text{const} dz^2$. This differential does not vanish because otherwise all points are umbilics, which is impossible for tori in \mathbb{R}^3 . Assume that

$$\omega = \frac{1}{2} dz^2, \quad H = 1,$$

which is achieved by rescaling z and by a homothety in \mathbb{R}^3 . Now, the Gauss–Codazzi equations in terms of $u = 2\alpha$ read

$$u_{z\bar{z}} + \sinh u = 0, \tag{35}$$

which is the (elliptic) *sinh-Gordon equation*. This equation has a commutation representation of the form (13):

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z & -\frac{\lambda^2}{2} e^{-\alpha} \\ -\frac{1}{2} e^\alpha & 0 \end{pmatrix} \right] \psi = 0, \quad \left[\frac{\partial}{\partial \bar{z}} - \begin{pmatrix} 0 & \frac{1}{2} e^\alpha \\ \frac{\lambda^{-2}}{2} e^{-\alpha} & \alpha_{\bar{z}} \end{pmatrix} \right] \psi = 0 \tag{36}$$

($\lambda^2 = -1$ in (13)).

Note that

- (1) for any $\lambda \neq 0$, the compatibility condition for (36) is (35);
- (2) if $|\lambda| = 1$, then (35) are the Gauss–Codazzi equations (14) for the surface defined by $\psi(\lambda, z, \bar{z})$ via (10).

There is another representation of (35) that gives rise to the *spectral curve of a CMC torus* [34, 4]. Consider equation (35) as the compatibility condition for the system

$$\left[\frac{\partial}{\partial z} - \frac{1}{2} \begin{pmatrix} -u_z & -\lambda \\ -\lambda & u_z \end{pmatrix} \right] \varphi = 0, \quad \left[\frac{\partial}{\partial \bar{z}} - \frac{1}{2\lambda} \begin{pmatrix} 0 & e^{-u} \\ e^u & 0 \end{pmatrix} \right] \varphi = 0, \tag{37}$$

which contains the linear problem

$$L\varphi = \partial_z \varphi - \frac{1}{2} \begin{pmatrix} -u_z & 0 \\ 0 & u_z \end{pmatrix} \varphi = \frac{1}{2} \begin{pmatrix} 0 & -\lambda \\ -\lambda & 0 \end{pmatrix} \varphi$$

for a general Λ -periodic potential u . Since L is a first-order 2×2 -matrix operator, for every $\lambda \in \mathbb{C}$, system (37) has a two-dimensional space \mathcal{V}_λ of solutions and these spaces are invariant under the translation operators

$$\widehat{T}_j \varphi(z) = \varphi(z + \gamma_j), \quad j = 1, 2.$$

Since $\widehat{T}_1, \widehat{T}_2$, and L commute, they have common eigenvectors and these vectors are glued into a meromorphic function $\Psi(z, \bar{z}, P)$ on a two-sheeted covering

$$\Gamma(L) \rightarrow \mathbb{C}: \quad P \in \Gamma(L) \rightarrow \lambda \in \mathbb{C},$$

which ramifies at points where \widehat{T}_j and L are not diagonalized simultaneously. To each point $P \in \Gamma(L)$, there corresponds a unique (up to a constant multiple) Floquet function φ with multipliers $\mu(\gamma_1, P)$ and $\mu(\gamma_2, P)$.

By the same reasoning as in example 3 in Section 3.2, it is shown that there are Floquet functions defined on a $\Gamma(L)$ such that $\Gamma(L)$ is compactified by four infinities ∞_{\pm}^1 and ∞_{\pm}^2 such that ∞_{\pm}^1 are mapped into $\lambda = \infty$ and ∞_{\pm}^2 are mapped into $\lambda = 0$, and there are asymptotics

$$\begin{aligned} \psi(z, P) &\approx \exp\left(\mp \frac{\lambda z}{2}\right) \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} && \text{as } P \rightarrow \infty_{\pm}^1, \\ \psi(z, P) &\approx \exp\left(\mp \frac{\bar{z}}{2\lambda}\right) \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} && \text{as } P \rightarrow \infty_{\pm}^2 \end{aligned}$$

(see [4]).

If $\varphi = (\varphi_1, \varphi_2)^\top$ satisfies (37) for $\lambda = \mu$, then

$$\sigma(\varphi) = (\varphi_1, -\varphi_2) \tag{38}$$

satisfies (37) for $\lambda = -\mu$, and this generates an involution $\sigma: \Gamma(L) \rightarrow \Gamma(L)$ that descends to an involution of $\mathbb{C}: \lambda \rightarrow -\lambda$. We also have $\sigma(\infty_{\pm}^1) = \infty_{\mp}^1$ and $\sigma(\infty_{\pm}^2) = \infty_{\mp}^2$. By (38), the immersion

$$\mathcal{M}: \Gamma(L) \rightarrow \mathbb{C}^2, \quad P \rightarrow (\mu(\gamma_1, P), \mu(\gamma_2, P))$$

descends through the quotient of σ ; i.e., $\mathcal{M}: \Gamma(L) \rightarrow \Gamma(L)/\sigma \rightarrow \mathbb{C}^2$.

Definition 6. The complex curve $\Gamma(L)/\sigma$ is called the *spectral curve of a CMC torus* Σ . It is said that $\mathcal{M}(\Gamma(L)/\sigma)$ is the *spectrum* of this torus.

By straightforward computations, we obtain

Proposition 6. $\varphi = (\varphi_1, \varphi_2)^\top$ satisfies (37) if and only if $\psi = (\lambda\varphi_2, e^\alpha\varphi_1)^\top$ satisfies (36).

This proposition implies that the Floquet functions of L and \mathcal{D} have the same multipliers.

