

# FADDEEV EIGENFUNCTIONS FOR TWO-DIMENSIONAL SCHRÖDINGER OPERATORS VIA THE MOUTARD TRANSFORMATION

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We demonstrate how the Moutard transformation of two-dimensional Schrödinger operators acts on the Faddeev eigenfunctions on the zero-energy level and present some explicitly computed examples of such eigenfunctions for smooth rapidly decaying potentials of operators with a nontrivial kernel and for deformed potentials corresponding to blowup solutions of the Novikov–Veselov equation.

**Keywords:** Schrödinger operator, Faddeev eigenfunction, Moutard transformation, scattering data

## 1. Introduction

In [1], we used the Moutard transformation to construct examples of two-dimensional Schrödinger operators with rapidly decaying smooth potentials and nontrivial kernels and of blowup solutions of the Novikov–Veselov equation. Here, we show how the Moutard transformation acts on the Faddeev eigenfunctions corresponding to the zero-energy level. In particular, we construct some explicit examples corresponding to the operators constructed in [1]. It turns out that such Faddeev eigenfunctions have very interesting analytic properties and dynamics under the Novikov–Veselov flow. A list of two-dimensional Schrödinger operators for which the Faddeev eigenfunctions are explicitly computed for at least one energy level can be extended in an essentially new way using the presented procedure.

For a multidimensional Schrödinger operator  $L = -\Delta + u$ , the Faddeev eigenfunctions  $\psi(x, k)$ ,  $k \in \mathbb{C}^n$ ,  $x \in \mathbb{R}^n$ , are formal solutions of the equation

$$L\psi = E\psi, \quad E = \langle k, k \rangle,$$

satisfying the condition

$$\psi = e^{i\langle k, x \rangle} (1 + o(1)) \quad \text{as } |x| \rightarrow \infty. \tag{1}$$

Here,  $\langle x, y \rangle = \sum_{j=1}^n x_j y_j$  is the standard inner product in  $\mathbb{C}^n$ . These functions were defined in [2] and play a crucial role in the multidimensional inverse scattering problem [3], [4].

In what follows, we consider two-dimensional Schrödinger operators

$$L = -\Delta + u = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + u = -4\partial\bar{\partial} + u,$$

where

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

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in the right-hand side,  $(x_1, x_2) = (x, y) \in \mathbb{R}^2$ , and  $z = x + iy \in \mathbb{C}$ . In this case and also in higher dimensions, there are not many examples for which these eigenfunctions have been computed explicitly. In fact, they are as follows:

1. For the multipoint potentials

$$u(x) = \sum_{k=1}^N \varepsilon_k \delta(x - x_k),$$

the Faddeev eigenfunctions were computed for all energy levels in [5], [6].

2. For the Grinevich–Zakharov potentials introduced by Grinevich [7] and Zakharov (these potentials are reflectionless on the energy level  $E$  and decay as  $|x|^{-2}$ ), explicit formulas for the Faddeev eigenfunctions on the energy level  $E$  can be extracted from computations in [7]).

It was shown in [8] that for negative energy levels  $E$ , the functions

$$\chi(x, k) = e^{-i\langle k, x \rangle} \psi(x, k) = \exp\left(-\frac{i}{2} \sqrt{-E} \left(\lambda z - \frac{\bar{z}}{\lambda}\right)\right) \psi(x, k),$$

where

$$k_1 = \frac{i\sqrt{-E}}{2} \left(\lambda + \frac{1}{\lambda}\right), \quad k_2 = -\frac{\sqrt{-E}}{2} \left(\lambda - \frac{1}{\lambda}\right),$$

satisfy the generalized Cauchy–Riemann equation

$$\frac{\partial \chi}{\partial \lambda} = T \bar{\chi},$$

i.e., they are generalized analytic functions, and if  $T$  is nonsingular, then this level is below the ground state. Moreover, in the last case, the inverse scattering problem of reconstructing the potential (without assuming that the potential is sufficiently small) is solved by methods of generalized analytic function theory [9].

