Amer. Math. Soc. Transl.
(2) Vol. **224**, 2008

Spectral Conservation Laws for Periodic Nonlinear Equations of the Melnikov Type

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We dedicate this article to our teacher S. P. Novikov on the occasion of his 70th birthday.

In the seminal paper [**24**] in 1974, S. P. Novikov established, in particular, that the spectral curve of the one-dimensional periodic Schrödinger operator

$$H = -\frac{d^2}{dx^2} + u(x)$$

is preserved when the real-valued potential $u(x,t)$ evolves via the Korteweg–de Vries (KdV) equation and that for finite-zone (finite gap) potentials the classical conservation laws, i.e. the Kruskal–Miura integrals, are described in terms of branch points for this curve. The spectral curve Γ is a hyperelliptic

$$\lambda^2 = Q(E),$$

where

$$Q(E) = (E - E_0) \dots (E - E_{2N})$$

is a polynomial of degree $2N+1$ for N-zone potentials. It was proved in [**24**] that finite-zone potentials are exactly solutions of the Novikov equations, i.e., stationary points of higher KdV flows and their linear combinations, and that the KdV flow on the set of N-zone potentials reduces to a completely integrable finite-dimensional Hamiltonian system for which the ends of the stability zones, i.e., $E_0, \dots, E_{2N}$, supply the necessary family of first integrals.

The article [**24**] was the starting point for the development of the finite gap integration theory in which the spectral curves play the main role.

In this article we consider the deformation of the spectral curve via the periodic equations of Melnikov type and we show that although the spectral curve is not preserved it is deformed in such a manner that it still gives many conservation laws for the system.

1. Introduction

We recall that the KdV equation

$$u_t = 6uu_x - u_{xxx}$$

The work was supported by RFBR (grants 05-01-01032 (P.G.G.) and 06-01-00094 (I.A.T.)). The first author (P.G.G.) was also supported by the Program of Russian Academy of Sciences "Mathematical methods in nonlinear dynamics" and the second author (I.A.T.) was supported by the Max Planck Institüt für Mathematik in Bonn.

has the Lax form

$$H_t = [H, A], \tag{1}$$

the *spectral curve* of H parameterizes the Bloch (–Floquet) functions which are formal eigenfunctions of H (here we do not mean that they lie in some nice functional space), and the monodromy operator $\widehat{T}f(x) = f(x+T)$ where T is the period of $u(x)$:

$$H\psi = E\psi, \quad \widehat{T}\psi(x) = e^{i\mu T}\psi(x) \tag{2}$$

and where μ is the quasimomentum which is defined on the spectral curve: $\mu = \mu(\lambda, E)$. The t-deformation of u results in the deformation of ψ via the flow

$$\psi_t = A\psi.$$

Another form of soliton equations instead of the Lax form is the Manakov triple:

$$H_t = [H, A] + BH, \tag{3}$$

where A and B are differential operators. The main example is given by the Novikov–Veselov (NV) equations [**27**] for which H is a two-dimensional Schrödinger operator: $H = \partial\bar{\partial} + u$. The *spectral curve of H on the zero energy level* Γ parameterizes only Floquet functions corresponding to the zero energy level:

$$\begin{aligned} &H\psi = 0, \\ &\psi(x + T_1, y) = e^{i\mu_1 T_1}\psi(x, y), \\ &\psi(x, y + T_2) = e^{i\mu_2 T_2}\psi(x, y), \end{aligned} \tag{4}$$

where T_1 and T_2 are the periods of u. This curve was first introduced by Dubrovin, Krichever, and Novikov in [**4**] where the inverse problem at one energy level for two-dimensional Schrödinger operators was posed and solved for finite-zone operators (the spectral data for potential operators, i.e. with no magnetic field, were later distinguished in [**26**]). Therewith the Floquet functions are deformed again via $\psi_t = A\psi$ and hence the spectral curve is again preserved and may be considered itself as a conservation law. Another equation of such a triple form is the modified Novikov–Veselov equation for which H is a two-dimensional Dirac operator and which, being introduced by Bogdanov, found applications in surface theory [**29**].

Another generalizations of the Lax equations was proposed by Melnikov [**18**] and later was also derived by Kuznetsov and Zakharov [**32**]. The general form of these equations is the following extension of the Lax form:

$$H_t = [H, A] + C,$$

where

$$C = \sum_{n=1}^{N} C_n$$

is the sum of differential operators C_i with coefficients depending on solutions $\phi_{i1}, \dots, \phi_{ik_i}$ of the auxiliary linear problems

$$H\psi_{ik} = \lambda_i\psi_{ik}, \quad k = 1, \dots, k_i.$$

Very frequently these equations are called the equations with self-consistent sources, each of them has a soliton predecessor of the form $H_t = [H, A]$ and, for example, the KdV equation with self-consistent force takes the form

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x + 2\partial_x \sum_{k=1}^{N} \psi_k(x,t)\psi_k^*(x,t), \tag{5}$$

where $\psi_k(x,t)$, $\psi_k^*(x,t)$ are some solutions of the auxiliary linear problem

$$(-\partial_x^2 + u)\psi_k = E_k\psi_k,$$
$$(-\partial_x^2 + u)\psi_k^* = E_k\psi_k^*.$$

To obtain a well-defined dynamics it is natural to assume that the products of the eigenfunctions in (5) are bounded. The simplest choice for the periodic problem is the following: $\psi_k(x)$ is a Bloch eigenfunction and $\psi_k^*(x)$ is the Bloch eigenfunction with the inverse Bloch multipliers.