Theorem 5. *The spectrum of a CMC torus Σ is a component of the spectrum of the surface defined as in Section 4.1 that contains both asymptotic ends, where $\mu(\gamma_j) \approx e^{\lambda+\gamma_j}$ and $\mu(\gamma_j) \approx e^{\lambda-\bar{\gamma}_j}$ as $\lambda_{\pm} \rightarrow \infty$.*

Therefore, the spectral curve of the CMC torus Σ is an irreducible component of the spectral curve defined in Section 4.1 for a general torus.

Assuming that the Pretheorem is valid, we have more; namely,

- *the spectrum and the spectral curve of a CMC torus coincide with the spectrum and the spectral curve defined in Section 4.1 for a general torus.*

In fact, for this conclusion, we only need the irreducibility of the spectral curve for $U \neq 0$.

Note that the genus of the spectral curve of a CMC torus is finite [34].

5.2. Isothermic tori. A surface is called *isothermic* if, near every point, there is a conformal parameter $z = x + iy$ such that the fundamental forms are

$$\mathbf{I} = e^{2\alpha}(dx^2 + dy^2), \quad \mathbf{II} = e^{2\alpha}(k_1 dx^2 + k_2 dy^2).$$

To each isothermic surface $F: \mathcal{U} \rightarrow \mathbb{R}^3$, there corresponds a *dual isothermic surface* $F^*: \mathcal{U} \rightarrow \mathbb{R}^3$ defined up to translations by the formulas

$$F_z^* = e^{-2\alpha} F_{\bar{z}}, \quad F_{\bar{z}}^* = e^{-2\alpha} F_z.$$

The fundamental forms of the dual surface are

$$\mathbf{I} = e^{-2\alpha}(dx^2 + dy^2), \quad \mathbf{II} = -k_1dx^2 + k_2dy^2;$$

the Gauss maps of F and F^* are antipodal: $N = -N^*$, and $F^{**} = F$ (modulo translations).

The following proposition is obtained by straightforward computations.

Proposition 7. *If an isothermic surface F is represented via (10) by a vector function $\psi = (\psi_1, \psi_2)^\top$, then the dual surface F^* is represented via (10) by the function $\psi^* = (ie^{-\alpha}\psi_2, ie^{-\alpha}\psi_1)$. The potentials of these surfaces are*

$$U = \frac{k_1 + k_2}{4}e^\alpha, \quad U^* = \frac{k_2 - k_1}{4}e^\alpha,$$

and the Hopf differentials $A dz^2$ and $A^* dz^2$ are

$$A = \frac{k_1 - k_2}{4}e^{2\alpha}, \quad A^* = -\frac{k_1 + k_2}{4}.$$

Corollary 1.

$$Ae^{-\alpha} = \frac{k_1 - k_2}{4}e^\alpha = -U^*.$$

The simplest examples of isothermic surfaces are given by surfaces of revolution and constant mean curvature surfaces.

Let Σ be an isothermic plane, which may be converted into an immersed torus, whose Gauss map descends through \mathbb{C}/Λ with $\Lambda = \gamma_1\mathbb{Z} + \gamma_2\mathbb{Z}$.

By Proposition 7, the Gauss–Weingarten equations (13) are written as

$$\psi_z = \mathbf{U}\psi, \quad \psi_{\bar{z}} = \mathbf{V}\psi,$$

where

$$\mathbf{U} = \begin{pmatrix} \alpha_z & \frac{k_1 - k_2}{4}e^\alpha \\ -\frac{k_1 + k_2}{4}e^\alpha & 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 0 & \frac{k_1 + k_2}{4}e^\alpha \\ \frac{k_2 - k_1}{4}e^\alpha & \alpha_{\bar{z}} \end{pmatrix},$$

and their compatibility conditions are

$$\alpha_{xx} + \alpha_{yy} + k_1k_2e^{2\alpha} = 0, \quad k_{2x} - (k_1 - k_2)\alpha_x = k_{1y} + (k_1 - k_2)\alpha_y = 0. \tag{39}$$

Equations (39) are also the compatibility conditions for linear problems with a spectral parameter [8, 5]:

$$\widehat{\varphi}_z = \widehat{\mathbf{U}}(\lambda)\widehat{\varphi}, \quad \widehat{\varphi}_{\bar{z}} = \widehat{\mathbf{V}}(\lambda)\widehat{\varphi}, \tag{40}$$

where

$$\widehat{\mathbf{U}}(\lambda) = \begin{pmatrix} \mathbf{U} & \lambda\mathbf{J}^- \\ \lambda\mathbf{J}^+ & \mathbf{U} + \alpha_z\mathbf{E} \end{pmatrix}, \quad \widehat{\mathbf{V}}(\lambda) = \begin{pmatrix} \mathbf{V} & \lambda\mathbf{J}^+ \\ \lambda\mathbf{J}^- & \mathbf{V} + \alpha_{\bar{z}}\mathbf{E} \end{pmatrix},$$

$$\mathbf{J}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{J}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, the reasonings for describing the spectrum of an isothermic torus are the same as those for a one-dimensional Dirac operator (see example 3 in Section 3.2) and CMC tori.

For every $\lambda \in \mathbb{C}$, system (40) has a four-dimensional space \mathcal{V}_λ of solutions. On each such space, the translation operators act as

$$\widehat{T}_j \widehat{\varphi}(z) = \widehat{\varphi}(z + \gamma_j), \quad j = 1, 2.$$

Since $\widehat{T}_1, \widehat{T}_2$, and L commute, they have common eigenvectors, and these vectors are glued into a meromorphic function $\widehat{\varphi}(z, \bar{z}, P)$ on a four-sheeted covering

$$\Gamma(\mathbf{U}) \rightarrow \mathbb{C}: \quad P \in \Gamma(\mathbf{U}) \rightarrow \lambda \in \mathbb{C},$$

which ramifies at points where \widehat{T}_j and L are not diagonalized simultaneously.