Here, we consider the Faddeev eigenfunctions on the zero-energy level  $E = 0$ . They satisfy the condition

$$\psi(z, \bar{z}, \lambda) = e^{\lambda z} \left(1 + \frac{A(\lambda, \bar{\lambda})}{z} + e^{\lambda \bar{z} - \lambda z} \frac{B(\lambda, \bar{\lambda})}{\bar{z}} + O\left(\frac{1}{|z|^2}\right)\right) \quad \text{as } |z| \rightarrow \infty. \quad (2)$$

In fact, this is one branch of these functions, and another branch is given by solutions of the equation  $(-\Delta + u)\psi = 0$  with the asymptotic behavior  $e^{\mu \bar{z}}(1 + o(1))$  as  $|z| \rightarrow \infty$ . For nonzero-energy levels, analogous asymptotic expansions were given in [10], and for the zero-energy level, they are derived similarly from the analytic properties of the Green–Faddeev function of the Laplace operator (see, in particular, [11] for the details).

## 2. The Moutard transformation

**2.1. The Moutard transformation and its double iteration.** Let  $\omega$  satisfy the equation

$$L\omega = (-\Delta + u)\omega = 0.$$

The *Moutard transformation* of  $L$  is defined as

$$L \rightarrow \tilde{L} = -\Delta + u - 2\Delta \log \omega = -\Delta - u + 2 \frac{\omega_x^2 + \omega_y^2}{\omega^2}.$$

The following property of this transformation is easily verified.

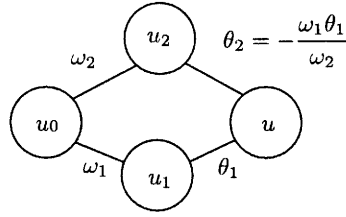


Fig. 1. Rhombic superposition formula.

**Property 1.** If  $\varphi$  satisfies the equation  $L\varphi = 0$ , then the function  $\theta$  defined up to the term  $C/\omega$ , where  $C = \text{const}$ , solving the system of equations

$$(\omega\theta)_x = -\omega^2 \left( \frac{\varphi}{\omega} \right)_y, \quad (\omega\theta)_y = \omega^2 \left( \frac{\varphi}{\omega} \right)_x \quad (3)$$

satisfies the transformed equation  $\tilde{L}\theta = 0$ .

If the potential  $u$  depends only on  $x$ , then a special reduction of the Moutard transformation gives the famous Darboux transformation of the one-dimensional Schrödinger operator:

$$L = -\frac{d^2}{dx^2} + u = \left( \frac{d}{dx} + v \right) \left( -\frac{d}{dx} + v \right) \rightarrow \tilde{L} = \left( -\frac{d}{dx} + v \right) \left( \frac{d}{dx} + v \right).$$

This is the case where

$$\omega = f(x)e^{\sqrt{E}y}, \quad \left( -\frac{d^2}{dx^2} + u \right) f = Ef.$$

In [1], we used a double iteration of the Moutard transformation starting from the Laplace operator  $L_0 = -\Delta$ . Given two harmonic functions  $\omega_1$  and  $\omega_2$ ,  $\Delta\omega_1 = \Delta\omega_2 = 0$ , we construct a pair of Moutard transformations corresponding to  $\omega_1$  and  $\omega_2$ :

$$L_1 = -\Delta + u_1 \xleftarrow{\omega_1} L_0 = -\Delta \xrightarrow{\omega_2} L_2 = -\Delta + u_2.$$

Both the potentials  $u_1 = -2\Delta \log \omega_1$  and  $u_2 = -2\Delta \log \omega_2$  have singularities at zeroes of the respective functions  $\omega_1$  and  $\omega_2$ , and we must iterate the Moutard transformation to obtain a smooth (i.e., bounded and differentiable) potential, if possible.