The theory of such equations was developed in a series of papers by Melnikov [**19, 20, 21, 22, 23**] and others, mostly for the case of functions fast decaying at infinity.

In this article we show that, unlike with soliton equations, the spectral curve is not preserved by these systems; however, it still gives many conservation laws.

2. The spectral curve

The systems (2) and (4) do not have solutions for all possible values of the constants, i.e. for all E and μ in the former case and for all μ_1, μ_2 in the latter case. In fact, such solutions exist if and only if these constants satisfy some analytical condition ("the dispersion laws"):

$$F(E,\mu) = 0, \quad G(\mu_1,\mu_2) = 0.$$

Each equation describes a complex curve $\Gamma \subset \mathbb{C}^2$ and to each point of Γ there corresponds a linear space of solutions to the corresponding equation, (2) or (4). This picture was drawn in physical terms in [**25**] and two different methods for the justification of it were proposed by Krichever and the second author (I.A.T.) (see [**15, 31, 8**]).

Now to obtain the spectral curve Γ_ψ we have to consider the ψ-bundles formed by solutions to (2) or (4) and normalize Γ in such a manner that the pull-back of the ψ-bundle onto Γ_ψ under the projection

$$\Gamma_\psi \to \Gamma$$

forms a bundle with fibers of constant dimension. We refer for details to [**31**] and here demonstrate this procedure by an important original example.

EXAMPLE ([**24**]). For the one-dimensional Schrödinger operator with a real-valued potential the multipliers of $\widehat{T}$ are defined on a Riemann surface (a complex curve) Γ,

$$\lambda^2 = \widehat{Q}(E),$$

where $\widehat{Q}(E)$ is an entire function with infinitely many zeroes. All zeroes lie on the real line. To every point $P = (E,\lambda) \in \Gamma$ where $\widehat{Q} \neq 0$ there corresponds a one-dimensional space of solutions to (2). Let $E' \in \mathbb{R}$ satisfy the following conditions:

(1) $\widehat{Q}$ has a zero at E' of multiplicity two;
(2) to the point $(E',0)$ there corresponds a two-dimensional space of solutions to (2). (This, in particular, implies that this is a double point on Γ.)

Let us unglue this double point and obtain another Riemann surface Γ'. Then the ψ-bundle over Γ is pulled back to a bundle ψ' over Γ' with one-dimensional fibers at the preimages of $(E', 0)$. Moreover this bundle is holomorphic near these points. We have

- if all zeroes of $\widehat{Q}$ except finitely many satisfy conditions 1 and 2 above, then, after ungluing all corresponding double points, we obtain a Riemann surface Γ_ψ of finite genus and the one-dimensional ψ-bundle over it. The surface Γ_ψ is defined by the equation

$$\lambda^2 = Q(E),$$

where Q is a polynomial of odd degree, say $2N+1$.

It is said that this operator is *finite-zone* (or *finite gap*), and if all zeroes of Q are simple, it is said that it has N zones (gaps). There is a function $\psi(P, x)$ meromorphic in $P \in \Gamma_\psi$ with the following asymptotics:

$$\psi \approx e^{i\sqrt{E}x} \quad \text{as } E \to \infty. \tag{6}$$

Therewith the complex curve Γ_ψ is compactified to an algebraic curve by adding the point $E = \infty$ and ψ becomes a meromorphic function on $\Gamma_\psi \setminus \{E = \infty\}$ with the essential singularity (6) at $E = \infty$.

Here we remark that Γ_ψ itself may have singularities and, in fact, there is a tower of projections

$$\Gamma_{\text{norm}} \to \Gamma_\psi \to \Gamma,$$

where Γ_{norm} is the normalization of Γ. The *multiplier mapping* which corresponds to a point, i.e., the set of "multipliers":

$$\mathcal{M} : \Gamma \to \mathbb{C}^2, \quad \mathcal{M}(E, \lambda) = (E, \mu)$$

can be naturally lifted to this tower.

Above we explain how the spectral curve arises from the spectral theory of differential operators.

