

## THE MOUTARD TRANSFORMATION: AN ALGEBRAIC FORMALISM VIA PSEUDODIFFERENTIAL OPERATORS AND APPLICATIONS

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ABSTRACT. In this paper we consider the Moutard transformation [9] which is a two-dimensional version of the well-known Darboux transformation. We give an algebraic interpretation of the Moutard transformation as a conjugation in an appropriate ring and the corresponding version of the algebro-geometric formalism for two-dimensional Schrödinger operators. An application to some problems of the spectral theory of two-dimensional Schrödinger operators and to the  $(2 + 1)$ -dimensional Novikov–Veselov equation is sketched.

### INTRODUCTION

The Moutard transformation plays a fundamental role in projective–differential geometry of surfaces and was extensively investigated by Bianchi, Darboux, Demoulin, Guichard and others.

In recent publications [15, 16, 17] we gave an application of the Moutard transformation to the explicit construction of two-dimensional Schrödinger operators

$$H = -\Delta + u = -(\partial_x^2 + \partial_y^2) + u(x, y)$$

with fast decaying smooth rational potentials such that their  $L_2$ -kernels contain at least two-dimensional subspaces spanned by rational eigenfunctions as well as a  $(2 + 1)$ -dimensional extension of the Moutard transformation which was able to produce explicit rational blowing-up solutions to the Novikov–Veselov (NV) equation with fast decaying smooth rational Cauchy data.

Purely algebraic constructions of the aforementioned papers imply that one should look for an algebraic interpretation of the Moutard transformation as a simple transformation in the ring of partial differential operators. The one-dimensional case of Darboux transformation was studied in [7]. Here we give an analogue of their results; as it turned out, for the Moutard transformation one needs to recourse to a more complicated ring of *rational* pseudodifferential operators (Ore localization of the ring of partial differential

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operators). As a by-product, we obtain a more natural version of the Dubrovin-Krichever-Novikov formalism [5, 8] of algebro-geometric two-dimensional Schrödinger operators. In the last Section we briefly sketch our previous results [15, 16, 17].

First we give an account of the algebraic theory of the Darboux transformations.

**0.1. The Darboux transformation.** Let

$$H = -\frac{d^2}{dx^2} + u(x)$$

be a one-dimensional Schrödinger operator and let  $\omega$  satisfy the equation

$$H\omega = 0.$$

The function  $\omega$  determines a factorization of  $H$ :

$$H = A^\top A, \quad A = -\frac{d}{dx} + v, \quad A^\top = \frac{d}{dx} + v, \quad v = \frac{\omega_x}{\omega}. \quad (1)$$

Indeed we have

$$A^\top A = \left(\frac{d}{dx} + v\right) \left(-\frac{d}{dx} + v\right) = -\frac{d^2}{dx^2} + v^2 + v_x$$

and the equation

$$v_x + v^2 = u$$

is equivalent to  $H\omega = 0$ . If  $v$  is real-valued we have  $A^* = A^\top$ .

The Darboux transformation<sup>1</sup> is the swapping of  $A^\top$  and  $A$ :

$$H = A^\top A \rightarrow \tilde{H} = AA^\top,$$

or in terms of  $u$ :

$$u = v^2 + v_x \rightarrow \tilde{u} = v^2 - v_x.$$

It is easy to check by simple computations that

**Proposition 1.** *If  $\varphi$  satisfies the equation  $H\varphi = E\varphi$  with  $E = \text{const}$  then  $\tilde{\varphi} = A\varphi$  satisfies the equation  $\tilde{H}\tilde{\varphi} = E\tilde{\varphi}$ .*

REMARK. In general the Darboux transformation is defined for any solution to the equation  $H\omega = c\omega$  with  $c = \text{const}$  (see, for instance, [7]). In this case it reduces to the transformation of  $H' = H - c$  for which  $H'\omega = 0$ .

**0.2. The Moutard transformation.** This transformation was invented by Th. Moutard for the hyperbolic equation of the form  $\psi_{xy} - u(x, y)\psi = 0$  in the context of local differential geometry of surfaces and was extensively studied and used in many areas of the local surface theory [4]. Here we give an elliptic version [15, 16, 17] of this transformation suitable for our purposes.

Let  $H$  be a two-dimensional potential Schrödinger operator and let  $\omega$  be a solution to the equation

$$H\omega = (-\Delta + u)\omega = 0.$$

Then THE MOUTARD TRANSFORMATION of  $H$  is defined as

$$\tilde{H} = -\Delta + u - 2\Delta \log \omega = -\Delta - u + 2\frac{\omega_x^2 + \omega_y^2}{\omega^2}.$$

<sup>1</sup>Sometimes it is also called the Crum transformation due to [3]. Burchnell and Chaundy called it transference [2].

**Proposition 2.** *If  $\varphi$  satisfies the equation  $H\varphi = 0$ , then the function  $\theta$  defined from the system*

$$(\omega\theta)_x = -\omega^2 \left(\frac{\varphi}{\omega}\right)_y, \quad (\omega\theta)_y = \omega^2 \left(\frac{\varphi}{\omega}\right)_x \quad (2)$$

satisfies  $\tilde{H}\theta = 0$ .

By this definition, if  $\theta$  satisfies (2) then

$$\theta + \frac{t}{\omega}, \quad t = \text{const}, \quad (3)$$

satisfies (2) for any constant  $t$ .