We let  $\theta_1$  and  $\theta_2$  denote the respective transforms of  $\omega_2$  and  $\omega_1$  via (3) for  $\omega = \omega_1$  and  $\omega = \omega_2$ . Such transforms are defined up to  $C/\omega$ , and we set  $\theta_2 = -(\omega_1/\omega_2)\theta_1$  for a fixed  $\theta_1$  (see Fig. 1). It is known that the functions  $\theta_1$  and  $\theta_2$  define the respective Moutard transforms of  $L_2$  and  $L_1$ , which give the same operator

$$L_1 = -\Delta + u_1 \xrightarrow{\theta_1} L = L_{12} = -\Delta + u_{12} \xleftarrow{\theta_2} L_2 = -\Delta + u_2.$$

Moreover, the functions  $\varphi_1 = 1/\theta_1$  and  $\varphi_2 = 1/\theta_2$  satisfy the equation  $L\psi = 0$ .

The potential  $u = u_{12}$  has the form

$$u = -2\Delta \log i \left( (p_1 \bar{p}_2 - p_2 \bar{p}_1) + \int ((p_1' p_2 - p_1 p_2') dz + (\bar{p}_1 \bar{p}_2' - \bar{p}_1' \bar{p}_2) d\bar{z}) \right), \quad (4)$$

where  $\omega_1 = p_1(z) + \overline{p_1(z)}$  and  $\omega_2 = p_2(z) + \overline{p_2(z)}$ ,  $p_1(z)$  and  $p_2(z)$  are holomorphic functions of  $z$ , and the free scalar parameter mentioned above appears as the integration constant in (4).

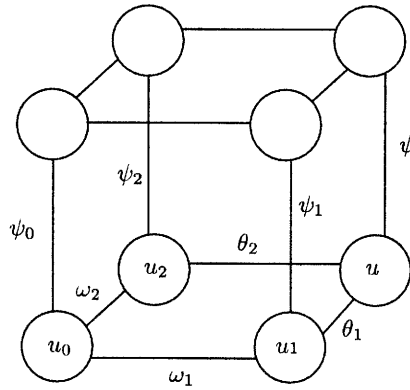


Fig. 2. Cubic superposition formula.

**2.2. The Moutard transformation of Faddeev eigenfunctions.** For  $L_0 = -\Delta$ , Faddeev eigenfunctions (2) are  $\psi_0(z, \lambda) = e^{\lambda z}$  and are transformed via (3) to  $\psi_1$  and  $\psi_2$ :

$$\psi_1 \xleftarrow{\omega_1} \psi_0 = e^{\lambda z} \xrightarrow{\omega_2} \psi_2.$$

As mentioned above, these transformations are not uniquely defined, but at least for functions  $\omega_1$  and  $\omega_2$  that are polynomial in  $z$  and  $\bar{z}$ , we can choose the branches of  $\psi_1$  and  $\psi_2$  such that they satisfy condition (1). Then the “cubic superposition formula” (see [1] and Fig. 2) implies that

$$\psi(z, \bar{z}, \lambda) = e^{\lambda z} + \frac{\omega_2}{\theta_1}(\psi_2 - \psi_1)$$

satisfies the equation  $L\psi = 0$ . It seems that for smooth rapidly decaying potentials  $u = u_{12}$ , the function  $\psi$  is the Faddeev eigenfunction on the zero-energy level. We do not state that here as a rigorous mathematical result, but this is the case for the potentials found in [1].

**Example 1.** Let

$$p_1(z) = \left(1 - \frac{i}{4}\right)z^2 + \frac{z}{2}, \quad p_2(z) = \frac{1}{4}(3 - 5i)z^2 + \frac{1 - i}{2}z.$$

For some appropriate constant  $C$  in  $\theta_1$ , we obtain

$$u = -\frac{5120|(4 - i)z + 1|^2}{(160 + |z|^2|(4 - i)z + 2|^2)^2}.$$

Multiplying  $\varphi_1$  and  $\varphi_2$  by some constants, we obtain

$$\varphi_1 = \frac{2(z + \bar{z}) + (4 - i)z^2 + (4 + i)\bar{z}^2}{160 + |z|^2|(4 - i)z + 2|^2},$$

$$\varphi_2 = \frac{2(1 - i)z + 2(1 + i)\bar{z} + (3 - 5i)z^2 + (3 + 5i)\bar{z}^2}{160 + |z|^2|(4 - i)z + 2|^2}.$$