However the strongest method for constructing exact periodic (and also quasi-periodic) solutions of solitons equation, i.e., the *Baker–Akhiezer function method* [**13**], starts with an introduction of an algebraic curve Γ and of a function ψ (which may be a vector or even a matrix function) with asymptotics of the kind of (6) at several points of Γ. It is assumed that ψ is defined uniquely by some additional data. The function ψ is a formal eigenfunction of some operator H which is uniquely reconstructed in terms of algebraic functions corresponding to Γ from ψ. Such a function ψ is called the *Baker–Akhiezer function* of H, and the soliton dynamics (1) or (3) is linearized in terms of some data coming in the definition of ψ and this leads to explicit algebra-geometrical formulas for so-called finite gap solutions of soliton equations. The spectral curve Γ is preserved by the flow, i.e., the flow is *isospectral.*

Therewith for operators H with periodic coefficients and with a nice spectral theory (i.e., for which the existence of the dispersion laws may be established), $\Gamma = \Gamma_\psi$ and ψ is the section of the ψ-bundle.

In this article we show that

- in contrast with soliton equations the periodic equations of Melnikov type may be *almost isospectral,* i.e., may preserve $\mathcal{M}(\Gamma_\psi)$ and deform Γ_ψ.

The first example of this effect was found by us in [**8**], and we expose it in the next section.

3. The conformal flow for the (Weierstrass) potentials of tori in $\mathbb{R}^3$ and $\mathbb{R}^4$

The authors' interest in the study of Melnikov-type equations was partially motivated by the problem of conformal invariance of the higher Willmore functionals.

By the generalized Weierstrass method, any torus in $\mathbb{R}^3$ is described in terms of the zero-eigenfunction ψ:

$$\mathcal{D}\psi = 0,$$

of a two-dimensional periodic operator

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix},$$

where the potential U is real-valued and any torus in $\mathbb{R}^3$ is described in terms of two solutions φ, ψ to the equations

$$\mathcal{D}\psi = 0, \quad \mathcal{D}^\vee \varphi = 0,$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}, \quad \mathcal{D}^\vee = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \bar{U} & 0 \\ 0 & U \end{pmatrix}$$

are two conjugate periodic operators (see, for instance, [**31**]). The spectral curve Γ_ψ of $\mathcal{D}$ is naturally defined (see (4) and §2) and contains in itself the information of the Willmore functional which is defined for all closed surfaces immersed in $\mathbb{R}^4$ as follows:

$$\mathcal{W}(M) = \int_M |\mathbf{H}|^2 d\mu,$$

where $\mathbf{H}$ is the mean curvature vector and $d\mu$ is the induced volume.

This functional is invariant with respect to conformal transformations of the ambient space; i.e., if we have a conformal transformation $f : \bar{\mathbb{R}}^4 \to \bar{R}^4$ which maps a compact surface without boundary M into a compact surface, then

$$\mathcal{W}(M) = \mathcal{W}(f(M)).$$

This follows from the conformal invariance of the form $(|\mathbf{H}|^2 - K)d\mu$, where K is the Gaussian curvature, and the Gauss–Bonnet theorem by which $\int K d\mu$ equals $2\pi\chi(M)$, i.e. the topological quantity.

The soliton local deformations of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ via the modified Novikov–Veselov (mNV) equation and the Davey–Stewartson (DS) equation were introduced by Konopelchenko [**9, 10**]. It appears that they preserve the tori globally and therewith preserve the Willmore functional as well as the spectral curve [**29, 31**]. Hence it is natural to treat higher conservation laws of these hierarchies as higher Willmore functionals.

The conformal invariance of the Willmore functional led the second author (I.A.T.) to the conjecture that these higher Willmore functionals and the spectral curve for tori in $\mathbb{R}^3$ themselves are conformally invariant [**30**].

This was rather soon established by the first author (P.G.G.) and M.U. Schmidt [7], who considered the conformal flow, i.e. the Melnikov-type flow, induced on the potential U by continuous conformal transformations:

$$U_\tau = |\psi_2|^2 - |\psi_1|^2,$$

where the torus is defined via the Weierstrass formulas by $\psi = (\psi_1, \psi_2)^\top$. Under this deformation the ψ-function on the spectral curve evolves in such a manner that the quasimomenta are preserved.

In [8], we analyzed carefully this situation for the more general case of tori in $\mathbb{R}^4$. It appears that the conformal flow on U which corresponds to the following generator of the conformal group

$$\partial_\tau x^1 = 2x^1x^3, \quad \partial_\tau x^2 = 2x^2x^3,$$
$$\partial_\tau x^3 = (x^3)^2 - (x^1)^2 - (x^2)^2 - (x^4)^2,$$
$$\partial_\tau x^4 = 2x^4x^3$$

has the Melnikov form:

$$\begin{aligned} \partial_\tau U &= \varphi_1\bar{\psi}_1 - \bar{\varphi}_2\psi_2, \\ \partial_\tau \bar{U} &= \bar{\varphi}_1\psi_1 - \varphi_2\bar{\psi}_2, \end{aligned} \tag{7}$$

where $\psi = (\psi_1, \psi_2)^\top$ and $\varphi = (\varphi_1, \varphi_2)^\top$ define a torus in $\mathbb{R}^4$ via the generalized Weierstrass formulas. It appears to be isospectral in the sense that all multipliers are preserved. However we knew about several explicitly computed examples of the Weierstrass representations of tori, which are the Clifford torus in S^3:

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2}$$

and its stereographic projection into $\mathbb{R}^3$ [31]. In these cases the spectral curves Γ_ψ are different: the complex projective line $\mathbb{C}P^1$ in the former case, and $\mathbb{C}P^1$ with two pairs of points glued into two double points. However both tori are connected by a continuous conformal transformation of $\bar{\mathbb{R}}^4$. A detailed analysis led us to the following conclusions:

- The conformal flow (7), i.e. a particular case of Melnikov deformations of periodic operators, is only *almost isospectral*, i.e. preserves the multipliers — the complex curve $\mathcal{M}(\Gamma_\psi)$ — and deforms the spectral curve Γ_ψ. In this particular case, the deformation of Γ_ψ consists in gluing and ungluing double points.
- Since the higher integrals of the mNV and the DS hierarchies are described in terms of $\mathcal{M}(\Gamma_\psi)$, these integrals are preserved and give us *spectral conservation laws* of the conformal flow.

4. The Baker–Akhiezer function and kernel and the (ψ, ψ^*)-representation of equations

Let us recall the definition of the Baker–Akhiezer function for the KP equation [13].

Let Γ be a smooth Riemann surface of genus g with the following data:

(1) a divisor of poles $D = \gamma_1 + \cdots + \gamma_g$;
(2) a distinguished point P with a local parameter $z = 1/\lambda$.

The Baker–Akhiezer function $\psi(\gamma, \vec{t})$ depends on the spectral parameter $\gamma \in \Gamma$ and on an infinite set of real variables $x = t_1$, $y = t_2$, $t = t_3$, t_4, t_5, $\dots$, $\vec{t} = (x, y, t, t_4, t_5, \dots)$. To avoid analytic problems it is convenient to assume that $\vec{t}$ has only a finite number of nonzero entries.

For generic $\vec{t}$ there exists a unique function of $\gamma \in \Gamma$ such that:

(1) $\psi(\gamma, \vec{t})$ is meromorphic in γ outside P with simple poles at $\gamma_1, \dots, \gamma_g$;

(2) $\psi(\gamma, \vec{t}) = \exp\left[\sum_{k>0} \lambda^k t_k\right]\left(1 + \sum_{k>0} \frac{\chi_k(\vec{t})}{\lambda^k}\right)$ as $\gamma \sim P$.

Let us define the potential $u(\vec{t})$ by

$$u(\vec{t}) = 2\partial_x \chi_1(\vec{t}). \tag{8}$$

Then $u(\vec{t})$ satisfy the KP hierarchy, and $\psi(\gamma, \vec{t})$ is the common eigenfunction for all auxiliary linear problems. In particular,

$$-\psi_{xx}(\lambda, \vec{t}) + \psi_y(\lambda, \vec{t}) + u(\vec{t})\psi(\lambda, \vec{t}) = 0. \tag{9}$$

If u is periodic in x and y, then Γ is the spectral curve (on the zero energy level) of the operator $\partial_y - \partial_x^2 + u(x, y)$ [**15**].

Let us assume that Γ is a hyperelliptic surface such that λ^2 is a global meromorphic function on Γ with exactly one second-order pole at P. Then

$$\psi(\gamma, \vec{t}) = \exp\left[\sum_{k>0} \lambda^{2k} t_{2k}\right] \tilde{\psi}(\gamma, x, t, t_5, t_7, \dots)$$

and $\tilde{\psi}(\gamma, \tilde{t})$, $\tilde{t} = (x, t, t_5, t_7, \dots)$ is the Baker–Akhiezer function of the KdV hierarchy [**5**]. We shall omit the tilde sign in the KdV formulas. In the KdV case we have

$$-\psi_{xx}(\lambda, \vec{t}) + u(\vec{t})\psi(\lambda, \vec{t}) = -\lambda^2\psi(\lambda, \vec{t}) \tag{10}$$

instead of (9).

In [**3**], Cherednik has shown that all flows from the KdV hierarchy are obtained as the expansion coefficients in λ^{-1} near $\lambda^{-1} = 0$ for the following λ-dependent nonlocal equation:

$$u_\tau = 2\partial_x(\psi_k(\lambda, x)\psi_k(-\lambda, x)). \tag{11}$$

Here we assume that all times except x are equal to 0.

THEOREM 1. *Let the source functions ψ_k and ψ_k^* in the right-hand side of (5) be the restrictions of the Baker–Akhiezer function at some points of Γ:*

$$\psi_k = \psi(\lambda_k), \quad \psi_k^* = \psi(-\lambda_k).$$

Then (5) can be represented as the following linear combination of the flows (11):

$$u_\tau = 2\partial_x\left[-\operatorname{res}\Big|_{\gamma=P}(\lambda^3\psi(\lambda, x, \tau)\psi(-\lambda, x, \tau)d\lambda) + \sum_{k=1}^{N} \psi(\lambda_k, x, \tau)\psi(-\lambda_k, x, \tau)\right].$$

All the higher KdV flows are isospectral and form a commutative algebra. Typically the complete algebra of symmetries for soliton equations is noncommutative and contains both isospectral and nonisospectral flows (see [**28**] for further references).