We shall use the following notation for the Moutard transformation:

$$M_\omega(u) = \tilde{u} = u - 2\Delta \log \omega, \quad M_\omega(\varphi) = \left\{ \theta + \frac{t}{\omega}, t \in \mathbb{C} \right\}.$$

For one-dimensional potentials the Moutard transformation reduces to the Darboux transformation. Indeed, let  $u = u(x)$  depend on  $x$  only and  $\omega = f(x)e^{\sqrt{c}y}$ . Then  $f$  satisfies the one-dimensional Schrödinger equation

$$H_0 f = \left( -\frac{d^2}{dx^2} + u \right) f = c f$$

and the Moutard transformation reduces to the Darboux transformation of  $H_0$  defined by  $f$ :

$$H = H_0 - \frac{\partial^2}{\partial y^2} \longrightarrow \tilde{H} = \tilde{H}_0 - \frac{\partial^2}{\partial y^2}.$$

If  $g = g(x)$  satisfies  $H_0 g = E g$ , then  $H\varphi = 0$  with  $\varphi = e^{\sqrt{E}y}g(x)$ . We derive from (2) that  $\theta = e^{\sqrt{E}y}h(x)$  satisfies  $\tilde{H}\theta = 0$  if  $h = \frac{1}{\sqrt{c+\sqrt{E}}} \left( \frac{d}{dx} - \frac{f_x}{f} \right) g$ , i.e.  $h$  is a multiple of the Darboux transform of  $g$ :  $h = -\frac{1}{\sqrt{c+\sqrt{E}}} A g$  where  $H_0 - c = A^\top A$  is the factorization of  $H_0 - c$  defined by  $f$ . The inverse Darboux transformation is given by  $g = \frac{1}{\sqrt{c-\sqrt{E}}} \left( \frac{d}{dx} + \frac{f_x}{f} \right) h$ .

REMARK. We can rewrite (2) as

$$\left( \bar{\partial} + \frac{\omega_{\bar{z}}}{\omega} \right) \theta = i \left( \bar{\partial} - \frac{\omega_{\bar{z}}}{\omega} \right) \varphi, \quad \left( \partial + \frac{\omega_z}{\omega} \right) \theta = -i \left( \partial - \frac{\omega_z}{\omega} \right) \varphi \quad (4)$$

which implies

$$\omega^{-1} \cdot \bar{\partial} \cdot \omega(\theta) = i \omega \cdot \bar{\partial} \cdot \omega^{-1}(\varphi), \quad \omega^{-1} \cdot \partial \cdot \omega(\theta) = -i \omega \cdot \partial \cdot \omega^{-1}(\varphi), \quad (5)$$

where, as usual,  $z = x + iy$ ,  $\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ ,  $\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ . This representation will be used later.

There is another two-dimensional generalization of the Darboux transformation called the Laplace transformation. Its relation to integrable systems was recently studied in [13].

## 1. THE SPECTRAL CURVES OF ALGEBRAIC SCHRÖDINGER OPERATORS

Since weakly algebraic two-dimensional Schrödinger operators are defined in terms of their spectral curves on the zero energy level we recall the definitions of the spectral curves of one- and two-dimensional Schrödinger operators.

**1.1. One-dimensional Schrödinger operators.** Let a one-dimensional Schrödinger operator

$$H = -\frac{d^2}{dx^2} + u(x)$$

commute with an ordinary differential operator  $L$  of order  $2n + 1$ . Then the Burchall–Chaundy theorem [2] guarantees that these two operators satisfy a polynomial equation

$$Q(H, L) = 0, \quad Q(\lambda, E) = \lambda^2 - P_{2n+1}(E) = 0,$$

where  $P_{2n+1}(E)$  is a polynomial of degree  $2n + 1$ . This equation defines *the spectral curve*  $\Gamma$  of the operator  $H$ . Its points parameterize the joint “eigenfunctions”, i.e. solutions of the equations

$$H\psi = E\psi, \quad L\psi = \lambda\psi.$$

Moreover these functions are glued (after some normalization) into a function  $\psi(P, x)$  which is meromorphic in  $P \in \Gamma$  and after completing  $\Gamma$  by adding an infinity point  $E = \infty$  the eigenfunction  $\psi$  gets a singularity at  $\infty$ :

$$\psi(P, x) \approx e^{ikx}$$

where  $k^{-1} = \frac{1}{\sqrt{E}}$  is a local parameter on  $\Gamma$  near the infinity point  $\infty$  [10, 6]. Therewith it is said that the operator  $H$  is *algebraic* (or *algebro-geometric*).

EXAMPLE. *The spectral curve of a constant potential.* Let  $u(x) = c$  be a constant potential, then  $\Gamma$  is given by  $\lambda^2 = E - c$ ,

$$\psi_{\pm} = e^{\pm i\lambda x},$$

and the covering  $(\lambda, E) \rightarrow E$  ramifies at  $E = c$  where  $\psi_{\pm}(x, c) = 1$ .

**1.2. Two-dimensional Schrödinger operators.** Given a partial differential operator  $H$ , one says that operators  $L_1, \dots, L_n$  generate a commutative (mod  $H$ ) algebra  $\mathcal{A}$  if they satisfy the following commutation relations:

$$[L_i, L_j] = D_{ij}H, \quad [L_i, H] = D_iH$$

where  $D_{ij}, D_i, 1 \leq i, j \leq n$ , are partial differential operators. This definition was introduced in [5] in which they considered in detail the particular case when  $H$  is the two-dimensional Schrödinger operator (probably with an electromagnetic field) in two variables and introduced the following definition:

- a two-dimensional operator

$$H = \partial\bar{\partial} + v\bar{\partial} + u$$

is called (*weakly*) *algebraic* if it is included in a nontrivial commutative (mod  $H$ ) algebra  $\mathcal{A}$  generated by operators  $L_1$  and  $L_2$  in two variables, the operators  $L_1$  and  $L_2$  satisfy the polynomial relation

$$Q(L_1, L_2) = 0(\text{mod } H)$$

and to a generic point of the algebraic curve  $Q(\lambda_1, \lambda_2) = 0$  there corresponds a  $k$ -dimensional space of functions  $\psi$  which satisfy the equations

$$H\psi = 0, \quad L_i\psi = \lambda_i\psi, \quad i = 1, 2.$$

The dimension  $k$  is called the rank of  $\mathcal{A}$ .

## 2. THE CONJUGATION REPRESENTATION

**2.1. The Darboux transformation.** The Darboux transformation admits the well-known representation as a conjugation in the ring of pseudodifferential operators which implies interesting corollaries concerning algebraic operators.

