Generalized Hermitian Matrix Model: Statistical Physics Insights

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Background and Motivation

- Hermitian matrix models are a significant topic in mathematical physics, with important applications in the study of Gromov-Witten theory, mirror symmetry, integrable systems theory, and more.

Background and Motivation

- Gromov-Witten Theory

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves of genus g $(g \geq 0)$ with n $(n \geq 1)$ marked points $x_1, x_2, \ldots, x_n$. Let $\mathscr{L}_i$ be the tautological line bundle on $\overline{\mathcal{M}}_{g,n}$ corresponding to the i-th marked point. The intersection number on $\overline{\mathcal{M}}_{g,n}$ is given by:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} c_1 \left(\mathscr{L}_1\right)^{d_1} \wedge \cdots \wedge c_1 \left(\mathscr{L}_n\right)^{d_n}, \tag{1}$$

where $\{\tau_i\}_{i=1,\ldots,n}$ are the tautological classes.
The generating series of these intersection numbers, denoted as

$$F := \sum_{g=0}^{\infty} \sum_{n_0, n_1, n_2 \cdots} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle_g \frac{t_0^{n_0}}{n_0!} \frac{t_1^{n_1}}{n_1!} \frac{t_2^{n_2}}{n_2!} \cdots,$$

is called the free energy of two-dimensional topological gravity, and

$$Z := e^F$$

is called the partition function of two-dimensional topological gravity.

Background and Motivation

- Witten Conjecture/Kontsevich Theorem

The free energy F of two-dimensional topological gravity is determined by two conditions:
(1) $U = \frac{\partial^2 F}{\partial t_0^2}$ satisfies the KdV hierarchy:

$$\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1}\left(U, U_{t_0}, \ldots\right), \tag{2}$$

where R_{n+1} is the Gelfand-Dickey polynomial.
(2) F satisfies the string equation:

$$\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{n=0}^{\infty} t_{n+1} \frac{\partial F}{\partial t_n}. \tag{3}$$

- Dijkgraaf, Verlinde and Verlinde: Virasoro constraints for 2-d topological gravity

  The partition function of two-dimensional gravity, denoted as Z, uniquely determined by an infinite set of equations

$$L_m Z = 0, \quad m \geq -1, \tag{4}$$

  where,

$$L_{-1} = \frac{t_0^2}{2} + \sum_{n \geqslant 1} (t_n - \delta_{n,1}) \frac{\partial}{\partial t_{n-1}}, \tag{5}$$

$$L_0 = \frac{1}{8} + \sum_{n \geqslant 0} (2n+1)(t_n - \delta_{n,1}) \frac{\partial}{\partial t_n}, \tag{6}$$

$$L_m = \sum_{n \geqslant 0} \frac{(2n+2m+1)!!}{(2n-1)!!} (t_n - \delta_{n,1}) \frac{\partial}{\partial t_{m+n}} + \frac{1}{2} \sum_{k+l=m-1} (2k+1)!!(2l+1)!! \frac{\partial^2}{\partial t_k \partial t_l}, \quad m \geq 1, \tag{7}$$

  satisfying the commutation relations of the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n}, \quad m, n \geq -1. \tag{8}$$

Background and Motivation

- Grothendieck's Dessins d'Enfants

Grothendieck's theory of dessins d'enfants considers the weighted counting problem of dessins.

- Belyi's Theorem. A smooth complex algebraic curve C is defined over $\bar{\mathbb{Q}}$ if and only if there exists a holomorphic branched cover $f : C \to \mathbb{P}^1$ that is ramified only over $0, 1, \infty$.

- Grothendieck's Correspondence. There is a one-to-one correspondence between the isomorphism classes of Belyi pairs (C, f) and connected bicolored ribbon graphs (called Grothendieck's dessin denfant), given by $f^{-1}([0,1])$.