Orlov and Schulman suggested a generic approach for studying the symmetry algebra based on the so-called infinitesimal dressing [**28**]. It particular, in [**28**] it was shown that the generators $K_{mn}[u]$ of the algebra of all KdV and KP symmetries are obtained by expanding the flow

$$u_\tau = 2\partial_x(\psi_k(\lambda,\vec{t})\psi_k^*(\mu,\vec{t})) \tag{12}$$

near the diagonal $\lambda = \mu$:

$$2\partial_x(\psi_k(\lambda,\vec{t})\psi_k^*(\mu,\vec{t})) = \sum_{m,n} K_{mn}[u]\left(\frac{1}{\lambda}\right)^m\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)^n$$

at the point P where $\lambda = \mu = \infty$. For $n = 0$ we have the standard KP (KdV) hierarchy. The $n = 1$ coefficients generate the Virasoro algebra of nonisospectral symmetries; the $n > 2$ symmetries are not compatible with the KdV reduction.

Here $\psi(\lambda,\vec{t})$ is the wave function for all auxiliary linear operators of the KP (KdV) hierarchy, $\vec{t} = (x = t_1, t = t_3, t_5, \ldots)$ or $\vec{t} = (x = t_1, y = t_2, t = t_3, t_4, \ldots)$ denotes the full set of KdV (KP) times, and $\psi^*(\lambda,\vec{t})$ satisfy the formal conjugate linear problems. In the KdV case all auxiliary problems are selfadjoint; therefore,

$$\psi^*(\lambda,\vec{t}) = \psi(-\lambda,\vec{t}).$$

An arbitrary source function may be expanded in terms of products of eigenfunctions. Such expansions play a critical role in the perturbation theory for soliton equations. The periodic perturbation theory for 1-dimensional finite-gap potentials and for the 2-dimensional finite-gap at one-energy potentials was developed by Krichever [**14, 15**]. In particular, he pointed out that it is natural to treat the conjugate Baker–Akhiezer function $\psi^*(\lambda,\vec{t})$ as a holomorphic 1-form in the spectral parameter γ on $\Gamma\backslash P$. It is defined by the following analytic properties:

(1) $\psi^*(\gamma,\vec{t})$ is a 1-form in γ; i.e., in local coordinates it reads as $\psi^*(\lambda,\vec{t}) = \tilde{\psi}^*(\lambda,\vec{t})d\lambda$, where $\tilde{\psi}^*(\lambda,\vec{t})$ is an analytic function.

(2) $\psi^*(\gamma,\vec{t})$ is holomorphic in γ outside P with simple zeroes at $\gamma_1,\ldots,\gamma_g$.

(3) $\psi^*(\gamma,\vec{t}) = \exp\left[-\sum_{k>0}\lambda^k t_k\right](1+o(1))d\lambda$ as $\gamma \sim P$.

The action of the Virasoro algebra symmetries on the finite-gap KP solutions (these symmetries generically result in nontrivial deformations of the complex structures on the spectral curves) was studied by Orlov and the first author in [**6**]. In particular, in [**6**] it was shown that the infinitesimal deformations of the Baker–Akhiezer function corresponding to the generators (12) (infinitesimal Darboux transformations of the finite-gap KP solutions) are naturally written in terms of the so-called Cauchy–Baker–Akhiezer kernel $\omega(\lambda,\mu,\vec{t})$:

$$\delta\psi(\gamma,\vec{t}) = -\frac{\omega(\gamma,\mu,\vec{t})}{d\mu}\psi(\lambda,\vec{t}).$$

The kernel $\omega(\lambda,\mu,\vec{t})$ is defined by the following analytic properties:

(1) $\omega(\lambda,\mu,\vec{t})$ is a meromorphic function in λ and a meromorphic 1-form in μ on $\Gamma\backslash P$.

(2) For a fixed μ, the function $\omega(\lambda,\mu,\vec{t})$ has simple poles $\gamma_1,\ldots,\gamma_g$ at the points μ.

(3) For a fixed λ, the 1-form $\omega(\lambda,\mu,\vec{t})$ has simple zeroes at $\gamma_1,\ldots,\gamma_g$ and a simple pole at λ.

(4) $\omega(\lambda, \mu, \vec{t}) = \frac{d\mu}{\mu - \lambda} + O(1)$ near the diagonal $\lambda = \mu$.

(5) For a fixed μ, the function $\omega(\lambda, \mu, \vec{t})\lambda \exp\left[-\sum_{k>0} \lambda^k t_k\right]$ is regular in λ at the point $\lambda = P$.

(6) For a fixed λ, the 1-form $\omega(\lambda, \mu, \vec{t})\mu^{-1} \exp\left[\sum_{k>0} \mu^k t_k\right]$ is regular in μ at the point $\mu = P$.