Let  $H\omega = 0$  and  $A = -\frac{d}{dx} + \frac{\omega_x}{\omega}$ . Then in the ring of pseudodifferential operators in  $x$  we have

$$A = -\omega \cdot \frac{d}{dx} \cdot \omega^{-1}, \quad A^\top = \omega^{-1} \cdot \frac{d}{dx} \cdot \omega, \quad H = A^\top A,$$

where the functions  $\omega$  and  $\omega^{-1}$  are identified with the operator of multiplication by them. From that we conclude

**Proposition 3.**

$$\tilde{H} = A \cdot H \cdot A^{-1}. \quad (6)$$

Fixing  $\omega$  let us denote by

$$\tilde{M} = A \cdot M \cdot A^{-1}$$

the conjugation of a pseudodifferential operator  $M$ . It is clear that if  $L$  commutes with  $H$ , then  $\tilde{L}$  commutes with  $\tilde{H}$ :

$$[H, L] = 0 \quad \Rightarrow \quad [\tilde{H}, \tilde{L}] = 0.$$

However, given an arbitrary differential operator  $M$ , we have

$$M \cdot \omega = M\omega + M' \cdot \frac{d}{dx}$$

where  $M\omega$  is the function obtained by applying  $M$  to  $\omega$  and  $M'$  is a differential operator. Hence

$$\begin{aligned} \tilde{M} &= \omega \cdot \frac{d}{dx} \cdot \omega^{-1} \cdot M \cdot \omega \cdot \left(\frac{d}{dx}\right)^{-1} \cdot \omega^{-1} = \\ &= \omega \cdot \frac{d}{dx} \cdot \omega^{-1} \cdot M\omega \cdot \left(\frac{d}{dx}\right)^{-1} \cdot \omega^{-1} + \omega \cdot \frac{d}{dx} \cdot \omega^{-1} \cdot M' \cdot \omega^{-1} \end{aligned}$$

and we conclude that

$\tilde{M}$  is a differential operator if and only if  $\omega^{-1} \cdot M\omega = \lambda = \text{const}$ , which means

$$M\omega = \lambda\omega.$$

Let  $H\omega = 0$  and let  $L$  be a differential operator of odd order which commutes with  $H$ . We assume that its order is minimal with respect to this property. Let, by the Burchnell–Chaundy theorem,

$$Q(H, L) = 0, \quad Q(E, \lambda) = \lambda^2 - P(E).$$

Then we have (see [7]):

- (1) if  $L\omega = \lambda\omega$  and  $P(E)$  has no a multiple root at  $E = 0$ , then  $\tilde{H}$  and  $\tilde{L}$  generate a commutative ring of differential operators and

$$Q(\tilde{H}, \tilde{L}) = 0;$$

- (2) if  $\omega^{-1} \cdot L\omega \neq \text{const}$ , then  $\tilde{H}$  and  $L_0 = \widetilde{LH}$  generate a commutative ring of differential operators and

$$Q_+(\tilde{H}, L_0) = 0, \quad Q_+(E, \lambda) = \lambda^2 - E^2 P(E);$$

- (3) if  $L\omega = \lambda\omega$  and  $P(E)$  has a multiple root at  $E = 0$ , then the action of the Darboux transformation is inverse to one described in the previous statement:  $\tilde{H}$  and some operator  $L_0$  generate commutative ring of differential operators with

$$Q_-(\tilde{H}, L_0) = 0, \quad Q_-(E, \lambda) = \lambda^2 - E^{-2} P(E).$$

In particular, this implies that

- The Darboux transformation preserves the class of algebro-geometric one-dimensional Schrödinger operators. Moreover it always preserves the normalization of the spectral curve.

In [7] it is noted that some facts mentioned above were discovered by Burchnell, Chaundy and Drach.

The first two cases are demonstrated by the following

EXAMPLE. *The Darboux transformation of the constant potential.* Let  $u = c = \text{const}$  and  $L = \frac{d}{dx}$ .

- (1) Let  $\omega = e^{\sqrt{c-c_0}x}$ . Then  $v = \frac{\omega_x}{\omega} = \sqrt{c-c_0}$ ,  $v_x = 0$ , and the Darboux transformation is even trivial.
- (2) Let  $\omega = \frac{1}{2}(e^{\sqrt{c-c_0}x} + e^{-\sqrt{c-c_0}x}) = \cos(\sqrt{c-c_0}x)$ . Then

$$v = -\sqrt{c-c_0} \tan(\sqrt{c-c_0}x), \quad v_x = \frac{c_0 - c}{\cos^2(\sqrt{c-c_0}x)},$$

$$u = c \rightarrow \tilde{u} = c + \frac{2(c-c_0)}{\cos^2(\sqrt{c-c_0}x)}.$$

If  $c - c_0 < 0$ , then  $\tilde{u}$  is not periodic.

An example of the third case is easily derived from the following explicitly computable examples which are interesting in themselves.

EXAMPLE. *Rational solitons via the Darboux transformation.* Let  $u = 0$  and  $\omega = x$ . Then

$$v = \frac{1}{x}, \quad v_x = -\frac{1}{x^2}, \quad \tilde{u} = \frac{2}{x^2}.$$