To each point $P \in \Gamma(\mathbf{U})$, there corresponds a unique (up to a constant factor) Floquet function $\widehat{\varphi}$ with multipliers $\mu(\gamma_1, P)$ and $\mu(\gamma_2, P)$. If $\Gamma(\mathbf{U})$ is of finite genus, it is compactified by four ‘‘infinities’’ at which $\widehat{\varphi}$ has exponential singularities.

Note that, if $\widehat{\varphi} = (\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4)^\top$ satisfies (40) for $\lambda = \mu$, then

$$\sigma(\widehat{\varphi}) = (\widehat{\varphi}_1, \widehat{\varphi}_2, -\widehat{\varphi}_3, -\widehat{\varphi}_4)^\top \tag{41}$$

satisfies (40) for $\lambda = -\mu$, and this generates an involution $\sigma: \Gamma(\mathbf{U}) \rightarrow \Gamma(\mathbf{U})$ that descends to an involution of $\mathbb{C}: \lambda \rightarrow -\lambda$.

By (41), the immersion

$$\mathcal{M}: \Gamma(\mathbf{U}) \rightarrow \mathbb{C}^2, \quad P \rightarrow (\mu(\gamma_1, P), \mu(\gamma_2, P))$$

descends through the quotient of σ ; i.e., $\mathcal{M}: \Gamma(\mathbf{U}) \rightarrow \Gamma(\mathbf{U})/\sigma \rightarrow \mathbb{C}^2$.

Definition 7. The complex curve $\Gamma(\mathbf{U})/\sigma$ is called the *spectral curve of an isothermic surface* Σ . It is said that $\mathcal{M}(\Gamma(\mathbf{U})/\sigma)$ is the *spectrum* of this surface.

The following proposition is again verified by straightforward computations.

Proposition 8. *If $\widehat{\varphi}$ satisfies (40), then*

- (1) $\psi = (e^{-\alpha}\varphi_3, e^{-\alpha}\varphi_4)$ satisfies the Dirac equation $\mathcal{D}\psi = 0$ with the potential

$$U = \frac{k_1 + k_2}{4} e^\alpha;$$

- (2) $\psi^* = (e^{-\alpha}\varphi_2, e^{-\alpha}\varphi_1)$ satisfies the Dirac equation $\mathcal{D}\psi^* = 0$ with the potential

$$U^* = \frac{k_2 - k_1}{4} e^\alpha.$$

We see that the Floquet functions of $(\partial_z - \mathbf{U})$ and the Dirac operators \mathcal{D} with potentials U and U^* have the same multipliers and obtain the following theorem.

Theorem 6. *The spectral curve and the spectrum of an isothermic surface Σ coincide with*

- (1) *a component, of the spectrum of Σ defined as in Section 4.1, that contains both asymptotic ends, where $\mu(\gamma_j) \approx e^{\lambda+\gamma_j}$ and $\mu(\gamma_j) \approx e^{\lambda-\bar{\gamma}_j}$ as $\lambda_\pm \rightarrow \infty$;*
- (2) *a component, of the spectrum of the dual surface Σ^* defined as in Section 4.1, that contains both asymptotic ends, where $\mu(\gamma_j) \approx e^{\lambda+\gamma_j}$ and $\mu(\gamma_j) \approx e^{\lambda-\bar{\gamma}_j}$ as $\lambda_\pm \rightarrow \infty$.*

The spectral curve of the isothermic surface is an irreducible component of the spectral curves of this surface and its dual defined as in Section 4.1.

Of course, when speaking about irreducibility, we exclude the case $U \equiv 0$.

Now, the Pretheorem or, more precisely, its statement about the irreducibility of the spectral curve implies that

- *the spectrum and the spectral curve of an isothermic surface coincide with the spectrum and the spectral curve of this surface and its dual defined as in Section 4.1.*

Here, we consider a general case when the surface may be an immersed plane rather than only a torus because usually the dual surface of an isothermic torus is not closed.

In [39], we introduced a particular case of the conjecture on the isospectrality of an isothermic surface and its dual. We did it for surfaces of revolution, for which the isospectrality is equivalent to the coincidence of all Kruskal–Miura integrals. Theorem 6 proves the general conjecture modulo the Pretheorem and, since the Pretheorem holds for surfaces of revolution (see example 3 in Section 3.2 and [39]), implies the following theorem.

Theorem 7. *A torus of revolution Σ and its dual surface Σ^* have the same values of the Kruskal–Miura invariants \mathcal{K}_l .*

The spectra of one-dimensional Schrödinger operators related by the Miura transformation (28) coincide because they are both doubly covered by the spectrum of L [10, 42]. Moreover, the Floquet function ψ of L consists of two components η_1 and η_2 , which are the Floquet functions of the Schrödinger operators. Similarly, the spectra of Dirac operators corresponding to an isothermic surface and its dual surface are doubly covered by the spectrum of $\widehat{L} = [\partial_z - \widehat{U}(\lambda)]$ for $\lambda = 0$, and the Floquet function of \widehat{L} consists of the Floquet functions ψ and ψ^* of the Dirac operators (Proposition 7).

Recall the formulas for the Kruskal–Miura invariants for surfaces of revolution. Choose a conformal parameter $z = x + iy$ such that x is a parameter on the rotating curve and y is the angle of rotation. Let T be the minimal period of the potential $U(x)$ of the surface.

The densities of the Kruskal–Miura integrals for the KdV equation are

$$R_1 = -q, \quad R_{n+1} = -R_{nx} - \sum_{k=1}^{n-1} R_k R_{n-k}.$$

Since R_{2n} are the derivatives, only the integrals

$$H_n(q) = \int_0^T R_{2n-1} dx$$

do not vanish identically. The simplest integrals are

$$H_1(q) = \int q dx, \quad H_2(q) = \int q^2 dx, \quad H_3(q) = \int (2q^3 - q_x^2) dx.$$

Put $q = 2iU_x - U^2$ and define the *Kruskal–Miura* invariants as

$$\mathcal{K}_l(U) = 2\pi H_l(q).$$

Note that \mathcal{K}_1 is the Willmore functional and others are multiples of \mathcal{W}_l .

