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MATHEMATICS

The Integrability of the n-Center Problem at High Energies

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We consider the n-center problem of celestial mechanics in dimensions $d = 2$ and $d = 3$. We show that, for a general configuration of centers at high energies, the system is completely integrable by means of C^∞-integrals of motion, but it is not integrable by means of real-analytic integrals.

The Hamiltonian

$$\hat{H}: T^*\hat{M} \to \mathbb{R}, \quad \hat{H}(\vec{p}, \vec{q}) = \frac{1}{2}\vec{p}^2 + V(\vec{q})$$

with potential

$$V: \hat{M} \to \mathbb{R}, \quad V(\vec{q}) = -\sum_{k=1}^{n} \frac{Z_k}{\|\vec{q} - \vec{s}_k\|}$$

on the cotangent bundle $T^*\hat{M}$ of the configuration space

$$\hat{M} := \mathbb{R}^d \backslash \{\vec{s}_1, \vec{s}_2, \dots, \vec{s}_n\}$$

always generates an incomplete flow. Here, $\vec{s}_k \in \mathbb{R}^d$ is the position of the kth center, $\vec{s}_k \neq \vec{s}_l$ for $k \neq l$, and $Z_k \in \mathbb{R}\backslash\{0\}$ for $k = 1, 2, \dots, n$. If $Z_k > 0$ for all k, then the problem describes the motion of a massive particle in the gravitational field of n fixed centers.

In [1], it was shown, in particular, that this system admits a smooth extension (P, ω, H) such that the corresponding flow is complete.

The following facts have been known.

(i) For $n = 1$, the system is integrable, and for $d = 3$, the angular momentum determines a real-analytic constant of motion (for $Z_1 > 0$, this is the Kepler problem);

(ii) For $n = 2$, the system is integrable in elliptic coordinates (this was shown by Euler);

(iii) For $n \geq 3$ and $d = 2$, there exists no analytic integral of motion which is not constant on the energy level surface $H^{-1}(E)$, where $E > 0$ [2];

(iv) For $d = 3$ and a collinear configuration of centers, the angular momentum with respect to the axis determines an additional constant of motion for any number of centers;

(v) For $d = 3$, the topological entropy of the flow restricted to the set of bounded orbits b_E is positive (see [1] for sufficiently high energies $E > E_{\text{th}}$ and [3] for nonnegative energies $E \geq 0$), and $h_{\text{top}} = 0$ if the set b_E is empty. Thus, $h_{\text{top}}(E)$ vanishes at $n = 1$ and 2, and $h_{\text{top}}(E) > 0$ if $n \geq 3$ and all centers are repulsive or no more than two centers $\vec{s}_k$ belong to one straight line [for collinear configurations with $Z_1, Z_2, \dots, Z_n < 0$, we have $h_{\text{top}}(E) = 0$ for $E > 0$].

The orbits of the flow are classified into bounded, scattering, and trapped. The subsets formed by these orbits are denoted by b, s, and t, respectively. The limits of scattering orbits are described by comparing them with the Kepler flow generated by the flow extension

$$\hat{H}_\infty: T^*(\mathbb{R}^d \backslash \{0\}) \to \mathbb{R}, \quad \hat{H}_\infty(\vec{p}, \vec{q}) := \frac{1}{2}\vec{p}^2 - \frac{Z_\infty}{\|\vec{q}\|},$$

$$Z_\infty = \sum_{k=1}^{n} Z_k.$$

In [1], it was proved that the set of trapped orbits has measure zero and,

- if $d = 2$ and the centers are repulsive ($Z_k > 0$) or
- if $d = 3$, $Z_k \neq 0$, the configuration of centers is noncollinear,

then there exists a threshold energy value $E_{\text{th}} \geq 0$ such that,

- for $E > E_{\text{th}}$, many estimates hold; in particular, the set b_E of bounded orbits has measure zero;
- therefore, for energies higher than the threshold value, almost each point x of the phase space belongs to a scattering orbit, and on the set of scattering orbits, the following smooth functions are defined:

(a) the momentum asymptotic limits

$$\vec{p}^{\pm}: s \to \mathbb{R}^d;$$

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(b) the time delay

$$\tau: s \to \mathbb{R},$$

which is the asymptotic difference between the time during which the orbit passing through the point x and its Kepler limit are contained in a ball of sufficiently large radius. This function diverges near $b \cup t$.

We use these results and assume in what follows that $d = 2$ and $Z_k > 0$ or $d = 3$ and the configuration of centers is noncollinear.

It turns out that the momentum asymptotic limits lead to integrals of motion. The following theorem is valid.

Theorem 1. *For any $E_1, E_2 > E_{\text{th}}$ such that $E_1 \le E_2$, there exists a constant $C > 0$ such that, for any $g > 1$, the functions $f_k^g: H^{-1}([E_1, E_2]) \to \mathbb{R}$ of the form*

$$f_k^g(x) := \begin{cases} p_k^+(x)\exp\left(-e^{\frac{C}{g-1}\sqrt{1+\tau^2(x)}}\right), & x \in s \\ 0, \quad x \notin s, \end{cases}$$

are functionally independent on a set of full measure, are integrals of motion, and belong to Gevrey classes of index g.

Corollary 1. *On any submanifold of the form $H^{-1}((E_1, E_2))$ with $E_2 > E_1 > E_{\text{th}}$, the n-center problem is completely integrable.*

Since we employ the same trick as that used in [4, 5], obviously, a similar result about integrability with the use of functions from Gevrey classes can be obtained for these systems too. In particular, in [5], an example of an integrable geodesic flow of a real-analytic metric on a compact manifold with positive topological entropy was constructed. Note that, in the situation of Theorem 1, the constraint of the n-center problem on the set of bounded orbits does not change the positive value of the topological entropy [1, 3]. Moreover, at large values of the energy E, the n-center problem is not analytically integrable either, as in the example given in [5]. The following theorem is valid.

Theorem 2. *On any energy level surface $H^{-1}(E)$ with $E \ge E_{\text{th}}$, the n-center problem does not admit a pair of functionally independent real-analytic integrals of motion.*

The obstruction to such an integrability is as follows. Suppose that two real-analytic integrals of motion as above exist. Let $P_E = H^{-1}(E)$, and let S be a subset in P_E on which these integrals are functionally dependent. It contains the set $b_E = P_E \cap b$, i.e., the set of bounded orbits at this energy level. We take a generic point $x \in S$ and denote the intersection of S with the unstable submanifold of the Poincaré surface by γ. Take some Riemannian metric on P_E. Using the results of [1], we can prove that, for $E \ge E_{\text{th}}$, there must exist a vector v_∞ tangent to γ at the point x and a sequence of vectors $\{v_n\}$ tangent to P_E at the point x such that

$$\exp(x, v) \in \gamma, \quad \lim_{n \to \infty} v_n = 0,$$

and

$$\frac{\pi}{2} \ge \angle(v_\infty, v_n) \ge O(r_n^{1+\alpha}), \quad r_n = |v_n|$$

for some constant $\alpha \in (0, 1)$. But analytic integrability at the level P_E implies that, for a generic point $x \in S$, the set γ must be a one-dimensional manifold. The Taylor expansion shows that the angles must converge faster than $r_n^{1+\alpha}$ in this case, namely,

$$\angle(v_\infty, v_n) \sim O(r^2), \quad r_n = |v_n|.$$

This contradiction proves Theorem 2.

The proofs of the theorems will be published elsewhere.

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