The spectral curve  $\Gamma = \{\lambda^2 = E\}$  is transformed to  $\tilde{\Gamma} = \{\lambda^2 = E^3\}$  and

$$\psi = \left(1 - \frac{1}{i\sqrt{E}x}\right) e^{i\sqrt{E}x}$$

(we normalize it by condition  $\psi \approx e^{i\sqrt{E}x}$  as  $E \rightarrow \infty$ ). The iterations of the Darboux transformation initially applied to the trivial potential  $u = 0$  give all rational solitons discovered in [1]. The spectral curve of the potential

$$u_n = \frac{n(n+1)}{2x^2},$$

obtained after  $n$  iterations, is given by the equation  $\lambda^2 = E^{2n+1}$ . The spectral curves of these are singular: topologically they are spheres but at  $E = 0$  they have the following singularities:

any rational function  $f$  on  $\Gamma_n$ , the spectral curve of  $u_n$ , which is holomorphic near  $E = 0$  satisfies the condition

$$f' = f''' = \dots = f^{(2n-1)} = 0. \tag{7}$$

This is easily explained by the normalization mapping  $\mathbb{C} \rightarrow \Gamma = \{\lambda^2 = E^{2n+1}\}$  which has the form

$$t \rightarrow (\lambda = t^{2n+1}, E = t^2).$$

Indeed any rational function  $f$  which is holomorphic at  $E = 0$  is a ratio of polynomials  $\frac{P(\lambda, E)}{Q(\lambda, E)}$  such that  $Q(0, 0) \neq 0$  and any such function written in terms of  $t$  satisfies the conditions (7).

**2.2. The Ore localization.** In order to give an analogous interpretation of the Moutard transformation as a conjugation we first recall some basic facts from the Ore theory of localization of noncommutative rings and introduce the rings  $F(\partial_y)$  and  $F(\partial_y)[\partial_x]$  which we shall use.

We recall that a ring  $R$  is an algebra with two operations, the addition and the multiplication, such that  $R$  is a commutative group with respect to the addition and the distribute laws

$$a(b + c) = ab + ac, \quad (a + b)c = ac + bc \quad \text{for all } a, b, c \in R$$

hold. It is said that a ring  $R$  is regular if it satisfies *the Ore conditions*:

- for  $a \neq 0, b \neq 0$  there exist  $r \neq 0, s \neq 0$  such that  $ar = bs$ ;
- if  $ab = ac$  or  $ba = ca$  for  $a \neq 0$  then  $b = c$  (so it has no zero divisors).

Ore showed in [14] that

any regular ring can be embedded as a subring into a non-commutative field (skew field of fractions)

as follows. Let us consider the set  $S$  of all formal fractions

$$ab^{-1} = \left(\frac{a}{b}\right), \quad b \neq 0.$$

Note that due to non-commutativity of  $R$  one shall always keep in mind that in our construction the denominators  $b^{-1}$  are always on the *right* of the numerators  $a$ ! It is easy to propose an analogous construction with denominators standing on the left, but the resulting skew field will be isomorphic to the field we are to construct.

We say that two such fractions are equal:

$$\left(\frac{a}{b}\right) = \left(\frac{c}{d}\right)$$

if and only if

$$ar = cs$$

where  $r$  and  $s$  satisfy the equality (see the first Ore condition)

$$br = ds. \tag{8}$$

It is easy to show that the equality is independent on the choice of  $r$  and  $s$  satisfying the last equality. Then the addition is defined by

$$\left(\frac{a}{b}\right) + \left(\frac{c}{d}\right) = \left(\frac{ar + cs}{br}\right) = \left(\frac{ar + cs}{ds}\right)$$

where  $r$  and  $s$  satisfy (8). The multiplication is given by

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{at}{du}\right)$$

where

$$bt = cu.$$

The unit element is defined as

$$\left(\frac{a}{a}\right) = 1, \quad a \neq 0,$$

this definition is independent on the choice of  $a$  and with such a law of multiplication we have  $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1$ .

We say that  $S$  (with these operations) is the *Ore localization* of  $R$ . One can routinely check all the usual properties of the addition and multiplication operations defined on  $S$ , which becomes a *skew field*. This means that we can use expressions like  $A^{-1}$  for arbitrary elements of this skew field. All operations in  $S$  given above are constructive, unlike the case of the ring of formal pseudodifferential operators defined as infinite series customary in solitons.

For a commutative ring  $R$  without divisors of zero the Ore localization coincides with the standard localization. Let us introduce the main example with non-commutative ring  $R$  which we shall use.

EXAMPLE. Let  $F = \mathbf{k}(x_1, \dots, x_n)$  be a field formed by all  $\mathbf{k}$ -valued functions of the form

$$\frac{P(x_1, \dots, x_n)}{Q(x_1, \dots, x_n)}$$

where  $P$  and  $Q$  are analytical functions of  $x_1, \dots, x_n$  or even formal power series in these variables. We consider two cases which are

- 1)  $\mathbf{k} = \mathbb{R}$  and  $x_1, \dots, x_n \in \mathbb{R}$ ;
- 2)  $\mathbf{k} = \mathbb{C}$ ,  $n = 2m$  and  $(x_1, \dots, x_n) = (z_1, \bar{z}_1, \dots, z_m, \bar{z}_m)$ ,

and, for simplicity and our needs, we even assume that  $n = 2$ ,  $x_1 = x$ , and  $x_2 = y$ . Let us consider  $F$  as an operator algebra  $R$  which acts on itself by multiplications

$$f(g) = fg, \quad f, g \in F,$$

take the partial derivative operators  $\partial_x = \frac{\partial}{\partial x}$ ,  $\partial_y = \frac{\partial}{\partial y}$  and consider the operator algebra  $F[\partial_y]$  which is generated by elements of  $R$  and by the operator  $\partial_y$  that acts on  $F$ . The ring  $F[\partial_y]$  is noncommutative because

$$[\partial_y, f] = \partial_y \cdot f - f \cdot \partial_y = \frac{\partial f}{\partial y} = f_y \quad \text{for } f \in F. \tag{9}$$

It is straightforward to check

**Proposition 4.**  $F[\partial_y]$  and  $F[\partial_x, \partial_y]$  satisfy the Ore conditions.

Let us take the Ore localization  $F(\partial_y)$  of  $F[\partial_y]$ . We have

**Proposition 5.** For any  $f \in F$  we have

$$[\partial_y^{-1}, f] = -\partial_y^{-1} \cdot \frac{\partial f}{\partial y} \cdot \partial_y^{-1}.$$



PROOF. Let us take the equality (9), multiply every its term by  $\partial_y^{-1}$  from both sides and obtain  $[f, \partial_y] = \partial_y^{-1} \cdot f_y \cdot \partial_y^{-1}$ . Proposition is proved.