6. SURFACES IN THE THREE-SPHERE

6.1. The Dirac equation for surfaces in the three-sphere. Let G be the Lie group $SU(2)$ and \mathcal{G} be its Lie algebra $su(2)$ identified with the tangent space $T_e G$ to G at the unity e . This Lie algebra is spanned by the elements

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{42}$$

which satisfy the commutation relations $[e_j, e_k] = 2\varepsilon_{jkl}e_l$. Take a biinvariant metric on G :

$$\langle \xi, \eta \rangle = -\frac{1}{2} \text{Tr}(\xi\eta), \quad \xi, \eta \in \mathcal{G} = T_e G,$$

in which the basis $\{e_1, e_2, e_3\}$ is orthonormal and G is isometric to the unit 3-sphere in \mathbb{R}^4 .

Let Σ be a surface immersed into G , let $z = x + iy$ be a conformal parameter on Σ , let $f: \Sigma \rightarrow G$ be the immersion, and let $\mathbf{I} = e^{2\alpha} dz d\bar{z}$ be the induced metric.

Take the pullback of TG to a \mathcal{G} -bundle over Σ : $\mathcal{G} \rightarrow E = f^{-1}(TG) \xrightarrow{\pi} \Sigma$ and the differential

$$d_{\mathcal{A}}: \Omega^1(\Sigma; E) \rightarrow \Omega^2(\Sigma; E),$$

which acts on E -valued 1-forms on Σ as follows. Let

$$\omega = f \cdot v dz + f \cdot v^* d\bar{z},$$

where $f \cdot: T_e G \rightarrow T_{f(p)} G$ is the left translation by $f(p)$. Then,

$$d_{\mathcal{A}}\omega = d'_{\mathcal{A}}\omega + d''_{\mathcal{A}}\omega,$$

where

$$d'_{\mathcal{A}}\omega = d'_{\mathcal{A}}(f \cdot v dz) = f \cdot \left(-\bar{\partial}v - \frac{1}{2}[f^{-1} \cdot \bar{\partial}f, v] \right) dz \wedge d\bar{z},$$

$$d''_{\mathcal{A}}\omega = d''_{\mathcal{A}}(f \cdot v^* dz) = f \cdot \left(\partial v^* + \frac{1}{2}[f^{-1} \cdot \partial f, v^*] \right) dz \wedge d\bar{z}.$$

By straightforward computations, we derive that

$$d_{\mathcal{A}}(df) = 0. \tag{43}$$

Since $*dz = -i dz$ and $*d\bar{z} = i d\bar{z}$, we have

$$d_{\mathcal{A}}(*df) = f \cdot (i\bar{\partial}(f^{-1} \cdot \partial f) + i\partial(f^{-1} \cdot \bar{\partial}f)) dz \wedge d\bar{z}.$$

By the definition of the mean curvature H , we have

$$d_{\mathcal{A}}(*df) = f \cdot (e^{2\alpha}\tau(f)) dx \wedge dy = \frac{i}{2}f \cdot (e^{2\alpha}\tau(f)) dz \wedge d\bar{z},$$

where $\tau(f)$ is the tension vector, $f \cdot \tau(f) = 2HN$, and N is the normal vector: $f^{-1} \cdot N = -ie^{-2\alpha} \times [f^{-1} \cdot \partial f, f^{-1} \cdot \bar{\partial}f]$. Finally, we obtain

$$d_{\mathcal{A}}(*df) = f \cdot (H[f^{-1} \cdot \partial f, f^{-1} \cdot \bar{\partial}f]) dz \wedge d\bar{z}. \tag{44}$$

The case $H = 0$ is described by the harmonicity equation

$$d_{\mathcal{A}}(*df) = 0.$$

Put

$$df = f \cdot (\Psi dz + \Psi^* d\bar{z}) \tag{45}$$

and rewrite (43) and (44) as

$$\bar{\partial}\Psi - \partial\Psi^* + [\Psi^*, \Psi] = 0, \tag{46}$$

$$\bar{\partial}\Psi + \partial\Psi^* = iH[\Psi^*, \Psi]. \tag{47}$$

Since $f^{-1} \cdot f_x = a^j e_j$ and $f^{-1} \cdot f_y = b^k e_k$ with $a^j, b^k \in \mathbb{R}$, we have

$$\Psi = \sum_{j=1}^3 Z_j e_j, \quad \Psi^* = \sum_{j=1}^3 \bar{Z}_j e_j,$$

where $Z_j = (a^j - ib^j)/2$ and $\Psi, \Psi^* \in su(2) \otimes \mathbb{C}$. The induced metric is

$$\begin{aligned} e^{2\alpha} dz d\bar{z} &= -\frac{1}{2} \operatorname{Tr}[(\Psi dz + \Psi^* d\bar{z})^2] \\ &= (Z_1^2 + Z_2^2 + Z_3^2)(dz)^2 + 2(|Z_1|^2 + |Z_2|^2 + |Z_3|^2) dz d\bar{z} + (\bar{Z}_1^2 + \bar{Z}_2^2 + \bar{Z}_3^2)(d\bar{z})^2, \end{aligned}$$

and we conclude that

$$|Z_1|^2 + |Z_2|^2 + |Z_3|^2 = \frac{1}{2} e^{2\alpha}, \quad Z_1^2 + Z_2^2 + Z_3^2 = 0.$$

Representing solutions to the latter equation as in Section 2.2,

$$Z_1 = \frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2), \quad Z_2 = \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2), \quad Z_3 = \psi_1 \bar{\psi}_2, \tag{48}$$

we derive that

$$e^\alpha = |\psi_1|^2 + |\psi_2|^2.$$

Now, rewriting (46) in terms of ψ_j and expanding it over the basis $\{e_j\}$, we show that (46) is equivalent to the system

$$\bar{\partial}(\psi_1 \bar{\psi}_2) - \partial(\bar{\psi}_1 \psi_2) = i(|\psi_2|^4 - |\psi_1|^4), \quad \bar{\partial}(\psi_1^2) + \partial(\psi_2^2) = 2i\psi_1 \psi_2 e^\alpha.$$