Faddeev eigenfunction (2) for  $L = -\Delta + u$  has the form

$$\psi(z, \bar{z}, \lambda) = e^{\lambda z} \left( 1 + \frac{1}{\lambda} \frac{(8i - 32)z\bar{z} - 8\bar{z} - (16 + 4i)\bar{z}^2 - 68z\bar{z}^2}{160 + |z|^2|(4 - i)z + 2|^2} + \frac{1}{\lambda^2} \frac{(32 - 8i)\bar{z} + 68\bar{z}^2}{160 + |z|^2|(4 - i)z + 2|^2} \right),$$

$$\psi(z, \bar{z}, \lambda) = e^{\lambda z} \left( 1 - \frac{4}{\lambda z} + O\left(\frac{1}{|z|^2}\right) \right) \quad \text{as } |z| \rightarrow \infty.$$

The functions  $u$ ,  $\varphi_1$ , and  $\varphi_2$  are smooth and real-valued, and the function  $\psi$  is also smooth in  $z$  and  $\bar{z}$ . The potential  $u$  decays as  $1/|z|^6$  as  $|z| \rightarrow \infty$ , and  $\varphi_1$  and  $\varphi_2$  decay as  $1/|z|^2$ . Hence,  $\varphi_1$  and  $\varphi_2$  span a two-dimensional subspace in the kernel of  $L$ , and because  $u \leq 0$  and  $u$  decays sufficiently fast, there are also negative eigenvalues of  $L$ .

We also see that the “scattering data” take a very simple form:

$$A(\lambda, \bar{\lambda}) = -\frac{4}{\lambda}, \quad B(\lambda, \bar{\lambda}) = 0.$$

Similar formulas for the Faddeev eigenfunctions can be easily derived for another smooth rapidly decaying potential with a nontrivial kernel that was found in [1]; it corresponds to

$$p_1(z) = (i-1)z^3 + \left(\frac{1}{10} + \frac{3}{20}i\right)z^2 + \frac{z}{2}, \quad p_2(z) = 2iz^3 + \left(\frac{1}{4} + \frac{i}{20}\right)z^2 + \frac{1-i}{2}z.$$

In this case, the “scattering data” are  $A(\lambda, \bar{\lambda}) = -6/\lambda$  and  $B(\lambda, \bar{\lambda}) = 0$ .

### 3. The extended Moutard transformation and the Novikov–Veselov equation

**3.1. The extended Moutard transformation.** Using the traditional notation associated with the Novikov–Veselov equation, we renormalize the Schrödinger operator as

$$H = \partial\bar{\partial} + U = \frac{1}{4}\Delta - \frac{u}{4}.$$

The Moutard transformation becomes

$$U \rightarrow U + 2\partial\bar{\partial}\log\omega, \\ \left(\bar{\partial} + \frac{\omega_{\bar{z}}}{\omega}\right)\theta = i\left(\bar{\partial} - \frac{\omega_{\bar{z}}}{\omega}\right)\varphi, \quad \left(\partial + \frac{\omega_z}{\omega}\right)\theta = -i\left(\partial - \frac{\omega_z}{\omega}\right)\varphi.$$

We assume that  $\varphi$  also depends on the time  $t$  and satisfies the equations

$$H\varphi = 0, \quad \partial_t\varphi = (\partial^3 + \bar{\partial}^3 + 3V\partial + 3\bar{V}\bar{\partial})\varphi, \quad \bar{\partial}V = \partial U, \quad (5)$$

where  $U = \bar{U}$ . It is easily verified by direct computation that if real-valued functions  $\omega$  and  $U$  are given, then system (5) is invariant under the extended Moutard transformation [1], [12]