- Let (C, f) be a Belyi pair of genus g and degree d, where $\Gamma = f^{-1}[0,1]$ is the corresponding dessin. Denote $k = |f^{-1}(0)|$, $l = |f^{-1}(1)|$, and $m = |f^{-1}(\infty)|$. Then by Riemann-Hurwitz's formula:

$$2g - 2 = d - (k + l + m).$$

Grothendieck's Dessins d'Enfants

- The poles of f are labeled by an ordered sequence of positive integers $\mu = (\mu_1, ..., \mu_m)$, and $d = \sum_{i=1}^{m} \mu_i$. The type of the dessin Γ is denoted by (k, l, μ).
- The set of all dessins of type (k, l, μ) is denoted as $\mathcal{D}_{k,l;\mu}$.
- The weighted count of all dessins of type (k, l, μ) is defined as

$$N_{k,l}(\mu) = N_{k,l}(\mu_1, \ldots, \mu_m) = \sum_{\Gamma \in \mathcal{D}_{k,l,\mu}} \frac{1}{|\mathrm{Aut}\Gamma|}, \tag{9}$$

- The free energy of this theory is defined as the generating function for the count of all connected bicolor ribbon graphs:

$$F(s, u, v, p_1, p_2, \ldots) = \sum_{k,l,m \geq 1} \frac{1}{m!} \sum_{\mu \in \mathbb{Z}_+^m} N_{k,l}(\mu) s^d u^k v^l p_{\mu_1} \ldots p_{\mu_m}. \tag{10}$$

Grothendieck's Dessins d'Enfants

- Zograf proved that the partition function $Z^D = e^{F(s,u,v,p_1,p_2,\dots)}$ of Grothendieck's dessins d'enfants is a τ-function of the KP hierarchy with three parameters.

- Kazarian and Zograf provided the Virasoro constraints satisfied by Z^D:

$$L_n Z^D = 0, \tag{11}$$

$$\begin{aligned} L_n = & -\frac{n+1}{s}\frac{\partial}{\partial p_{n+1}} + (u+v)n\frac{\partial}{\partial p_n} + \sum_{j=1}^{\infty} p_j(n+j)\frac{\partial}{\partial p_{n+j}} \\ & + \lambda^2 \sum_{i+j=n} ij\frac{\partial^2}{\partial p_i \partial p_j} + \delta_{n,0}uv\lambda^{-2}, \quad n \geq 0. \end{aligned} \tag{12}$$

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{13}$$

More models related to integrable systems and Virasoro constraints

- Brezin-Gross-Witten Model: The Virasoro constraints for this model were derived by Gross and Newman. Mironov, Morozov, and Semenoff demonstrated that its partition function provides the τ-function of the KdV hierarchy. They also provided a description of this model as a generalized Kontsevich model.

- Modified Gaussian Hermitian Matrix Model with Even Coupling Constants: Dubrovin, Liu, Yang, and Zhang introduced a transformation when proving Dubrovin and Yang's proposed Hodge-GUE conjecture. This led to a modified version of the Gaussian Hermitian Matrix Model with Even Coupling Constants, and the Virasoro constraints for the partition function of this version were obtained.

- Zhou calculated the fermionic representation of the partition functions for the Brezin-Gross-Witten Model, the Grothendieck's Dessins d'Enfants theory and Modified Gaussian Hermitian Matrix Model with Even Coupling Constants, establishing duality between them and the Gaussian Hermitian 1-matrix.

A quick introduction to matrix model: Gaussian Unitary Ensemble (GUE)

Denote by $\mathcal{H}_N$ the space of Hermitian matrices of size N. Define the *Gaussian Unitary Ensemble (GUE) partition function of size N* by

$$Z_N^{\mathsf{GUE}}(\mathbf{t}) = \frac{\int_{\mathcal{H}_N} e^{-\mathrm{tr}(M^2)+\mathrm{tr}V(M;\mathbf{t})}\mathrm{d}M}{\int_{\mathcal{H}_N} e^{-\mathrm{tr}(M^2)}\mathrm{d}M}, \tag{14}$$

where $\mathbf{t} = (t_1, t_2, ...)$, $V(y;\mathbf{t})$ is a formal power series in y of the form

$$V(y;\mathbf{t}) = \sum_{j\geq 1} t_j y^j, \tag{15}$$

and

$$\mathsf{d}M = \prod_{1\leq i\leq n} \mathsf{d}M_{ii} \prod_{1\leq i\leq j\leq n} \mathsf{dRe}M_{ij}\mathsf{dIm}M_{ij}. \tag{16}$$