Let us consider  $F(\partial_y)[\partial_x]$ , the ring of formal differential operators in  $x$  with coefficients from  $F(\partial_y)$ . This ring is embedded into its Ore localization  $F(\partial_x, \partial_y)$ .

We need to define the commutation of  $\partial_x^{\pm 1}$  with elements of  $F(\partial_y)[\partial_x]$ . Since

$$[\partial_x^{-1}, f] = -\partial_x^{-1}[\partial_x, f]\partial_x^{-1},$$

it is enough to define the commutators of  $f \in F(\partial_y)[\partial_x]$  with  $\partial_x$ . Here we do that even for the Ore localization ring  $F(\partial_x, \partial_y)$ .

If  $L \in F[\partial_x, \partial_y]$  is a differential operator in  $x$  and  $y$  with coefficients from  $F$ , then its derivative in  $x$  (i.e. the operator with coefficients being the derivatives of the respective coefficients of  $L$ ) may be defined by the commutation formula

$$L_x = [\partial_x, L].$$

Let us take this formula as a definition of the derivative of any element of  $P \in F(\partial_x, \partial_y)$  w.r.t.  $x$ . Using the Leibnitz identity

$$(LM)_x = L_x M + L M_x, \quad L, M \in F(\partial_x, \partial_y),$$

we derive the following proposition by straightforward computations.

**Proposition 6.** *Given  $P = M \cdot L^{-1}$  with  $L, M \in F[\partial_x, \partial_y]$ , we have*

$$P_x \stackrel{\text{def}}{=} [\partial_x, P] = M(L^{-1})_x + M_x L^{-1} = -M L^{-1} L_x L^{-1} + M_x L^{-1}.$$

**Corollary 1.** *If  $P \in F(\partial_y)$ , then  $P_x = [\partial_x, P] \in F(\partial_y)$ , i.e.  $P_x$  contains no derivations in  $x$ .*

**2.3. The Moutard transformation.** Let us consider the general Moutard transformation which is applied to an operator:

$$H = \partial_r \partial_s - u(r, s)$$

where

$$r = x, \quad s = y, \quad x, y \in \mathbb{R} \quad (\text{the hyperbolic case}),$$

or

$$r = z, \quad s = \bar{z}, \quad z \in \mathbb{C} \quad (\text{the elliptic case}).$$

Via the formulas

$$(\omega\theta)_r = -\omega^2 \left(\frac{\varphi}{\omega}\right)_r, \quad (\omega\theta)_s = \omega^2 \left(\frac{\varphi}{\omega}\right)_s, \tag{10}$$

it relates solutions  $\varphi$  and  $\theta$  to the equations

$$H\varphi = 0, \quad \tilde{H}\theta = 0$$

where

$$\tilde{H} = \partial_r \partial_s - \tilde{u}, \quad \tilde{u} = u - 2\partial_r \partial_s \log \omega = -u + 2\frac{\omega_r \omega_s}{\omega^2}.$$

Let us consider the Ore localization  $F(\partial_s)$  of the ring of differential operators in  $s$  and the ring  $F(\partial_s)[\partial_r]$  of differential operators in  $r$  with coefficients from (noncommutative) ring  $F(\partial_s)$ . Given  $\omega$ , a solution to the equation  $H\omega = 0$ , we consider the differential operators  $A, B \in F[\partial_s]$ :

$$A = \omega^{-1} \cdot \partial_s \cdot \omega, \quad B = \omega \cdot \partial_s \cdot \omega^{-1}$$

and their “ratio”, the operator  $\Omega$  of the form

$$\Omega = A^{-1}B = \left(\omega^{-1} \cdot \partial_s \cdot \omega\right)^{-1} \cdot \left(\omega \cdot \partial_s \cdot \omega^{-1}\right).$$

We denote by  $M$  and  $\widetilde{M}$  the following formal operators of the first order from  $F(\partial_s)[\partial_r]$ :

$$M = \partial_s^{-1} \cdot H = \partial_r - \partial_s^{-1} \cdot u, \quad \widetilde{M} = \partial_s^{-1} \cdot \widetilde{H} = \partial_r - \partial_s^{-1} \cdot \widetilde{u}.$$

**Theorem 1.** *Given  $\omega$  satisfying  $H\omega = (\partial_r \partial_s + u)\omega = 0$  and the Moutard transform  $\widetilde{u}$  of  $u$  (we assume that the transformation is generated by  $\omega$ ), the operators  $M$  and  $\widetilde{M}$  are conjugated in  $F(\partial_s)[\partial_r]$  by  $\Omega$ :*

$$\widetilde{M} = \Omega \cdot M \cdot \Omega^{-1}.$$

REMARK. It is easy to check that  $H$  and  $\widetilde{H}$  are not conjugated by  $\Omega$ :  $\widetilde{H} \neq \Omega \cdot H \cdot \Omega^{-1}$ .  
PROOF. First we expose some auxiliary facts.

**Lemma 1.**

$$\Omega = A^{-1}B = 1 - \frac{2}{\omega} \cdot \partial_s^{-1} \cdot \omega_s. \quad (11)$$

PROOF OF LEMMA. It is enough to show that  $B = A \left(1 - \frac{2}{\omega} \cdot \partial_s^{-1} \cdot \omega_s\right)$ . Let us write down the right-hand side of this equality as

$$\begin{aligned} A - \omega^{-1} \cdot \partial_s \cdot \omega \frac{2}{\omega} \cdot \partial_s^{-1} \cdot \omega_s &= \omega^{-1} \cdot \partial_s \cdot \omega - 2\omega^{-1} \omega_s = \\ &= \omega^{-1}(\omega_s + \omega \cdot \partial_s) - 2\omega^{-1} \omega_s = -\omega^{-1} \omega_s + \partial_s = B. \end{aligned}$$

This proves the lemma.