For $\vec{t} = \vec{0}$ this kernel coincides with the Cauchy kernel on Riemann surfaces used by Koppelman [**11**]. For data generating regular potentials $u(\vec{t})$, the following explicit formula was suggested in [**6**]:

$$\omega(\lambda, \mu, x, y, t_3, t_4, \ldots) = \int_x^{\pm\infty} \psi(\lambda, x', y, t_3, t_4, \ldots)\psi^*(\mu, x', y, t_3, t_4, \ldots)dx'.$$

The upper limit of the integral depends on the quasimomenta at the points λ, μ and is chosen to make the integral convergent. An analogous representation for the Cauchy kernels on Riemann surfaces for systems with discrete x was suggested earlier by I. M. Krichever and S. P. Novikov in [**16**].

5. The periodic Kadomstev–Petviashvili equation with a self-consistent source

To integrate the KP equation with the self-consistent sources we have to consider spectral curves with additional double points. Such curves correspond to the solitons on the finite-gap background [**12**]. We assume that we have the same spectral data as in Section 4 plus $2N$ marked points R_+^k, R_-^k, $k = 1, \ldots, N$. Denote the local parameters near these points by λ. The Baker–Akhiezer function depends on N extra real parameters $\tau_1, \ldots, \tau_N$, $\vec{\tau} = (\tau_1, \ldots, \tau_N)$ and has the following analytic properties:

(1) $\psi(\gamma, \vec{t}, \vec{\tau})$ is meromorphic in γ outside P with $g + N$ simple poles at $\gamma_1, \ldots, \gamma_g$, $R_+^1, \ldots, R_+^N$.

(2) $\operatorname{res}\big|_{\lambda = R_+^k} \Psi(\lambda, \vec{t}, \vec{\tau})d\lambda = \tau_k \Psi(R_-^k, \vec{t}, \vec{\tau})$.

(3) $\psi(\gamma, \vec{t}, \vec{\tau}) = \exp\left[\sum_{k>0} \lambda^k t_k\right](1 + o(1))$ as $\gamma \sim P$.

The properties of the conjugate Baker–Akhiezer 1-form are as follows:

(1) $\psi^*(\gamma, \vec{t}, \vec{\tau})$ is meromorphic in γ outside P with simple zeroes at $\gamma_1, \ldots, \gamma_g$ and simple poles at $R_-^1, \ldots, R_-^N$.

(2) $\operatorname{res}\big|_{\lambda = R_-^k} \Psi^*(\lambda, \vec{t}, \vec{\tau}) = -\tau_k \Psi^*(\lambda, \vec{t}, \vec{\tau})/d\lambda\big|_{\lambda = R_+^k}$.

(3) $\psi^*(\gamma, \vec{t}, \vec{\tau}) = \exp\left[-\sum_{k>0} \lambda^k t_k\right](1 + o(1))d\lambda$ as $\gamma \sim P$.

The corresponding potential $u(\vec{t}, \vec{\tau})$ is defined by the formula (8).

THEOREM 2. *Let Γ be a Riemann surface of algebraic genus g with the following KP data:*

(1) *a divisor of poles $D = \gamma_1 + \cdots + \gamma_g$;*

(2) *a distinguished point P with a local parameter $z = 1/\lambda$;*

(3) *an additional collection of* $2N$ *points* R_+^k, R_-^k, $k = 1, \dots, N$. *Denote the local parameters near* R_+^k, R_-^k *by* λ.

Then the potential $u(\vec{t}, \vec{\tau})$ *defined above satisfies the following equations with self-consistent sources:*

$$\frac{\partial u(\vec{t},\vec{\tau})}{\partial \tau_k} = 2\partial_x \frac{\psi(R_-^k,\vec{t},\vec{\tau})\psi^*(\lambda,\vec{t},\vec{\tau})}{d\lambda}\bigg|_{\lambda=R_+^k}. \tag{13}$$

PROOF. The Cauchy–Baker–Akhiezer kernel $\omega(\lambda,\mu,\vec{t},\vec{\tau})$ corresponding to this spectral data has the following analytic properties:

(1) $\omega(\lambda,\mu,\vec{t},\vec{\tau})$ is a meromorphic function in λ and a meromorphic 1-form in μ on $\Gamma\backslash P$.
(2) For a fixed μ the function $\omega(\lambda,\mu,\vec{t},\vec{\tau})$ has simple poles at the points μ, $\gamma_1,\dots,\gamma_g$, $R_+^1,\dots,R_+^N$.
(3) $\mathrm{res}\big|_{\lambda=R_+^k} \omega(\lambda,\mu,\vec{t},\vec{\tau})d\lambda = \tau\omega(R_-^k,\mu,\vec{t},\vec{\tau})$.
(4) For a fixed λ the 1-form $\omega(\lambda,\mu,\vec{t},\vec{\tau})$ has simple zeroes at $\gamma_1,\dots,\gamma_g$ and simple poles at λ, $R_-^1,\dots,R_-^N$.
(5) $\mathrm{res}\big|_{\mu=R_-^k} \omega(\lambda,\mu,\vec{t},\vec{\tau}) = -\tau_k\omega(\lambda,\vec{t},\vec{\tau})/d\mu\big|_{\mu=R_+^k}$.
(6) $\omega(\lambda,\mu,\vec{t},\vec{\tau}) = \frac{d\mu}{\mu-\lambda} + O(1)$ near the diagonal $\lambda=\mu$.
(7) For a fixed μ, the function $\omega(\lambda,\mu,\vec{t},\vec{\tau})\lambda\exp\left[-\sum_{k>0}\lambda^k t_k\right]$ is regular in λ at the point $\lambda = P$.
(8) For a fixed λ, the 1-form $\omega(\lambda,\mu,\vec{t},\vec{\tau})\mu^{-1}\exp\left[\sum_{k>0}\mu^k t_k\right]$ is regular in μ at the point $\mu = P$.

It is easy to check (see [**6**]):

$$\begin{aligned} &\partial_x\omega(\lambda,\mu,\vec{t},\vec{\tau}) = -\psi(\lambda,\vec{t},\tau)\psi^*(\mu,\vec{t},\vec{\tau}),\\ &\partial_y\omega(\lambda,\mu,\vec{t},\vec{\tau}) = -\psi_x(\lambda,\vec{t},\vec{\tau})\psi^*(\mu,\vec{t},\vec{\tau}) + \psi(\lambda,\vec{t},\vec{\tau})\psi_x^*(\mu,\vec{t},\vec{\tau}).\end{aligned} \tag{14}$$

We use the following formula:

$$\partial_{\tau_k}\psi(\lambda,\vec{t},\vec{\tau}) = -\frac{\omega(\lambda,\mu,\vec{t},\vec{\tau})}{d\mu}\bigg|_{\mu=R_+^k}\cdot\psi(R_-^k,\vec{t},\vec{\tau}). \tag{15}$$

To prove (15) it is sufficient to check that the right-hand side has the correct analytic properties. In particular, the relation

$$\partial_{\tau_k}\mathrm{res}\Big|_{\lambda=R_+^k}\psi(\lambda,\vec{t},\vec{\tau})d\lambda = \tau_k\partial_{\tau_k}\psi(R_-^k,\vec{t},\vec{\tau}) + \psi(R_-^k,\vec{t},\vec{\tau})$$

follows from the following expansion near the point $\lambda \sim R_+$:

$$\frac{\omega(\lambda,\mu,\vec{t},\vec{\tau})}{d\mu}\bigg|_{\mu=R_+^k} = \left[\tau\frac{\omega(R_-^k,\mu,\vec{t},\vec{\tau})}{d\mu}\bigg|_{\mu=R_+^k} - 1\right]\frac{1}{\lambda-R_+^k} + O(1).$$

This expansion can be easily derived from the properties 3 and 6 of the Cauchy–Baker–Akhiezer kernel.

Taking into account that

$$\psi_{xx}(\lambda) - \psi_y(\lambda) - u\psi(\lambda) = 0, \quad \psi_{xx}^*(\mu) + \psi_y^*(\mu) - u\psi^*(\mu) = 0, \tag{16}$$

we obtain the following formula for the variation of $u = u(\vec{t}, \vec{\tau})$:

$$u_{\tau_k} = \frac{\psi_{xx\tau_k}(\lambda) - \psi_{y\tau_k}(\lambda) - u\psi_{\tau_k}(\lambda)}{\psi(\lambda)}. \tag{17}$$

Substituting (15), (14) into (17) we obtain (13). We see that the right-hand side of (17) turns out to be λ-independent; therefore the deformation (15) of the Bloch function is admissible.

COROLLARY 1. *Denote by* $\widehat{u}(x,y,t)$, $\widehat{\psi}(\gamma,x,y,\tau)$, $\widehat{\psi}^*(\gamma,x,y,\tau)$ *the functions obtained from* $u(\vec{t},\vec{\tau})$, $\psi(\gamma,\vec{t},\vec{\tau})$, $\psi^*(\gamma,\vec{t},\vec{\tau})$ *by the following linear substitution:*

$$t_1 = x,\ t_2 = y,\ t_k = c_k\tau,\ k = 3,\dots,M,\ t_k = 0,\ k > M,$$
$$\tau_k = \alpha_k + \beta_k\tau,\ k = 1,\dots,N.$$

Then $\widehat{u}(x,y,t)$ *solves the following Melnikov-type equation:*

$$\frac{\partial\widehat{u}(x,y,\tau)}{\partial\tau} = \sum_{k=3}^{M} c_k K_k[\widehat{u}] + 2\partial_x \sum_{k=1}^{N} \beta_k \frac{\widehat{\psi}(R_-^k,x,y,\tau)\widehat{\psi}^*(\lambda,x,y,\tau)}{d\lambda}\Bigg|_{\lambda=R_+^k}, \tag{18}$$

where $K_k[\widehat{u}]$ *denotes the k-th flow from the standard KP hierarchy and the functions on the right-hand side satisfy* (16).