$$\varphi \rightarrow \theta = \frac{i}{\omega} \int (\varphi \partial\omega - \omega \partial\varphi) dz - (\varphi \bar{\partial}\omega - \omega \bar{\partial}\varphi) d\bar{z} + \\ + (\varphi \partial^3\omega - \omega \partial^3\varphi + \omega \bar{\partial}^3\varphi - \varphi \bar{\partial}^3\omega + 2(\partial^2\varphi \partial\omega - \partial\varphi \partial^2\omega) - \\ - 2(\bar{\partial}^2\varphi \bar{\partial}\omega - \bar{\partial}\varphi \bar{\partial}^2\omega) + 3V(\varphi \partial\omega - \omega \partial\varphi) + 3\bar{V}(\omega \bar{\partial}\varphi - \varphi \bar{\partial}\omega)) dt, \quad (6)$$

$$U \rightarrow U + 2\partial\bar{\partial}\log\omega,$$

$$V \rightarrow V + 2\partial^2\log\omega.$$

The compatibility condition for (5) is the well-known Novikov–Veselov equation [13]

$$U_t = \partial^3U + \bar{\partial}^3U + 3\partial(VU) + 3\bar{\partial}(\bar{V}U) = 0, \quad \bar{\partial}V = \partial U, \quad (7)$$

It is a two-dimensional generalization of the Korteweg–de Vries equation, to which it reduces in the case where  $U = U(x)$  and  $U = V$ .

We can obtain analogues of all constructions in Sec. 2 by replacing the Moutard transformation with its extended version (6) and thus construct nontrivial real-valued solutions of Novikov–Veselov equation (7) (cf. [1]). The analogue of (4) has the form

$$U(z, \bar{z}, t) = 2\partial\bar{\partial}\log i \left( (p_1\bar{p}_2 - p_2\bar{p}_1) + \int ((p_1'p_2 - p_1p_2') dz + (\bar{p}_1\bar{p}_2' - \bar{p}_1'\bar{p}_2) d\bar{z}) + \right. \\ \left. + \int (p_1'''p_2 - p_1p_2''' + 2(p_1'p_2'' - p_1''p_2') + \bar{p}_1\bar{p}_2''' - \bar{p}_1'''\bar{p}_2 + 2(\bar{p}_1''\bar{p}_2' - \bar{p}_1'\bar{p}_2'')) dt \right), \quad (8)$$

and if the functions  $p_1(z, t)$  and  $p_2(z, t)$  holomorphic in  $z$  satisfy the equation

$$\frac{\partial p}{\partial t} = \frac{\partial^3 p}{\partial^3 z},$$

then  $U(z, \bar{z}, t)$  satisfies Novikov–Veselov equation (7).

**3.2. An example of the Novikov–Veselov evolution of Faddeev eigenfunctions.** Let  $p_1(z) = iz^2$  and  $p_2 = (1+i)z + z^2$ . Although these functions are independent of  $t$ , formula (8) gives a nonstationary solution of the Novikov–Veselov equation. The Faddeev eigenfunction is obtained by applying iterations of (6) to  $\psi_0(z, \bar{z}, t) = e^{\lambda z + \lambda^3 t}$ . These data lead to the functions

$$U(z, \bar{z}, t) = \frac{F_U(z, \bar{z}, t)}{Q^2(z, \bar{z}, t)}, \quad V(z, \bar{z}, t) = \frac{F_V(z, \bar{z}, t)}{Q^2(z, \bar{z}, t)},$$

where

$$F_U(z, \bar{z}, T) = 6i(24t(i\bar{z} - iz + z + \bar{z}) + 2iz^4\bar{z} + 4iz^3\bar{z} - 2iz\bar{z}^4 - 4iz\bar{z}^3 + \\ + 60iz - 60i\bar{z} + 96tz\bar{z} + 2z^4\bar{z} + z^4 + 6z^2\bar{z}^2 + 2z\bar{z}^4 - 240z\bar{z} - 60z + \bar{z}^4 - 60\bar{z}), \\ F_V(z, \bar{z}, t) = 6i(24t(iz - i\bar{z} + z + \bar{z}) + iz^4 + 4iz^3\bar{z}^2 - 12iz^2\bar{z}^3 - 6iz^2\bar{z}^2 + \\ + 6iz\bar{z}^4 - 60iz + 2i\bar{z}^5 + 5i\bar{z}^4 + 60i\bar{z} + 48t\bar{z}^2 + 4z^3\bar{z}^2 + 4z^3\bar{z} + 12z^2\bar{z}^4 + 12z^2\bar{z}^3 + \\ + 6z\bar{z}^4 + 4z\bar{z}^3 - 60z - 2\bar{z}^5 - 120\bar{z}^2 - 60\bar{z}),$$