**Corollary 2.**

$$(A^{-1}B)_r = \left(1 - \frac{2}{\omega} \cdot \partial_s^{-1} \cdot \omega_s\right)_r = \frac{2\omega_r}{\omega^2} \cdot \partial_s^{-1} \cdot \omega_s - \frac{2}{\omega} \cdot \partial_s^{-1} \cdot \omega_{rs}. \quad (12)$$

Now we are ready to prove Theorem 1. It is enough to establish the equivalent equality

$$\widetilde{H} \cdot \Omega = \partial_s \cdot \Omega \cdot \partial_s^{-1} \cdot H, \quad (13)$$

i.e.

$$(\partial_r \partial_s - \widetilde{u}) \cdot A^{-1} \cdot B = \partial_s \cdot A^{-1} \cdot B \cdot \partial_s^{-1} \cdot H. \quad (14)$$

By Lemma 1, the left-hand side of (14) is

$$\begin{aligned} (\partial_r \partial_s - \widetilde{u}) \cdot A^{-1} \cdot B &= \partial_s \left[ (A^{-1}B)_r + A^{-1}B \cdot \partial_r \right] - \left( -u + \frac{2\omega_r \omega_s}{\omega^2} \right) \cdot \left( 1 - \frac{2}{\omega} \cdot \partial_s^{-1} \cdot \omega_s \right) = \\ &= \left[ \partial_s \cdot \left( \frac{2\omega_r}{\omega^2} \cdot \partial_s^{-1} \cdot \omega_s - \frac{2}{\omega} \cdot \partial_s^{-1} \cdot \omega_{rs} \right) + \partial_s \cdot A^{-1}B \cdot \partial_r \right] + \\ &\quad + u \cdot \left( 1 - \frac{2}{\omega} \cdot \partial_s^{-1} \cdot \omega_s \right) - \frac{2\omega_r \omega_s}{\omega^2} \cdot \left( 1 - \frac{2}{\omega} \cdot \partial_s^{-1} \cdot \omega_s \right) = \\ &= \left( \left( \frac{2\omega_r}{\omega^2} \right)_s \cdot \partial_s^{-1} \cdot \omega_s + \frac{2\omega_r}{\omega^2} \cdot \partial_s \cdot \partial_s^{-1} \cdot \omega_s - \left( \frac{2}{\omega} \right)_s \cdot \partial_s^{-1} \cdot \omega_{rs} - \frac{2}{\omega} \cdot \partial_s \partial_s^{-1} \cdot \omega_{rs} \right) + \\ &\quad + \partial_s \cdot A^{-1}B \cdot \partial_r + u - \frac{2u}{\omega} \cdot \partial_s^{-1} \cdot \omega_s - \frac{2\omega_r \omega_s}{\omega^2} + \frac{4\omega_r \omega_s}{\omega^3} \cdot \partial_s^{-1} \cdot \omega_s = \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{2\omega_{rs}}{\omega^2} \cdot \partial_s^{-1} \cdot \omega_s - \frac{4\omega_r\omega_s}{\omega^3} \cdot \partial_s^{-1} \cdot \omega_s + \frac{2\omega_r\omega_s}{\omega^2} + \left( \frac{2\omega_s}{\omega^2} \right) \cdot \partial_s^{-1} \cdot \omega_{rs} - \frac{2}{\omega} \omega_{rs} \right) + \\
&\quad + \partial_s \cdot A^{-1}B \cdot \partial_r + u - \frac{2u}{\omega} \cdot \partial_s^{-1} \cdot \omega_s - \frac{2\omega_r\omega_s}{\omega^2} + \frac{4\omega_r\omega_s}{\omega^3} \cdot \partial_s^{-1} \cdot \omega_s = \\
&= \left( \frac{2\omega_s}{\omega^2} \right) \cdot \partial_s^{-1} \cdot (u\omega) - 2u + \partial_s \cdot A^{-1}B \cdot \partial_r + u = \\
&= \left( \frac{2\omega_s}{\omega^2} \right) \cdot \partial_s^{-1} \cdot (u\omega) + \partial_s \cdot A^{-1}B \cdot \partial_r - u.
\end{aligned}$$

In the right-hand side of (14) we have

$$\begin{aligned}
&\partial_s \cdot A^{-1}B \cdot \partial_s^{-1} \cdot (\partial_r \partial_s - u) = \partial_s \cdot A^{-1}B \cdot \partial_r - \partial_s \cdot \left( 1 - \frac{2}{\omega} \cdot \partial_s^{-1} \cdot \omega_s \right) \partial_s^{-1} \cdot u = \\
&= \partial_s \cdot A^{-1}B \cdot \partial_r - u + \left( \frac{2}{\omega} \right)_s \cdot \partial_s^{-1} \cdot \omega_s \cdot \partial_s^{-1} \cdot u + \left( \frac{2}{\omega} \right) \cdot \partial_s \cdot \partial_s^{-1} \cdot \omega_s \cdot \partial_s^{-1} \cdot u = \\
&= \partial_s \cdot A^{-1}B \cdot \partial_r - u - \frac{2\omega_s}{\omega^2} \cdot \partial_s^{-1} \cdot \omega_s \cdot \partial_s^{-1} \cdot u + \frac{2\omega_s}{\omega} \cdot \partial_s^{-1} \cdot u.
\end{aligned}$$

Let us apply Proposition 5 to the third term  $\partial_s^{-1} \cdot \omega_s \cdot \partial_s^{-1} = \omega \cdot \partial_s^{-1} - \partial_s^{-1} \cdot \omega$  in the last formula and finally derive

$$\begin{aligned}
&\partial_s \cdot A^{-1}B \cdot \partial_r - u - \frac{2\omega_s}{\omega^2} \cdot (\omega \cdot \partial_s^{-1} - \partial_s^{-1} \cdot \omega) \cdot u + \frac{2\omega_s}{\omega} \cdot \partial_s^{-1} \cdot u = \\
&= \partial_s \cdot A^{-1}B \cdot \partial_r - u + \frac{2\omega_s}{\omega^2} \cdot \partial_s^{-1} \cdot \omega \cdot u,
\end{aligned}$$

which coincides with the left-hand side of (14).

