

Tame Integrals of Motion and o-Minimal Structures

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1. Introduction

Integrability of a Hamiltonian flow presumes that we can describe the flow in a simple way, and in particular, the decomposition of its phase space into invariant Liouville tori and the singular locus looks very simple geometrically. In this case we say that the integrals of motion are tame. Usually this is the case when the flow is integrable in terms of (real) analytic functions as it has already been shown in various examples. Here we would like to present some ideas taken from mathematical logics, namely from the theory of o-minimal structures, which clarify the analytic notion of a tame integral of motion, and to demonstrate this approach by the integrability problem for geodesic flows.

2. Different meanings of integrability

Let M^{2n} be a symplectic manifold with the symplectic form

$$\omega = \sum_{i < j} \omega_{ij} dx^i \wedge dx^j.$$

Denote by ω^{ij} the inverse matrix to a skew-symmetric matrix ω_{ij} . Then each smooth function H on this manifold defines a Hamiltonian flow by the equation which describes the evolution of any smooth function f along the trajectories of the flow:

$$\frac{df}{dt} = \{f, H\} = \omega^{ij} \frac{\partial f}{\partial x^i} \frac{\partial H}{\partial x^j}.$$

We recall that the function H is called the Hamiltonian function of the flow (or just the Hamiltonian) and the skew-symmetric operation $\{f, g\}$ on smooth functions is called the Poisson brackets.

It is said that a function f is a first integral, or an integral of motion, of the flow if it is preserved by the flow:

$$\{f, H\} = 0.$$

The Liouville or complete integrability of a Hamiltonian flow is defined as follows:

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- a flow on an n -dimensional symplectic manifold M^{2n} is called integrable if it admits a family of first integrals $I_1, \dots, I_n = H$ such that these integrals are in involution:

$$\{I_j, I_k\} = 0, \quad 1 \leq j, k \leq n,$$

and they are functionally independent almost everywhere, i.e., outside a certain nowhere dense set Σ , which is called the singular locus.

The mapping

$$\Phi : M^{2n} \rightarrow \mathbb{R}^n, \quad \Phi(q) = (I_1(q), \dots, I_n(q)),$$

is called the momentum map.

The condition that the first integrals are in involution means that the Hamiltonian flows generated by the functions I_1, \dots, I_n as Hamiltonians commute everywhere. If X is a compact component of the level surface $\Phi = c$ on which the integrals are functionally independent, then this component is diffeomorphic to a torus on which the Hamiltonian flows corresponding to I_1, \dots, I_n are linearized, and moreover such a linearization can be extended to a neighborhood of X .

The proofs of these statements on integrable flows are presented in [1].

But we see that there is a freedom in the definition when we speak about functional independence of first integrals almost everywhere. It could be that

- they are functionally independent on an open dense set;
- given a smooth measure on M^{2n} such that the measure of M^{2n} is finite, the first integrals are functionally independent on a subset of full measure.

Moreover, in classical mechanics the most popular situation is when M^{2n} and H are (real) analytic and

- the first integrals I_1, \dots, I_n are analytic.

In this case it is said that the flow is analytically integrable.

Another reasonable treatment of what one means by the “functional independence almost everywhere of integrals of motion” for a compact phase space is as follows:

- there is a finite smooth (or even analytic) simplicial decomposition of the phase space M^{2n} such that the singular locus Σ forms a subcomplex of this decomposition and the complement of it is cut by another subcomplex of positive codimension into a union of finitely many sets U_α which are foliated by invariant tori over their images under the momentum map.

We considered this notion in [13] and called it geometric simplicity.

It is reasonable to say that integrals of motion are tame if they lead to a geometrically simple behaviour of the flow.

Some important examples of Hamiltonian flows do not have a compact phase space. This is the case, for instance, for the geodesic flow of a Riemannian manifold M^n , which is a Hamiltonian flow on a cotangent bundle T^*M^n to this manifold. The symplectic structure on T^*M^n is given by the form

$$\omega = \sum_{j=1}^n dx^j \wedge dp_j,$$

where $p_j = g_{jk}\dot{x}^k$ and $g_{jk}dx^j dx^k$ is the Riemannian metric. The Hamiltonian of the geodesic flow is homogeneous in momenta:

$$H(x, p) = g^{jk}(x)p_j p_k,$$

where $g^{jk}g_{kl} = \delta_l^j$ and therefore the restrictions of the geodesic flow to different nonzero level surfaces of H are smoothly trajectory equivalent; moreover they are related by reparametrization of trajectories by a constant. Therefore it is reasonable to call the geodesic flow on M^n integrable if it satisfies a weaker condition (see [13]), which is

- there are $(n - 1)$ additional integrals of motion I_1, \dots, I_{n-1} which are in involution and almost everywhere independent on the unit covector bundle $SM^n = \{H = 1\}$.