Analogously, we show that (47) is equivalent to the system

$$\bar{\partial}(\psi_1 \bar{\psi}_2) + \partial(\bar{\psi}_1 \psi_2) = H(|\psi_2|^4 - |\psi_1|^4), \quad \bar{\partial}(\psi_1^2) - \partial(\psi_2^2) = 2H\psi_1 \psi_2 e^\alpha.$$

Introduce $V_1 = -\partial\psi_2/\psi_1$ and $V_2 = \bar{\partial}\psi_1/\psi_2$. It follows from (46) that $\operatorname{Re} V_1 = \operatorname{Re} V_2$, $\operatorname{Im} V_1 = -e^\alpha/2$, and $\operatorname{Im} V_2 = e^\alpha/2$, and (47) implies that $\operatorname{Re} V_1 = \operatorname{Re} V_2 = He^\alpha/2$.

Finally, we obtain the following theorem.

Theorem 8. *For any immersed surface Σ in S^3 , the spinor field ψ defined by (45) and (48) satisfies the Dirac equation*

$$\mathcal{D}^S \psi = 0$$

with

$$\mathcal{D}^S = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} V & 0 \\ 0 & \bar{V} \end{pmatrix}, \quad V = \frac{1}{2}(H - i)(|\psi_1|^2 + |\psi_2|^2). \tag{49}$$

This spinor field is unique by its construction, and we say that ψ is the generating spinor for a surface.

Note that not all solutions ψ to the equation $\mathcal{D}^S \psi = 0$ correspond to a surface but only those that satisfy the condition

$$|\psi_1|^2 + |\psi_2|^2 = -2 \operatorname{Im} V.$$

It is easy to verify that, if $\mathcal{D}^S \psi = 0$, then $\mathcal{D}^S \varphi = 0$ with $\varphi = (\bar{\psi}_2, -\bar{\psi}_1)^\top$.

Let us write out the complete system of the Gauss–Weingarten equations. Recall that the Hopf differential equals $A dz^2 = \langle f_{zz}, N \rangle dz^2$, and, since the metric is left-invariant, we have $A = \langle f^{-1} f_{zz}, f^{-1} N \rangle$. Now, $\Psi = f^{-1} f_z$ and

$$f^{-1} f_{zz} = \Psi_z + \Psi^2,$$

where

$$\Psi = \begin{pmatrix} iZ_1 & Z_2 + iZ_3 \\ -Z_2 + iZ_3 & -iZ_1 \end{pmatrix}, \quad \Psi^* = \begin{pmatrix} i\bar{Z}_1 & \bar{Z}_2 + i\bar{Z}_3 \\ -\bar{Z}_2 + i\bar{Z}_3 & -i\bar{Z}_1 \end{pmatrix}. \tag{50}$$

We have $\Psi^2 = (Z_1^2 + Z_2^2 + Z_3^2)e_1$ and, since z is a conformal parameter, $\Psi^2 = 0$. Therefore, as in Section 2.2, the Hopf differential takes the form

$$A dz^2 = (\psi_{1z}\bar{\psi}_2 - \bar{\psi}_{2z}\psi_1) dz^2.$$

We also have

$$\alpha_z e^\alpha = \bar{\psi}_1 \psi_{1z} + \psi_2 \bar{\psi}_{2z},$$

and we finally write down the Gauss–Weingarten equations for a surface in S^3 as

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z & Ae^{-\alpha} \\ -V & 0 \end{pmatrix} \right] \psi = \left[\frac{\partial}{\partial \bar{z}} - \begin{pmatrix} 0 & \bar{V} \\ -\bar{A}e^{-\alpha} & \alpha_{\bar{z}} \end{pmatrix} \right] \psi = 0. \tag{51}$$

The compatibility conditions are the Gauss–Codazzi equations

$$\alpha_{z\bar{z}} + |V|^2 - |A|^2 e^{-2\alpha} = 0, \quad A_{\bar{z}} = (\bar{V}_z - \alpha_z \bar{V}) e^\alpha. \tag{52}$$

Examples. 1. *The Clifford torus.* This torus in \mathbb{R}^4 is defined by the equations

$$(x^1)^2 + (x^2)^2 = (x^3)^2 + (x^4)^2 = \frac{1}{2},$$

where $(x^1, \dots, x^4) \in \mathbb{R}^4$. It is immersed into $SU(2)$ by the formula

$$f(x, y) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ix} & e^{iy} \\ -e^{-iy} & e^{-ix} \end{pmatrix}$$

and has a conformal type of the square torus, $(x, y) \in \mathbb{Z}^2/2\pi\mathbb{Z}^2$. We have

$$\begin{aligned} \Psi &= \frac{1}{4} \left((1+i)e_1 + (1-i)\sin(x-y)e_2 + (1-i)\cos(x-y)e_3 \right), \\ \psi_1 &= \sqrt{\frac{1-i}{2}} \sin\left(\frac{x-y}{2} - \frac{\pi}{4}\right), \quad \psi_2 = \sqrt{\frac{1+i}{2}} \cos\left(\frac{x-y}{2} - \frac{\pi}{4}\right), \\ e^\alpha &= \frac{1}{\sqrt{2}}, \quad V = -\frac{i}{2\sqrt{2}}, \quad A = \frac{1}{4}. \end{aligned}$$

2. *Minimal tori in S^3 .* In this case, we have $V = -ie^\alpha/2$ and derive from (52) that $A_{\bar{z}} = 0$. This means that the Hopf differential is holomorphic and, as in the case of CMC tori in \mathbb{R}^3 , we conclude that it is constant. By rescaling the conformal parameter, we arrive at a situation when $A = 1/2$. The case $A = 0$ is also excluded for tori: it is realized by the equatorial S^2 -spheres in S^3 (complete CMC surfaces in \mathbb{R}^3 with $A = 0$ are round spheres). The first equation in (52) is

$$u_{z\bar{z}} + \sinh u = 0, \quad u = 2\alpha.$$

For CMC tori in S^3 , the Hopf differential is also holomorphic, and they are described in the same manner.