The free energy of GUE is defined by

$$\mathcal{F}^{\mathrm{GUE}}(N,\mathbf{t}) := \log Z_N^{\mathrm{GUE}}(\mathbf{t}) \tag{17}$$

Gaussian Unitary Ensemble can be considered as a formal QFT:

- The basis of the space of observables is given by $\{\mathrm{tr}(M^m)\}_{m\geq 1}$.

- The correlators are given by

$$\langle \prod_{j=1}^{\ell} \mathrm{tr}\, M^{k_j} \rangle_{\mathsf{GUE}} := \frac{\int_{\mathcal{H}_N} \left(\prod_{j=1}^{\ell} \mathrm{tr}\, M^{k_j} \right) \exp^{-\mathrm{tr}(M^2)} \, \mathrm{d}M}{\int_{\mathcal{H}_N} \exp^{-\mathrm{tr}(M^2)} \, \mathrm{d}M} = \frac{\partial^l}{\partial_{t_{k_1}} \ldots \partial_{t_{k_l}}} Z_N^{\mathsf{GUE}}(\mathbf{t})|_{t_i=0} \tag{18}$$

- For the case $N = 1$,

$$Z^{\mathsf{GUE}}_{N=1}(\mathbf{t}) = \frac{\int_{\mathbb{R}} e^{-x^2+\sum_{i=1}^{\infty} t_i x^i} \mathrm{d}x}{\int_{\mathbb{R}} e^{-x^2} \mathrm{d}x}, \quad \langle \mathrm{tr}\, M^j \rangle_{\mathsf{GUE}} = \frac{\int_{\mathbb{R}} x^j e^{-x^2} \mathrm{d}x}{\int_{\mathbb{R}} e^{-x^2} \mathrm{d}x} \tag{19}$$

  $\langle \mathrm{tr}\, M^j \rangle_{\mathsf{GUE}}$ is exactly the jth-moment m_j of the Gaussian distribution.

- For the case $N > 1$, we have the famous formula to reduce the integral on the space of Hermitian matrices to the space of their eigenvalues:

$$Z^{\mathsf{GUE}}_{N}(\mathbf{t}) = \frac{\int_{\mathbb{R}^N} \prod_{i=1}^{N} e^{-\lambda_i^2} d\lambda_i \exp\left(\sum_{i=1}^{N} \sum_{k=1}^{\infty} t_k \lambda_i^k\right) \prod_{i>j=1}^{N} (\lambda_i - \lambda_j)^2}{\int_{\mathbb{R}^N} \prod_{i=1}^{N} e^{-\lambda_i^2} d\lambda_i \prod_{i>j=1}^{N} (\lambda_i - \lambda_j)^2}, \tag{20}$$

$$\langle \mathrm{tr}\, M^j \rangle_{\mathsf{GUE}} = \frac{\text{polymomial of } \{m_k\}_{k\geq 0}}{\text{polymomial' of } \{m_k\}_{k\geq 0}} \tag{21}$$

- Our aim is to study Hermitian matrix models in the broadest sense, exploring their general properties and interconnections. Hence, we formulate the following definition.