Theorem is proved.

Let us assume that  $H$  is a (weakly) algebraic operator, i.e., there are differential operators  $L_1$  and  $L_2$  such that

$$[L_1, L_2] = D_0H, \quad [L_1, H] = D_1H, \quad [L_2, H] = D_2H \quad (15)$$

and there is a polynomial  $Q$  in two variables with constant coefficients such that

$$Q(L_1, L_2) = 0 \pmod{H}.$$

By applying the Euclid algorithm (division with remainder in the ring of formal linear ordinary differential operators  $F(\partial_s)[\partial_r]$ ) we obtain  $R_1, R_2 \in F(\partial_s)$  such that

$$L_1 - Q_1 \cdot M = R_1 \in F(\partial_s), \quad L_2 - Q_2 \cdot M = R_2 \in F(\partial_s) \quad (16)$$

with  $Q_i \in F(\partial_s)[\partial_r]$ .

**Theorem 2.** (1)  $[R_1, R_2] = 0$ ;

(2)  $Q(R_1, R_2) = 0$ ;

(3)  $[R_1, M] = [R_2, M] = 0$ .

PROOF.

(1) Using (15) we compute that

$$[R_1, R_2] = [L_1 - Q_1H, L_2 - Q_2H] = S_1 \cdot H, \quad S_1 \in F(\partial_s)[\partial_r].$$

However in the left-hand side we have an element of  $F(\partial_s)[\partial_r]$  which contains no derivations in  $r$  which implies that  $S_1 = 0$ .

- (2) The equality  $Q(L_1, L_2) = 0 \pmod{H}$  means that  $Q(L_1, L_2) = U \cdot H$  in  $F[\partial_r, \partial_s]$  which, by (15), implies

$$Q(R_1, R_2) = S_2 \cdot H \in F(\partial_s)[\partial_r].$$

However in the left-hand side we have an element of  $F(\partial_s)[\partial_r]$  with no derivations in  $r$  which implies  $S_2 = 0$ .

- (3) We have

$$[R_1, H] = [L_1 - Q_1 H, H] = S_3 \cdot H,$$

$$[R_1, M] = [R_1, (\partial_s)^{-1} H] = [R_1, (\partial_s)^{-1} H] + (\partial_s)^{-1} [R_1, H] = S_4 \cdot H = S_5 \cdot M,$$

where  $S_i \in F(\partial_s)[\partial_r]$  for  $i = 3, 4, 5$ . By Corollary 1, this implies

$$[R_1, M] = [R_1, \partial_r - (\partial_s)^{-1} \cdot u(r, s)] = -(R_1)_r + [R_1, (\partial_s)^{-1} \cdot u(r, s)] \in F(\partial_s),$$

and again as above we see that  $S_5 = 0$ . The proof of the equality  $[R_2, M] = 0$  is completely the same.

Theorem is proved.

In view of Theorems 1 and 2 we conjecture that the Moutard transformation should preserve the class of weakly algebraic two-dimensional Schrödinger operators.

### 3. APPLICATIONS OF THE MOUTARD TRANSFORMATION

In [15, 16, 17] we gave a simple examples of fast decaying rational potentials for the two-dimensional Schrödinger operator which has a degenerated  $L_2$ -kernel. These examples are constructed by using the Moutard transformation as follows.

MAIN CONSTRUCTION. Let

$$H_0 = -\Delta = -\Delta + u_0$$

be an operator with a potential  $u_0(x, y)$  and let  $\omega_1(x, y)$  and  $\omega_2(x, y)$  satisfy the equations

$$H_0 \omega_1 = H_0 \omega_2 = 0.$$

We take the Moutard transformations  $M_{\omega_1}$  and  $M_{\omega_2}$  defined by  $\omega_1$  and  $\omega_2$  and obtain the operators

$$H_1 = -\Delta + u_1, \quad H_2 = -\Delta + u_2$$

where  $u_1 = M_{\omega_1}(u_0)$ ,  $u_2 = M_{\omega_2}(u_0)$ . By the construction, we have

$$H_1 M_{\omega_1}(\omega_2) = 0, \quad H_2 M_{\omega_2}(\omega_1) = 0.$$

Let us choose some function

$$\theta_1 \in M_{\omega_1}(\omega_2)$$

and put

$$\theta_2 = -\frac{\omega_1}{\omega_2} \theta_1 \in M_{\omega_2}(\omega_1).$$

These functions define the Moutard transformations of  $H_1$  and  $H_2$  and we obtain the operators  $H_{12}$  and  $H_{21}$  with the potentials

$$u_{12} = M_{\theta_1}(u_1), \quad u_{21} = M_{\theta_2}(u_2).$$

The following key lemma is checked by straightforward computations which we omit.

**Lemma 2.** (1)  $u_{12} = u_{21} = u$ ;

(2) For  $\psi_1 = \frac{1}{\theta_1}$  and  $\psi_2 = \frac{1}{\psi_2}$  we have

$$H\psi_1 = H\psi_2 = 0$$

where  $H = -\Delta + u$ .

We note that in this construction we have a free scalar parameter  $t$  (see (3)) for the choice of  $\theta_1 \in M_{\omega_1}(\omega_2)$ . This parameter can be used in some cases to build a non-singular potential  $u$  and functions  $\psi_1$  and  $\psi_2$ .