REMARK. It is easy to notice from the previous formulas that[1]

if at $\tau = 0$ *the function* $\widehat{u}$ *is periodic in* x *and* y, ψ *is the Floquet eigenfunction of the operator* $L = \partial_y - \partial_x^2 + u$, *and for* $k = 1,\dots,N$ *the products* $\widehat{\psi}(R_-^k,x,y,\tau)\widehat{\psi}^*(R_+^k,x,y,\tau)$ *are periodic, then the evolution* (18) *preserves the periodicity of* u *and the multipliers of* ψ.

We conclude that

- In fact, the Baker–Akhiezer function in Theorem 2 is defined on the spectral curve with double points. The double point (R_+^k, R_-^k) is unglued if and only if $\tau_k = 0$. For generic τ all double points are present, i.e., the spectral curve is obtained from Γ by pairwise gluing points $(R_+^1, R_-^1),\dots,(R_+^N, R_-^N)$, and the k-th double point is unglued when $\alpha_k + \beta_k\tau = 0$. If the initial spectral curve is regular for $\tau = 0$, then equations with self-consistent sources immediately generate double points, which remain unglued for almost all times.

  Another example of such an effect is given by the conformal flow (see Section 3).

This observation demonstrates the principal difference between the Melnikov-type equations and the standard hierarchies like KdV, nonlinear Schrödinger (NLS), sine-Gordon, or KP. The standard hierarchies are isospectral; therefore the evolution cannot generate double points or unglue the existing double points in a finite time.

In the case of the self-focusing NLS equation, the spectral curves with double points may correspond to regular space-periodic solutions, associated with so-called whiskered tori. All double points for such solutions remain glued for all values of t but become unglued in the limit $t \to \pm\infty$. In [**1, 2**] it was shown that for a periodic solution corresponding to a smooth curve, the generation of double points after arbitrarily small perturbations results in numerical chaos.

[1] The derivation of the similar fact for the conformal flow (see Section 3) is exposed in [**8**].

6. An annihilation of a soliton in a finite time

Let us discuss the simplest example, i.e., the one-soliton solution of the KdV equation.

The wave function $\psi(\lambda, x, c)$ of a one-dimensional Schrödinger operator

$$-\psi'' + u\psi = E\psi, \quad u = u(x,c), \;\; E = (i\lambda)^2$$

has the following form:

$$\psi(\lambda, x, c) = e^{\lambda x}\left(1 + \frac{\chi(x,c)}{\lambda + \kappa}\right).$$

The spectral curve Γ is the Riemann sphere with a double point: $\lambda = -\kappa$ and $\lambda = \kappa$ are glued together. We assume that the divisor point is located at $\lambda = -\kappa$; therefore we have

$$\text{res } \psi(\lambda, x, c)\Big|_{k=-\kappa} = -c\psi(\kappa, x, c)$$

(this relation coincides with property (2) of the Baker–Akhiezer function from Section 5) and

$$\chi(x,c) = \frac{-2c\kappa\, e^{\kappa x}}{2\kappa e^{-\kappa x} + ce^{\kappa x}}.$$

By (8) we obtain

$$u(x,c) = 2\partial_x \chi(x,c) = \frac{-16c\kappa^3}{(2\kappa e^{-\kappa x} + ce^{\kappa x})^2}. \tag{19}$$

The c-dynamics corresponds to the following choice of Baker–Akhiezer function solutions:

$$\psi_1(x,c) = \psi(\kappa, x, c) = \frac{2\kappa}{2\kappa e^{-\kappa x} + ce^{\kappa x}}.$$

The conjugate Baker–Akhiezer function is defined by:

$$\psi^*(\lambda, x, c) = \psi(-\lambda, x, c)dk.$$

A simple straightforward calculation shows that

$$\partial_c u(x,c) = -2\partial_x \psi^2(\kappa, x, c). \tag{20}$$

Formula (19) generates regular solitons for $c > 0$, singular solitons for $c < 0$ and the zero solution for $c = 0$. If $c \to 0$, the position of the soliton goes to $+\infty$.

The standard KdV evolution of a soliton is given by (20), where $c = c(t)$ is governed by the formula

$$\partial_t c(t) = \kappa^3 c(t).$$

Let us consider the following Melnikov-type flow:

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x + 2\partial_x\psi^2(\kappa, t).$$

The corresponding evolution of $c(t)$ is given by

$$\partial_t c = \kappa^3 c - 1.$$

We see that

- starting with sufficiently small c we reach the point $c = 0$ in finite time. At this moment the double point on the spectral curve vanishes and the soliton is annihilated.

By inverting the direction of evolution we obtain examples of such effects as

- a creation of a soliton (there is no soliton at $c = 0$ and it exists as soon as $c > 0$),
- a capture of a soliton ($c \to \kappa^{-3}$ as $t \to \infty$),

first observed by Melnikov [**19**].

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