Definition: Generalized Hermitian Matrix Model

The **generalized Hermitian matrix model** partition function is defined as

$$Z_N^w(\mathbf{t}) := \frac{1}{C_N^w N!} \int_{\mathbb{R}^N} \prod_{i=1}^{N} w(\lambda_i) d\lambda_i \exp\left(\sum_{i=1}^{N} \sum_{k=1}^{\infty} t_k \lambda_i^k \right) \prod_{i>j=1}^{N} (\lambda_i - \lambda_j)^2, \tag{22}$$

where $\mathbf{t} = (t_1, t_2, ..., t_k, ...)$, and the function $w(y) : \mathbb{R} \to \mathbb{R}$ is called the **weight function**, satisfying the condition: for any $n \geq 0$, the moments of the probability distribution with this function as the density function

$$m_n := \frac{\int_{\mathbb{R}} y^n w(y) dy}{\int_{\mathbb{R}} w(y) dy} \tag{23}$$

are all finite.

$$C_N^w := \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{i=1}^{N} w(\lambda_i) d\lambda_i \prod_{i>j=1}^{N} (\lambda_i - \lambda_j)^2 \tag{24}$$

is the normalization constant, ensuring $Z_N^w(0) = 1$.

Classical Examples

- Gaussian Unitary Ensemble (GUE)
$$w(y) = e^{-\frac{y^2}{2}}. \tag{25}$$
- Jacobi Unitary Ensemble (JUE)
$$w(y) = y^\alpha (1-y)^\beta \chi_{(0,1)}(y). \tag{26}$$
- Laguerre Unitary Ensemble (LUE)
$$w(y) = y^\alpha e^{-y} \chi_{(0,\infty)}(y). \tag{27}$$

II: Moment Problems

The moment problem in probability theory can be stated as follows: Given a sequence $\{m_n\}_{n\geq 0}$, does there exist a positive definite Borel measure $\mu(x)$ on $\mathbb{R}$ such that its moments are given by this sequence, i.e.,

$$m_n = \int_{-\infty}^{\infty} x^n d\mu(x). \tag{28}$$

There are three classical types of moment problems in the literature:

- Hamburger Moment Problem: Discusses the case where the support of μ is the entire $\mathbb{R}$.
- Stieltjes Moment Problem: Discusses the case where the support of μ is $[0, \infty)$.
- Hausdorff Moment Problem: Discusses the case where the support of μ is a finite interval, without loss of generality, assumed to be $[0, 1]$.

Solution to the Moment Problem

The conditions for the existence of a solution to the moment problem can be described using the properties of Hankel matrices. A Hankel matrix is formed by arranging the sequence $\{m_n\}$ as follows, creating an infinite-order symmetric matrix:

$$H = \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_k & \\ m_1 & m_2 & m_3 & \cdots & m_{k+1} & \\ m_2 & m_3 & m_4 & \cdots & m_{k+2} & \\ \vdots & \vdots & \vdots & & & \\ m_k & m_{k+1} & m_{k+2} & \cdots & m_{2k} & \\ & & & & & \ddots \end{pmatrix} \quad (29)$$

Let

$$\Delta_n = \det(m_{i+j})_{0\le i,j\le n}, \quad \Delta_n^{(1)} = \det(m_{i+j+1})_{0\le i,j\le n}. \quad (30)$$

- The Hamburger moment problem has a solution if and only if the Hankel matrix H is positive semi-definite, i.e.,

$$\Delta_n \geq 0, \forall n \geq 0.$$

- The Stieltjes moment problem has a solution if and only if

$$\Delta_n \geq 0, \Delta_n^{(1)} \geq 0, \forall n \geq 0.$$

- The conditions for the solution to the Hausdorff moment problem require a description using differences between $\{m_n\}$. Let Δ be the difference operator

$$(\Delta m)_n = m_{n+1} - m_n,$$

then the Hausdorff moment problem has a solution if and only if

$$(-1)^k (\Delta^k m)_n \geq 0, \forall n, k \geq 0.$$

Boas's Theorem

Boas's Theorem

For any given sequence of real numbers $\{m_n\}$, there exists a bounded variation function $\mu(x)$ such that

$$\int_{-\infty}^{\infty} x^n d\mu(x) = m_n, \quad n = 0, 1, 2, \dots \tag{31}$$

III: Statistical Physics and the Interpolating Statistical Space $\mathcal{S}$

- In statistical physics, the quantum states of different ensembles of indistinguishable particles at thermal equilibrium are described by the statistical model characterized by the **single-particle partition function**:

$$z(X) = \sum_{n \geq 0} W_n X^n. \tag{32}$$

  The expansion of its Nth power,

$$z(X)^N = \sum_{k \geq 0} W_k(N) X^k, \tag{33}$$

  has coefficients $W_k(N)$ representing the number of quantum states with N indistinguishable particles occupying k states,called the **occupation number**.