and

$$Q(z, \bar{z}, t) = (12 - 12i)t - z^3 - (3 - 3i)z^2\bar{z}^2 + 3iz^2\bar{z} - 3z\bar{z}^2 + i\bar{z}^3 - (30 - 30i) = \\ = \frac{1}{1+i} (24t - [6(x^2 + y^2)^2 + 8(x^3 + y^3) + 60]).$$

The Faddeev eigenfunctions for the potentials  $U(z, \bar{z}, t)$  become

$$\psi(z, \bar{z}, t, \lambda) = e^{\lambda z} \left( 1 + \frac{\mu_1(z, \bar{z}, t)}{\lambda} + \frac{\mu_2(z, \bar{z}, t)}{\lambda^2} \right) = \\ = e^{\lambda z} \left( 1 + \frac{1}{\lambda} \frac{6(-2iz\bar{z}^2 - 2iz\bar{z} + z^2 + 2z\bar{z}^2 + \bar{z}^2)}{Q(z, \bar{z}, t)} + \frac{1}{\lambda^2} \frac{12(i\bar{z}^2 + i\bar{z} - z - \bar{z}^2)}{Q(z, \bar{z}, t)} \right).$$

From the last formula, it is easy to derive the “scattering data” for  $U(z, \bar{z}, t)$ :

$$\psi(z, \bar{z}, t, \lambda) = e^{\lambda z} \left( 1 - \frac{4}{\lambda z} + O\left(\frac{1}{|z|^2}\right) \right) \quad \text{as } |z| \rightarrow \infty,$$

i.e., we have

$$A(\lambda, \bar{\lambda}, t) = -\frac{4}{\lambda}, \quad B(\lambda, \bar{\lambda}, t) = 0.$$

The function  $U(z, \bar{z}, t)$  is a solution of the Novikov–Veselov equation with smooth rapidly decaying (as  $1/|z|^3$  as  $|z| \rightarrow \infty$ ) initial data  $U(z, \bar{z}, 0)$  at  $t = 0$  that blows up at a finite time  $t = T_* > 0$  [1]. The blowup occurs when  $Q(z, \bar{z}, t)$  vanishes at some point.

We note that

1. the “scattering data”  $A(\lambda, \bar{\lambda}, t)$  and  $B(\lambda, \bar{\lambda}, t)$  are conserved and
2. the real and imaginary parts of  $\mu_2(z, \bar{z}, t)$  give eigenfunctions of  $H$  at the zero-energy level. By the formula for  $\mu_2(z, \bar{z}, t)$ , they stay nonsingular and square-integrable until the critical time  $t = T_*$ , when they have the same singularities as  $U$ .

#### 4. Final remarks

We believe that these examples are helpful in understanding the scattering transform for the two-dimensional Schrödinger equation at zero energy and the regularity of the Cauchy problem for the Novikov–Veselov equation. In connection with this, we must mention some results very recently obtained in this direction:

The inverse scattering scheme for the Novikov–Veselov equation was realized for initial data of the conductivity type [14]. We recall that a conductivity-type potential is just a potential of the form  $u = \Delta\psi/\psi$  with  $\psi > 0$  everywhere and  $\psi = 1$  near infinity and that operators with conductivity-type potentials have no eigenvalues in the space  $L_2$ .

In [8], the first example was found of a potential with a nontrivial exceptional set, i.e., a set consisting of  $\lambda \in \mathbb{C}$  for which Faddeev eigenfunction (2) is not uniquely defined. This was established for a point potential  $u(x) = \varepsilon\delta(x)$ . This study was recently continued in [15], where the first examples of similar smooth potentials were found and a dichotomy-type conjecture on the regularity of the Cauchy problem for the Novikov–Veselov equation was also introduced.

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