For example if we apply this construction to the situation when  $u_0 = 0$  and  $\omega_1$  and  $\omega_2$  are real-valued harmonic polynomials

$$\omega_1 = x + 2(x^2 - y^2) + xy, \quad \omega_2 = x + y + \frac{3}{2}(x^2 - y^2) + 5xy,$$

then for some appropriate constant of integration in  $\theta_1$  we obtain

$$\begin{aligned} u &= -\frac{5120(1 + 8x + 2y + 17x^2 + 17y^2)}{(160 + 4x^2 + 4y^2 + 16x^3 + 4x^2y + 16xy^2 + 4y^3 + 17(x^2 + y^2)^2)^2} = \\ &= -\frac{5120|1 + (4 - i)z|^2}{(160 + |z|^2|2 + (4 - i)z|^2)^2} \end{aligned} \quad (17)$$

and

$$\begin{aligned} \psi_1 &= \frac{x + 2x^2 + xy - 2y^2}{160 + 4x^2 + 4y^2 + 16x^3 + 4x^2y + 16xy^2 + 4y^3 + 17(x^2 + y^2)^2}, \\ \psi_2 &= \frac{2x + 2y + 3x^2 + 10xy - 3y^2}{160 + 4x^2 + 4y^2 + 16x^3 + 4x^2y + 16xy^2 + 4y^3 + 17(x^2 + y^2)^2} \end{aligned} \quad (18)$$

(here we simplify  $\psi_1$  and  $\psi_2$  by multiplying by some constant).

**Theorem 3** ([15, 16]). *The potential  $u$  given by (17) is smooth, rational, and decays like  $1/r^6$  for  $r \rightarrow \infty$  (here  $r = \sqrt{x^2 + y^2}$ ).*

*The functions  $\psi_1$  and  $\psi_2$  given by (18) are smooth, rational, decay like  $1/r^2$  for  $r \rightarrow \infty$  and span a two-dimensional space in the kernel of the operator  $L = -\Delta + u : L_2(\mathbb{R}^2) \rightarrow L_2(\mathbb{R}^2)$ .*

If one takes appropriate two harmonic polynomials of the third order  $\omega_1$  and  $\omega_2$  (cf. [15, 16]) then one can construct the potential  $u$  and the functions  $\psi_1$  and  $\psi_2$  which are smooth, rational, the potential decays like  $1/r^8$  for  $r \rightarrow \infty$ , the functions  $\psi_1$  and  $\psi_2$  decay like  $1/r^3$  for  $r \rightarrow \infty$  and span a two-dimensional space in the kernel of the operator  $L = -\Delta + u : L_2(\mathbb{R}^2) \rightarrow L_2(\mathbb{R}^2)$ .

REMARK. We conjecture that for every  $N > 0$  by applying this construction to other harmonic polynomials one can construct smooth rational potentials  $u$  and the eigenfunctions  $\psi_1$  and  $\psi_2$  decaying faster than  $\frac{1}{r^N}$ .

In [16, 17] we used a time-dependent extension of the Moutard transformation constructing explicit solutions of the Novikov–Veselov equation [11, 12]

$$\begin{aligned} U_t &= \partial^3 U + \bar{\partial}^3 U + 3\partial(VU) + 3\bar{\partial}(\bar{V}U) = 0, \\ \bar{\partial}V &= \partial U. \end{aligned} \quad (19)$$

Some of our solutions show a very special behavior: the initial data for  $t = 0$  are smooth decaying rational functions of  $x$  and  $y$ ; nevertheless for  $t \geq t_0 > 0$  the solutions to the NV equation (19) blow up (become singular).

In particular the following solution  $U(z, \bar{z}, t)$  of the Novikov–Veselov equation can be obtained using this technology:

$$U = \frac{H_1}{H_2},$$

with

$$H_1 = -12 \left( 24tx^2 + 12tx + 24ty^2 + 12ty + x^5 - 3x^4y + 2x^4 - 2x^3y^2 - 4x^3y - 2x^2y^3 - 60x^2 - 3xy^4 - 4xy^3 - 30x + y^5 + 2y^4 - 60y^2 - 30y \right),$$

$$H_2 = (3x^4 + 4x^3 + 6x^2y^2 + 3y^4 + 4y^3 + 30 - 12t)^2.$$

This solution decays as  $r^{-3}$  at infinity, it is nonsingular for  $0 \leq t < T_* = \frac{29}{12}$  and have singularities for  $t \geq T_* = \frac{29}{12}$ .

For  $t \rightarrow T_*$ , as we can see on Figure 1, the solution  $U(x, y, t)$  oscillates with growing amplitudes in the neighborhoods of the points  $P = (-1, 0)$  and  $Q = (0, -1)$  since the denominator  $H_2$  vanishes at these points for  $t = T_*$ . The numerator  $H_1$  has zeros of order 3 at the points  $P$  and  $Q$  so their respective neighborhoods are subdivided by smooth lines into 6 sectors with different signs of the numerator. The complicated behavior of the potential in the neighborhood of one of the singular points for  $t = T_*$  is shown on Figure 2.

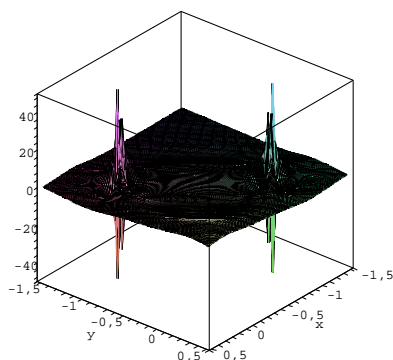


Figure 1: The potential  $U$  for  $t = \frac{29}{12}$ .

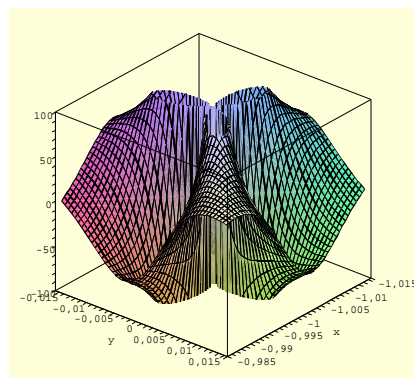


Figure 2: The potential  $U$  for  $t = \frac{29}{12}$  in the neighborhood of the point  $(-1, 0)$ .

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