- Building upon the single-particle partition function, we define the **average particle number**

$$w(X) = X \frac{\partial}{\partial X} \log z(X), \tag{34}$$

  and refer to $w(X)$ as the cluster series for this statistical model. The coefficients w_n in its expansion with respect to X are called cluster coefficients.

Basic Models: Bose-Einstein, Fermi-Dirac, and Boltzmann Statistics

Bose-Einstein (BE), Fermi-Dirac (FD), and Boltzmann (B) statistics are the three fundamental models in statistical physics. Their single-particle partition functions are given by:

$$z^{\mathsf{BE}} = \frac{1}{1-X}, \tag{35}$$

$$z^{\mathsf{FD}} = 1 + X, \tag{36}$$

$$z^{\mathsf{B}} = e^{X}. \tag{37}$$

The corresponding occupation numbers are:

$$W_k^{\mathsf{BE}}(N) = \frac{\prod_{i=0}^{k-1}(N+i)}{k!}, \tag{38}$$

$$W_k^{\mathsf{FD}}(N) = \frac{\prod_{i=0}^{k-1}(N-i)}{k!}, \tag{39}$$

$$W_k^{\mathsf{B}}(N) = \frac{N^k}{k!}, \tag{40}$$

The occupation numbers satisfy the following property with respect to particle numbers:

$$W_k(N_1 + N_2) = \sum_{i=0}^{k} W_i(N_1) W_{k-i}(N_2), \tag{41}$$

referred to as **deformed binomial coefficients**.

- Based on the single-particle partition function, we can define the average particle number as:

$$w(X) := X \frac{\partial}{\partial X} \log z(X), \tag{42}$$

where $w(X)$ is called the **cluster series**.

- The cluster series for Bose-Einstein, Fermi-Dirac, and Boltzmann statistics are, respectively:

$$w^{\mathsf{BE}}(X) = \frac{X}{1-X}, \tag{43}$$

$$w^{\mathsf{FD}}(X) = \frac{X}{1+X}, \tag{44}$$

$$w^{\mathsf{B}}(X) = X. \tag{45}$$

- The cluster series of a statistical model always has the following form:

$$w(X) = X + \sum_{n=2}^{\infty} w_n X^n. \tag{46}$$

Interpolation models for Bosons and Fermions Statistics

- Physicists have introduced various interpolation models for Bosons and Fermions statistics.
- Example: the **Acharya-Swamy statistics**. It is given by interpolating the occupation numbers as follows:
$$W_k(N;\alpha) = \frac{\prod_{i=0}^{k-1}(N+i\alpha)}{k!}, \tag{47}$$
When $\alpha = -1$, it yields $W_k^{\mathsf{FD}}(N)$; when $\alpha = 0$, it yields $W_k^{\mathsf{MB}}(N)$; when $\alpha = 1$, it yields $W_k^{\mathsf{B}}(N)$.

- This interpolation can be understood as follows: in the counting process, for the Fermi-Dirac model, as the number of particles increases by 1, the number of available single-particle states decreases by 1; in the Bose-Einstein case, it increases by 1. This interpolation allows the number of available single-particle states to change by any real number α (in physical models, some fraction) as the number of particles increases by 1.