Now, it is clear that the analogues of Theorems 1, 2, and 3 hold for surfaces in S^3 . We have the same spinor bundles over constant curvature surfaces.

Definition 8. For a torus represented in S^3 via ψ that satisfies (51), the spectral curve Γ^S of the operator \mathcal{D}^S with potential (49) is called the *spectral curve of the torus*. If, in addition, γ_1, γ_2 is a basis for Λ , the period lattice for U , then the image of the multiplier map

$$\mathcal{M}: Q_0(\mathcal{D}^S)/\Lambda^* \rightarrow \mathbb{C}^2, \quad \mathcal{M}(k) = (e^{2\pi i \langle k, \gamma_1 \rangle}, e^{2\pi i \langle k, \gamma_2 \rangle})$$

is called the *spectrum of the torus in S^3* .

Example. Let Σ be the Clifford torus. Then, $V = -i/2\sqrt{2}$ and the Floquet eigenfunctions $\psi = (\psi_1, \psi_2)^\top$ satisfy the equation

$$\left(\partial\bar{\partial} + \frac{1}{8}\right)\psi_j = 0, \quad j = 1, 2.$$

We derive that the general Floquet function is

$$\psi(z, \bar{z}, \lambda) = \left(e^{\lambda z - \frac{1}{8\lambda}\bar{z}}, \frac{i}{2\sqrt{2}\lambda} e^{\lambda z - \frac{1}{8\lambda}\bar{z}} \right)^\top,$$

and find the spectrum as the image of the multiplier map

$$\lambda \in \mathbb{C} \setminus \{0\} \rightarrow \left(e^{2\pi(\lambda - \frac{1}{8\lambda})}, e^{2\pi i(\lambda + \frac{1}{8\lambda})} \right).$$

The spectral curve is the two-sphere, i.e., the punctured plane $\mathbb{C} \setminus \{0\}$ compactified by the points $\lambda = 0, \infty$. This implies that the spectral genus of the Clifford torus equals zero.

We will not discuss the spectra of tori in S^3 in detail; we only mention that the Pretheorem must also hold for them.

6.2. The Hitchin system. Let us compare the previous computations with the Hitchin theory of harmonic tori in the 3-sphere [18]. For Riemannian manifolds N and M , a mapping $f: N \rightarrow M$ is called *harmonic* if it satisfies the equations

$$d_{\mathcal{A}}(df) = d_{\mathcal{A}}(*df) = 0,$$

where \mathcal{A} is the pullback of the Levi-Civita connection on TM and the Hodge operator $*$ is taken with respect to the metric on N . If f is an embedding and the metric on N is the induced metric, then $f(N)$ is a minimal submanifold.

Let N be an immersed surface Σ with the induced metric and $M = G$ be a Lie group with a biinvariant metric. We adopt the notation from Section 6.1.

The harmonicity equations take the form

$$\begin{aligned} d_{\mathcal{A}}(\Psi dz + \Psi^* d\bar{z}) &= 0, \\ d_{\mathcal{A}}*(\Psi dz + \Psi^* d\bar{z}) &= -i d_{\mathcal{A}}(\Psi dz - \Psi^* d\bar{z}) = 0. \end{aligned}$$

They describe minimal surfaces in S^3 and are rewritten as

$$\bar{\partial}\Psi - \partial\Psi^* + [\Psi^*, \Psi] = \bar{\partial}\Psi + \partial\Psi^* = 0. \tag{53}$$

Following [18], put

$$\Phi = \frac{1}{2}\Psi, \quad \Phi^* = -\frac{1}{2}\Psi^*$$

and rewrite (53) as the *Hitchin system*

$$d''_{\mathcal{A}}\Phi = 0, \quad F_{\mathcal{A}} = d_{\mathcal{A}}^2 = [\Phi, \Phi^*] = 0, \tag{54}$$

where $F_{\mathcal{A}}$ is the curvature of the connection,

$$d_{\mathcal{A}}: \Omega^p(\Sigma; f^{-1}(TG)) \rightarrow \Omega^{p+1}(\Sigma; f^{-1}(TG)),$$

and the formula means that $d_{\mathcal{A}}^2$ coincides with the multiplication by $F_{\mathcal{A}}$. System (54) describes general harmonic mappings of surfaces in S^3 (when the metric on the surface is not necessarily induced) in terms of a connection \mathcal{A} associated to the harmonic map and the Higgs field Φ .

The equation $d_{\mathcal{A}}df = 0$ is equivalent to

$$\bar{\partial}\Psi - \partial\Psi^* + [\Psi^*, \Psi] = 0$$

and means that the connection $\mathcal{A} = (\partial + \Psi, \bar{\partial} + \Psi^*)$ on $f^{-1}(TG)$ is flat, which is evident from its construction. However, the second of the equations (53) implies that this connection is extended to an analytic family of flat connections

$$\mathcal{A}_\lambda = \left(\partial + \frac{1 + \lambda^{-1}}{2}\Psi, \bar{\partial} + \frac{1 + \lambda}{2}\Psi^* \right),$$

where $\mathcal{A} = \mathcal{A}_1$ and $\lambda \in \mathbb{C} \setminus \{0\}$. This commutation representation with a spectral parameter was found by Pohlmeyer [35] for harmonic maps into $SU(2)$ and was later developed by Mikhailov and Zakharov for the case when the target space is not a Lie group but a symmetric space S^2 [46]. These two papers gave rise to the ‘‘integrability’’ part of the modern theory of harmonic maps [44, 18, 7, 15].