It appears that analytic integrability is the strongest assumption that implies, in particular, geometric simplicity as was shown in [13].

Some recent examples of integrable geodesic flows of analytic metrics show that using even mildly nonanalytic C^∞ functions such as, for instance,

$$f(x, p) = \exp(-Q(p)^{-2}) \sin(\mu \log |p_x - \tau p_y|),$$

where μ, τ are constants and $Q(p)$ is a polynomial in momenta p_x, p_y divided by $(p_x - \tau p_y)$, for integration of geodesic flows leads to geometrically nonsimple flows [2, 3].

Therefore, *for studying topological properties of integrable flows by means of topology of finite CW-complexes or tame subsets in \mathbb{R}^n we must restrict ourselves to geometrically simple flows or to tame integrals of motion.*

In the next section we present some background material from mathematical logics which gives the most transparent analytic approach to understanding what it means that an integral of motion is tame.

Before we do that we would like to note that an analogue of the Morse theory for integrable systems on four-dimensional symplectic manifolds developed by Fomenko and his collaborators also requires a certain analytic condition to be fulfilled. This condition says that an additional (to the Hamiltonian) integral of motion must be of Bott type, i.e., its restrictions to planes that are normal to critical level surfaces locally is a Morse function. This is another type of tameness condition, which was generalized in [11] to a geometric condition under which this theory works.

3. o-minimal structures and analytic-geometric categories

3.1. o-minimal structures. By definition, a family \mathcal{S} of subsets of Euclidean spaces \mathbb{R}^n is called an o-minimal structure (on $(\mathbb{R}, +, \cdot, <)$) if, graded by the dimensions of ambient Euclidean spaces,

$$\mathcal{S} = \bigcup_{n \geq 1} \mathcal{S}_n,$$

it satisfies the following conditions:

- 1) \mathcal{S}_n is a Boolean algebra of some subsets of \mathbb{R}^n with a standard union operation (in particular, this means that this algebra is closed with respect to complements and finite unions and intersections);
- 2) if $X \in \mathcal{S}_n$ and $Y \in \mathcal{S}_k$, then $X \times Y \in \mathcal{S}_{n+k}$;
- 3) let $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be a projection $(x^1, \dots, x^n, x^{n+1}) \rightarrow (x^1, \dots, x^n)$; then $X \in \mathcal{S}_{n+1}$ implies $\pi(X) \in \mathcal{S}_n$;

- 4) \mathcal{S}_n contains all algebraic sets in \mathbb{R}^n , i.e., if $P(x^1, \dots, x^n)$ is a polynomial, then its zero set $\{P = 0\}$ belongs to \mathcal{S} ;
- 5) \mathcal{S}_1 consists of all finite unions of open intervals and points.

We present the main known examples of o-minimal structures:

- \mathbb{R}_{alg} : **semialgebraic sets**. It consists of all semialgebraic sets, i.e., sets determined by finitely many equations $P_1 = \dots = P_k = 0$ and inequalities $Q_1 > 0, \dots, Q_l > 0$, where $P_1, \dots, P_k, Q_1, \dots, Q_l$ are polynomials. Such sets form an o-minimal structure by the Tarski–Seidenberg theorem.
- \mathbb{R}_{an} : **finite subanalytic sets**. It consists of intersections of subanalytic sets in \mathbb{R}^n with cubes $[-D, D]^n$, and their projections. Note that the family formed only by intersections of subanalytic sets with cubes is not closed under projections. It is a theorem of Gabrielov [7], which implies that \mathbb{R}_{an} is an o-minimal structure.
- \mathbb{R}_{exp} . To a polynomial $P(x^1, \dots, x^{2k})$ we associate an exponential polynomial $Q(x^1, \dots, x^k) = P(x^1, \dots, x^k, e^{x^1}, \dots, e^{x^k})$ and denote by \mathbb{R}_{exp} the family of all sets generated by the zero sets of such exponential polynomials under projections $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$. Wilkie proved that this family is closed under complements and forms an o-minimal structure [15].
- \mathbb{R}_f , where f is a Pfaffian function. This is a generalization of \mathbb{R}_{exp} . It is said that a chain of C^1 -functions $f_1, \dots, f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Pfaffian chain if for each $j = 1, \dots, k$ the first derivatives of f_j with respect to x^1, \dots, x^n are polynomials in $x^1, \dots, x^n, f_1, \dots, f_j$. In this case the function $f = f_k$ is called a Pfaffian function. The similarity between the zero sets of Pfaffian functions and algebraic sets was first pointed out by Khovanskii [8]. It was proved by Wilkie that if we replace exp by a Pfaffian function f in the definition of \mathbb{R}_{exp} and close this family with respect to Boolean operations, projections and products, we obtain an o-minimal structure which is denoted by \mathbb{R}_f [16].
- $\mathbb{R}_{\text{an,exp}}$. It was proved by van den Dries, Macintyre, and Marker [6], using the results of Wilkie [15] that sets from \mathbb{R}_{an} and \mathbb{R}_{exp} generate, by Boolean operations and projections, an o-minimal structure which is denoted by $\mathbb{R}_{\text{an,exp}}$.

These o-minimal structures are related by the following evident inclusions:

$$\mathbb{R}_{\text{alg}} \subset \mathbb{R}_{\text{an}} \subset \mathbb{R}_{\text{an,exp}}.$$

Given an o-minimal structure \mathcal{S} , we say that

- a subset $X \in \mathbb{R}^n$ is definable if $X \in \mathcal{S}_n$; ¹
- a mapping $f : X \rightarrow \mathbb{R}^k$ with $X \subset \mathbb{R}^n$ is definable if its graph $\{(x, f(x))\}$ is a definable set, i.e., belongs to \mathcal{S}_{n+k} ;

¹This terminology originates in mathematical logics and reflects the fact that definable sets are exactly the sets which are defined by the first order logics formulas involving addition “+”, multiplication “.” and linear ordering “<” plus some additional functions which lead to extensions of the smallest o-minimal structure on $(\mathbb{R}, +, \cdot, <)$, i.e., the subalgebraic sets. These are analytic functions restricted to cubes $[-1, 1]^n$ for \mathbb{R}_{an} , the exponential function exp for \mathbb{R}_{exp} , etc. If we drop the multiplication from the signature of our language (in the sense of mathematical logics) we must replace the fourth condition by another which reads that the graphs of some functions coming into the signature are definable. For instance, the smallest o-minimal structure which includes +, < and the multiplications by all real numbers $r \in \mathbb{R}$ is formed by all semilinear sets.

- a set X is \mathcal{S} -triangulable if $X \in \mathcal{S}$ and there is a definable mapping $f : X \rightarrow \mathbb{R}^n$ which maps X homeomorphically onto a union of open simplices of a finite simplicial complex $K \subset \mathbb{R}^n$. In this event we say that f defines an \mathcal{S} -triangulation of X .

By this definition, any \mathcal{S} -triangulable set is definable. The converse is also true:

THEOREM 1 (Triangulation Theorem). *Every definable set $X \in \mathcal{S}$ is \mathcal{S} -triangulable.*

This theorem is proved by a general method for all o-minimal structures [5]. The proof uses induction on the dimension of a definable set and starts with an evident statement that all sets from \mathcal{S}_1 are \mathcal{S} -triangulable. We sketched such a proof for sets from \mathbb{R}_{an} in [13].

We also notice that it follows from the definition of an o-minimal structure that images and preimages of definable sets under definable proper mappings are definable. Here we recall that a mapping is called proper if the preimage of a compact set is compact.

3.2. Geometric and analytic-geometric categories. For using the theory of o-minimal structures in topology and geometry one must develop its analog for subsets in manifolds. This was done in [4].