- Alternatively, expressing this interpolation with the single-particle partition function gives:

$$z(X,\alpha) = \frac{1}{(1-\alpha X)^{1/\alpha}}, \tag{48}$$

or in terms of the cluster series:

$$w(X,\alpha) = \frac{X}{1-\alpha X}. \tag{49}$$

	Single-Particle Partition Function	$W_k(N)$	Cluster Series $w(X)$
Boltzmann Statistics	e^X	$\frac{N^k}{k!}$	X
Bose-Einstein Statistics	$\frac{1}{1-X}$	$\frac{\prod_{i=0}^{k-1}(N+i)}{k!}$	$\frac{1}{1-X} = \sum_{n=1}^{\infty} X^n$
Fermi-Dirac Statistics	$1+X$	$\frac{\prod_{i=0}^{k-1}(N-i)}{k!}$	$\frac{1}{1+X} = \sum_{n=1}^{\infty}(-1)^{n-1}X^n$
Acharya-Swamy Statistics	$\frac{1}{(1-\alpha X)^{1/\alpha}}$	$\frac{\prod_{i=0}^{k-1}(N+i\alpha)}{k!}$	$\frac{X}{1-\alpha X}$

Space of Interpolating Statistics

- Jian Zhou introduced the concept of the **Space $\mathcal{S}$ of the Interpolating Statistics** :

$$\mathcal{S} := \{w(X) = X + \sum_{n\geq} w_n X^n\}, \tag{50}$$

- For any point in $\mathcal{S}$, expand its single-particle partition function $z(X)$ as follows:

$$e^{x \log z(t)} = \sum_{k=0}^{\infty} \frac{\gamma_k(x)}{k!} t^k. \tag{51}$$

  This defines a sequence of functions $\gamma_n(x)$ with properties:

$$\gamma_1 = x, \quad \gamma_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} \gamma_j(x)\gamma_{n-j}(y). \tag{52}$$

  Functions of this type $\gamma_n(x)$ are termed "binomial-type."
- **These $\gamma_n(x)$ will serve as the moments of some probability distributions that we use to build generalized matrix models.**

$\mathcal{S}$: Group Structure, Involution Transformation, and Generalized Bose-Fermi Duality

- For $w_1(X) = X + \sum_{n\geq} w_n^1 X^n$ and $w_2(Y) = Y + \sum_{n\geq} w_n^2 Y^n$, consider the composition:

$$\begin{aligned} w_1(w_2) =& Y + \sum_{n\geq} w_n^2 Y^n + \sum_{n\geq} w_n^1 (Y + \sum_{m\geq} w_m^2 Y^m)^n \\ =& Y + \sum_{n\geq 2} w_n^3 Y^n =: w_3(Y) \end{aligned} \tag{53}$$

  Define $\circ : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, $w_1 \circ w_2 = w_3$. It can be verified that $(\mathcal{S}, \circ)$ forms a group with the identity element $w^B(X) = X$.

- Define $\sigma : \mathcal{S} \to \mathcal{S}$, $w \mapsto w^{-1}$, satisfying $\sigma^2 = \text{id}$. σ has a unique fixed point $w(X) = X$, corresponding to Boltzmann statistics. Moreover, $\sigma(w^{BE}) = w^{FD}$ and $\sigma(w^{FD}) = w^{BE}$. This is referred to as the **Generalized Bose-Fermi Duality**.

Moment Problem Solving: Boltzmann Statistics

The single-particle partition function is given by

$$z(t) = e^t,$$

Computing the expansion of $e^{x \log z(t)}$ yields

$$m_n(x) = x^n.$$

The Hankel matrix is

$$\begin{pmatrix} 1 & x & x^2 & \cdots \\ x & x^2 & x^3 & \cdots \\ x^2 & x^3 & x^4 & \cdots \\ \cdots & \cdots & \cdots & \end{pmatrix}$$

In this case, $\Delta_0 = 1$, $\Delta_n = 0$ for all $n > 0$, and $x \in \mathbb{R}$. The corresponding probability distribution has a density function of $\delta(y - x)$.