For harmonic tori, Hitchin introduced spectral curves and showed that they are of finite genus [18]. Their construction is as follows.

Let Σ be a harmonic torus in S^3 . For any $\lambda \in \mathbb{C} \setminus \{0\}$, we have a flat $SL(2, \mathbb{C})$ connection. Fix a basis $\{\gamma_1, \gamma_2\}$ for $H_1(\Sigma)$. For γ_1 and γ_2 , define matrices $H(\lambda), \tilde{H}(\lambda) \in SL(2, \mathbb{C})$ that describe the monodromies of \mathcal{A}_λ along closed loops that realize γ_1 and γ_2 . These matrices commute and have common eigenfunctions $\varphi(\lambda, \mu)$, where μ is a root of the characteristic equation for $H(\lambda)$,

$$\mu^2 - \text{Tr } H(\lambda) + 1 = 0;$$

therefore, there exists a Riemann surface on which the eigenvalues

$$\mu_{1,2} = \frac{1}{2} \left(\text{Tr } H(\lambda) \pm \sqrt{\text{Tr}^2 H(\lambda) - 4} \right)$$

are defined. A complex curve Γ that is a two-sheeted covering of $\mathbb{C}P^1$ ramified at the odd zeros of the function $\text{Tr}^2 H(\lambda) - 4$ and at 0 and ∞ is called the *spectral curve of a harmonic torus* in S^3 .

On Γ , the eigenvalues of $H(\lambda)$ are pasted together to form a single-valued function μ with singularities at 0 and ∞ . Moreover, the common eigenfunctions of $H(\lambda)$ and $\tilde{H}(\lambda)$ are pasted into a vector function φ that is meromorphic on $\Gamma \setminus \{0, \infty\}$.

Let $f: \Sigma \rightarrow S^3$ be an immersion of a minimal torus, $\Psi = f^{-1}f_z$, and $\Psi^* = f^{-1}f_{\bar{z}}$. Let the surface be defined by a spinor ψ .

The Hitchin eigenfunction $\varphi(z, \bar{z}, \lambda, \mu)$ satisfies the equations

$$\left[\partial + \frac{1 + \lambda}{2}\Psi \right] \varphi = \left[\bar{\partial} + \frac{1 + \lambda^{-1}}{2}\Psi^* \right] \varphi = 0.$$

Take the matrix

$$L = \begin{pmatrix} \bar{a} & -\bar{b} \\ b & a \end{pmatrix} \tag{55}$$

with $a = (-i\bar{\psi}_1 + \psi_2)/\sqrt{2}$ and $b = (-i\psi_1 + \bar{\psi}_2)/\sqrt{2}$. Using (48) and (50), we compute that

$$L^{-1}\Psi L = e^\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L^{-1}\Psi^* L = e^\alpha \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

We also have

$$L^{-1}L_z = \begin{pmatrix} \alpha_z & -iV \\ -iAe^{-\alpha} & 0 \end{pmatrix}, \quad L^{-1}L_{\bar{z}} = \begin{pmatrix} 0 & -i\bar{A}e^{-\alpha} \\ -i\bar{V} & \alpha_{\bar{z}} \end{pmatrix}.$$

The vector function $L^{-1}\varphi$ satisfies the equations

$$\begin{aligned} \left[\partial + \begin{pmatrix} \alpha_z & -iV \\ -iAe^{-\alpha} & 0 \end{pmatrix} + \frac{1+\lambda}{2} e^\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] L^{-1}\varphi &= 0, \\ \left[\bar{\partial} + \begin{pmatrix} 0 & -i\bar{A}e^{-\alpha} \\ -i\bar{V} & \alpha_{\bar{z}} \end{pmatrix} + \frac{1+\lambda^{-1}}{2} e^\alpha \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right] L^{-1}\varphi &= 0. \end{aligned}$$

These two equations are compatible only for minimal tori, which are described by the condition

$$V = -\frac{ie^\alpha}{2}.$$

For $\tilde{\varphi} = e^\alpha L^{-1}\varphi$, we derive

$$\partial\tilde{\varphi}_1 + \frac{\lambda}{2}e^\alpha\tilde{\varphi}_2 = 0, \quad \bar{\partial}\tilde{\varphi}_2 - \frac{1}{2\lambda}e^\alpha\tilde{\varphi}_1 = 0.$$

Put $\tilde{\psi}_1 = i\lambda\tilde{\varphi}_2$ and $\tilde{\psi}_2 = \tilde{\varphi}_2$ and note that $\tilde{\psi}$ satisfies (49):

$$\left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} V & 0 \\ 0 & \bar{V} \end{pmatrix} \right] \tilde{\psi} = 0 \quad \text{with } V = -\frac{ie^\alpha}{2}.$$

As in the proofs of Theorems 5 and 6, we arrive at the following theorem.

Theorem 9. *For a minimal torus $f: \Sigma \rightarrow S^3$, the Hitchin eigenfunction $\varphi(z, \bar{z}, \lambda, \mu)$ is mapped to a Floquet function $\tilde{\psi}$ of \mathcal{D}^S by the transformation*

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = e^\alpha \begin{pmatrix} 0 & i\lambda \\ 1 & 0 \end{pmatrix} L^{-1} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

There is a mapping of the eigenvalues μ and $\tilde{\mu}$ of φ with respect to the monodromy operators $H(\lambda)$ and $\tilde{H}(\lambda)$ to the multipliers of $\tilde{\psi}$:

$$(\mu, \tilde{\mu}) \rightarrow ((-1)^{\varepsilon(\gamma_1)}\mu, (-1)^{\varepsilon(\gamma_2)}\tilde{\mu}), \quad (56)$$

where $(-1)^{\varepsilon(\gamma_1)}$ and $(-1)^{\varepsilon(\gamma_2)}$ are the multipliers of the spinor ψ that generates a minimal torus.