Given an o-minimal structure \mathcal{S} , we distinguish a class of \mathcal{S} -manifolds. We say that a smooth manifold M^n is an \mathcal{S} -manifold if it admits a finite \mathcal{S} -atlas $\{U_\alpha\}$, i.e., an atlas such that

- every coordinate mapping $\varphi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$ homeomorphically maps a chart U_α onto a definable set V_α ;
- for any intersection $U_\alpha \cap U_\beta$ the transition mapping

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$$

is definable.

Now we say that

- a subset $X \subset M^n$ is definable if for any chart U_α the set $\varphi_\alpha(X \cap U_\alpha) \subset \mathbb{R}^n$ is definable;
- for definable sets $X \subset M^n$ and $Y \subset N^k$ a mapping $f : X \rightarrow Y$ is definable if it is continuous and its graph $\{(x, f(x))\}$ is definable in $M^n \times N^k$.

Notice that these definitions are independent of the choice of \mathcal{S} -atlases for M^n and N^k .

Therefore an o-minimal structure \mathcal{S} defines a “geometric category” of \mathcal{S} -manifolds; their definable subsets are objects, and definable maps between them are morphisms. For instance, if $\mathcal{S} = \mathbb{R}_{\text{alg}}$, we have the category of so-called Nash manifolds [12].

Corresponding to an o-minimal structure \mathcal{S} which contains \mathbb{R}_{an} is an “analytic-geometric” category defined as follows:

- given an analytic manifold M^n , a subset $X \subset M^n$ is called definable if for any point $x \in X$ there exist an open neighborhood U of this point and an analytic isomorphism $\varphi : U \rightarrow V \subset \mathbb{R}^n$ such that $\varphi(X \cap U)$ is definable;
- a mapping $f : X \rightarrow Y$ of two definable sets $X \subset M^n$ and $Y \subset N^k$ is called definable if its graph is definable in $M^n \times N^k$.

Notice that by this definition the image of a definable set under a proper analytic mapping is a definable set.

The analytic-geometric category corresponding to \mathcal{S} has definable sets as objects and definable mappings as morphisms.

Introduction of these categories allows us to use the machinery developed for definable sets in \mathbb{R}^n , for instance the Triangulation Theorem and many other facts (see [4, 5]), for subsets in manifolds.

4. On obstructions to integrability

The first obstruction to integrability of geodesic flows was found by Kozlov, who proved that if the geodesic flow on a compact oriented analytic Riemannian two-manifold admits an additional analytic first integral, then the surface is diffeomorphic to a sphere or a torus [10].

The analyticity condition was strongly used in the proof, and it is still unknown whether the theorem is valid under only C^∞ assumptions for the manifold and first integrals. Using the theory of modular forms, Kolokoltsov extended the Kozlov theorem assuming that an additional first integral is C^∞ but polynomial in momenta [9].

High-dimensional obstructions were obtained by the author in [13] in two steps:

1) some obstructions to a nice geometric behaviour of the geodesic flow on a manifold were found, i.e., obstructions to its geometric simplicity;

2) some analytic properties of first integrals which imply geometric simplicity were established.

We remark here that the condition of geometric simplicity can be weakened, and by using the language of o-minimal structures the analytic condition can be clarified and slightly strengthened.

For realizing the first step we proved the following

THEOREM 2. *If the geodesic flow on a compact analytic manifold M^n is geometrically simple, then there is an invariant torus $T^n \subset SM^n$ such that the natural projection $\pi : SM^n \rightarrow M^n$ induces a homomorphism*

$$\pi_* : \pi_1(T^n) \rightarrow \pi_1(M^n)$$

whose image is a subgroup of finite index in $\pi_1(M^n)$.

COROLLARY 1. *If a geodesic flow on a compact manifold M^n is geometrically simple, then*

1) *the fundamental group $\pi_1(M^n)$ of M^n is almost commutative, i.e., contains a commutative subgroup of finite index;*

2) *the real cohomology ring $H^*(M; \mathbb{R})$ of M^n contains a subring A which is isomorphic to the real cohomology ring $H^*(T^k; \mathbb{R})$ of the k -dimensional torus, where k is the first Betti number of M^n : $b_1 = \dim H^1(M^n; \mathbb{R}) = k$;*

3) *moreover if the first Betti number of M^n equals its dimension, $b_1 = n$, then the ring $H^*(M^n; \mathbb{R})$ is isomorphic to $H^*(T^n; \mathbb{R})$.*