Mapping (56) establishes a biholomorphic equivalence between the Hitchin spectral curve of a minimal torus and the connected component of the spectrum of this torus defined in Section 6.1. This connected component contains both asymptotic ends near which $\tilde{\psi} \approx (e^{\lambda+z}, 0)^\top$ or $\tilde{\psi} \approx (0, e^{\lambda-\bar{z}})^\top$.

If the Pretheorem holds for \mathcal{D}^S , then the spectrum is irreducible, and, therefore, (56) establishes a biholomorphic equivalence of the spectra.

7. CONFORMAL INVARIANCE OF THE SPECTRA OF TORI

7.1. The Möbius group. We consider \mathbb{R}^4 as the space of matrices

$$a = \begin{pmatrix} x^4 + ix^1 & x^2 + ix^3 \\ -x^2 + ix^3 & x^4 - ix^1 \end{pmatrix}, \quad x^1, x^2, x^3, x^4 \in \mathbb{R}, \tag{57}$$

and \mathbb{R}^3 , as a subset described by $x^4 = 0$. The unit sphere $S^3 = SU(2)$ is defined by the equation $|x| = 1$. Take the north pole $P = (0, 0, 0, 1)$ and denote by π the stereographic projection of S^3 to $\mathbb{R}^3 = \{x^4 = 0\}$ from P :

$$\pi: a \rightarrow \frac{1}{1 - x^4} \begin{pmatrix} ix^1 & x^2 + ix^3 \\ -x^2 + ix^3 & -ix^1 \end{pmatrix} = (1 + a)(1 - a)^{-1}.$$

The inverse mapping is given by $\pi^{-1}: b \rightarrow (b - 1)(b + 1)^{-1}$. The mapping π establishes a conformal equivalence between S^3 and \mathbb{R}^3 compactified by a point at infinity, i.e., by $\pi(P) = \infty$.

The group of conformal transformations of $\bar{\mathbb{R}}^3 = \mathbb{R}^3 \cup \infty$ is isomorphic to $O^+(1, 4)$, the subgroup of $O(1, 4)$ formed by isochronic transformations. The geometric picture is as follows. Let $\mathbb{R}^{1,4}$ be a 5-dimensional pseudo-Euclidean space with the metric $\langle x, y \rangle_{1,4} = x^0 y^0 - \sum_{j=1}^4 x^j y^j$. The 4-dimensional hyperbolic space \mathcal{H}^4 is embedded into $\mathbb{R}^{1,4}$ as the upper half of a hyperboloid, $\langle x, x \rangle_{1,4} = 1$, $x^0 > 0$, with the metric $\langle \xi, \xi \rangle = -\langle \xi, \xi \rangle_{1,4}$. The group of isometries of \mathcal{H}^4 is $O^+(1, 4)$, and it acts on S^3 , the absolute of \mathcal{H}^4 , by conformal transformations.

By the Liouville theorem, the group $O^+(1, 4)$ is generated by isometries of \mathbb{R}^3 , inversions with centers at $x_0 \in \mathbb{R}^3: x \rightarrow \frac{x-x_0}{|x-x_0|^2}$, and the homotheties $x \rightarrow \lambda x$, $\lambda \in \mathbb{R} \setminus \{0\}$. Any conformal transformation of $\bar{\mathbb{R}}^3$ that preserves ∞ is a composition of isometries and homotheties.

We believe that

- *the spectrum of a torus in S^3 coincides with the spectrum of its image in \mathbb{R}^3 under the stereographic projection.*

We cannot prove this now but would like to note that this statement easily implies the conformal invariance of both spectra: it is clear that the spectrum of a torus in \mathbb{R}^3 is invariant under translations of the torus and the spectrum of a torus in S^3 is invariant under rotations. However, the stereographic projection converts rotations in S^3 into conformal transformations of \mathbb{R}^3 , which, together with translations and homotheties, generate the conformal group $O^+(1, 4)$. The same holds for the translations of \mathbb{R}^3 : they are mapped under the projection to translations, which, together with rotations, generate the group of conformal transformations of S^3 .

7.2. Conformal invariance of the spectra for isothermic tori in \mathbb{R}^3 .

Theorem 10. *Let Σ be an isothermic torus in \mathbb{R}^3 and let $F: \bar{\mathbb{R}}^3 \rightarrow \bar{\mathbb{R}}^3$ be a conformal transformation that maps Σ into a torus $F(\Sigma)$ lying in \mathbb{R}^3 . Then, the spectra of Σ and $F(\Sigma)$ coincide.*

Note that the spectrum of an isothermic torus is defined as a component of the general spectrum that contains the asymptotic ends where $\mu(\gamma_j) \approx e^{\lambda+\gamma_j}$ and $\mu(\gamma_-) \approx e^{\lambda-\bar{\gamma}_j}$ as $\lambda_{\pm} \rightarrow \infty$. The Pretheorem states that the general spectrum is irreducible and therefore coincides with this component.

Proof. By Theorem 6, the spectrum of an isothermic torus and its dual isothermic surface coincide. The potential of the dual surface equals

$$U^* = \frac{k_2 - k_1}{4} e^\alpha,$$

and, by the Blaschke theorem, the density of the Willmore functional

$$\left(\frac{k_2 - k_1}{2}\right)^2 d\mu = 4(U^*)^2 dx \wedge dy$$

is invariant under conformal transformations of $\overline{\mathbb{R}^3}$. Conformal transformations map isothermic surfaces into isothermic ones.

Let z be a conformal parameter on Σ that is mapped into a conformal parameter on $F(\Sigma)$ and V be the potential of $F(\Sigma)$ with respect to this parameter. By the Blaschke theorem, $V^2 = (U^*)^2$ and, therefore, $V = \pm U^*$.

It is clear that the spectra of the Dirac operators whose potentials differ by sign coincide. Now, we conclude that the spectra of the isothermic tori Σ and $F(\Sigma)$ coincide with the spectrum of the isothermic surface with the potential U^* . This proves the theorem.

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