To explain these results we recall that a geodesic flow on M^n is said to be geometrically simple if the unit cotangent bundle SM^n admits a decomposition

$$SM^n = \Gamma \cup \left(\bigcup_{\alpha=1}^d U_\alpha \right)$$

such that

- this decomposition is invariant under the flow;
- the set Γ is closed and the complement of it is everywhere dense;
- for each point $q \in SM^n$ and every neighborhood V of it there is another neighborhood W of q such that $W \subset V$ and $W \cap (M^n \setminus \Gamma)$ has finitely many connected components;
- any component U_α is diffeomorphic to a product of an n -dimensional torus and an $(n - 1)$ -dimensional disc.

In fact we have proved that if we omit the fourth condition, then there is a component U_α such that the image of its fundamental group under the projection homomorphism $\pi_*(\pi_1(U_\alpha))$ has a finite index in $\pi_1(M^n)$.

Moreover in this formulation the proof of the theorem works for a more general case where the flow is locally simple, i.e., there is a point $x \in M^n$ and its neighborhood U such that

- the universal covering $\widehat{M}^n \rightarrow M^n$ is trivial over U ;
- the preimage of U under the projection $\pi : SM^n \rightarrow M^n$ admits a decomposition

$$\pi^{-1}(U) = \widetilde{\Gamma} \cup \left(\bigcup_{\alpha=1}^d \widetilde{U}_\alpha \right),$$

where $\widetilde{\Gamma}$ is closed and its complement is dense, and each component \widetilde{U}_α is an intersection of $\pi^{-1}(U)$ with an invariant open set U_α .

The second step was realized in [13] by the following

THEOREM 3. *If a geodesic flow on a compact manifold is analytically integrable, then it is geometrically simple.*

In proving this theorem the basic point is to show that given analytic first integrals I_1, \dots, I_{n-1} (here we assume that I_n is the Hamiltonian of the flow, $I_n = g^{ij}(x)p_i p_j$) the set C of the critical values of the momentum map restricted to SM^n ,

$$\Phi : q \rightarrow (I_1(q), \dots, I_{n-1}(q)) \in \mathbb{R}^{n-1}$$

and its preimage in SM^n are analytically triangulable.

In modern terminology of §3, the proof consists in observing that the sets C and $\Phi^{-1}(C)$ are definable in \mathbb{R}_{an} -analytic-geometric category, and therefore are \mathbb{R}_{an} -triangulable. We proved their analytic triangulability directly by using the Gabrielov theorem [7]. We already mentioned that the proof of the Triangulation Theorem for general o-minimal structures follows the same scheme as we used; this scheme probably originates in Hironaka's proof of the Triangulation Theorem for semialgebraic sets.

After proving that C and $\Phi^{-1}(C)$ are analytically triangulable we completed the set C of Φ by adding some additional analytic $(n - 2)$ -dimensional simplices to the simplicial subcomplex K whose complement in $\Phi(SM^n)$ is a union of finitely many discs V_α , and denote $\Phi^{-1}(V_\alpha)$ by U_α thus proving geometric simplicity.

Now by using the general form of the Triangulation Theorem we can generalize Theorem 3 as follows.

THEOREM 4. *Let \mathcal{S} be an o-minimal structure. Let M^n be a compact Riemannian \mathcal{S} -manifold and assume that the geodesic flow on M^n is integrable in terms of \mathcal{S} -definable first integrals. Then this geodesic flow is geometrically simple.*

For $\mathcal{S} = \mathbb{R}_{\text{an}}$ this theorem reduces to Theorem 3.

It was first shown by Butler that assuming only integrability in terms of C^∞ first integrals we cannot conclude that the fundamental group of the manifold is almost commutative. He did that by constructing a C^∞ integrable geodesic flow on a three-dimensional nilmanifold [3]. Later Bolsinov and the author even managed to construct a C^∞ integrable geodesic flow on a solvmanifold whose fundamental group has exponential growth [2].

But as Theorem 4 shows, we can derive topological conclusions of Corollary 1 by assuming that the flow is integrated in terms of C^∞ first integrals which are definable in some analytic-geometric category. In this event the category corresponding to \mathbb{R}_{an} is the smallest possible category.

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