Moment Problem Solving: Bose-Einstein Statistics

The single-particle partition function is given by

$$z^{\mathsf{BE}}(X) = \frac{1}{1-X}, \tag{54}$$

Computing the expansion of $e^{x \log z(t)}$ yields

$$m_0^{\mathsf{BE}} = 1, \quad m_n^{\mathsf{BE}} = \prod_{k=0}^{n-1} (x+k) = x^{(n)}. \tag{55}$$

We checked that

$$\Delta_n^{\mathsf{BE}} = \prod_{m=1}^{n} m! \prod_{k=0}^{m-1} (x+k)^{n-k} = \prod_{k=0}^{n-1} (k+1)! x^{(k+1)}, \tag{56}$$

When $x > 0$, we have $\Delta_n^{\mathsf{BE}} > 0$ for all $n \geq 0$. The corresponding probability measure on $\mathbb{R}$, denoted as $\mu^{\mathsf{BE}} = w(y)dy$, is given by

$$w(y) = \frac{1}{\Gamma(x)} y^{x-1} e^{-y} \chi_{(0,\infty)}, \quad x > 0. \tag{57}$$

Examples that we find good

	MGF	Moments m_n	Cluster Series $w(X)$
Boltzmann	e^{xt}	x^n	X
Bose-Einstein	$e^{-x\log(1-t)}$	$\prod_{k=0}^{n-1}(x+k) =: x^{(n)}$	$\sum_{n=1}^{\infty} X^n$
Fermi-Dirac	$e^{x\log(1+t)}$	$\prod_{k=0}^{n-1}(x-k) =: (x)_n$	$\sum_{n=1}^{\infty}(-1)^{n-1}X^n$
Exponential Polynomial	$e^{x(e^t-1)}$	$\sum_{k=0}^{n} S(n,k)x^k$	$\sum_{n=0}^{\infty} \frac{X^{n+1}}{n!}$
Bessel Model	$e^{\frac{x}{2A}(1-\sqrt{1-4At})}$	$xA^{n-1}n^{(n-1)}{}_1F_1(1-n;2-2n;\frac{x}{A})$	$\sum_{n=0}^{\infty}\binom{2n}{n}A^nX^{n+1}$

*Here $S(n,k)$ is the Stirling numbers of the second kind.

(Continued)

	MGF	PDF $w(y)$	Distribution
Boltzmann	e^{xt}	$\delta(y-x)$	Dirac δ
Bose-Einstein	$e^{-x\log(1-t)}$	$\frac{1}{\Gamma(x)}y^{x-1}e^{-y}\chi_{(0,\infty)}, x>0$	Gamma
Fermi-Dirac	$e^{x\log(1+t)}$	$\frac{1}{\Gamma(-x)}(-y)^{-x-1}e^{y}\chi_{(-\infty,0)}(y), x<0$	negative Gamma
Exponential Polynomial	$e^{x(e^t-1)}$	$\sum_{k=0}^{\infty}\frac{x^k}{k!}e^{-x}\delta(y-k)$	Poisson
Bessel Model	$e^{\frac{x}{2A}(1-\sqrt{1-4At})}$	$\frac{x}{\sqrt{4A\pi y^3}}e^{-\frac{(y-x)^2}{4Ay}}\chi_{(0,\infty)}(y), x>0$	Inverse Gaussian

Summary of the Stage

- We formulate an extended definition of the Hermitian matrix model.
- We establish a map from the infinite-dimensional space $\mathcal{S}$ of the Interpolating Statistics to the infinite-dimensional space of formal QFT formulated by the generalized Hermitian Matrix Model, given by
$$\{\text{cluster coefficients}\} \mapsto \{\text{the moments series}\}$$
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Integrability of Generalized Hermitian Matrix Models

Theorem(Y.,Zhou)

The partition function of the generalized Hermitian matrix model, when multiplied by a factor independent of t, is the Dubrovin-Zhang type τ-function of the Toda lattice hierarchy.

Theorem(Y.,Zhou)

The partition function of the generalized Hermitian matrix model is the τ-function of the KP hierarchy.

Orthogonal Polynomial Method

Let $P_n(\lambda)$ be a family of monic polynomials of degree n, satisfying the orthogonal relation

$$\int_{\mathbb{R}} w(\lambda) d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) =: h_n \delta_{nm}, \tag{58}$$

where $V(\lambda) = \sum_{k=1}^{\infty} t_k \lambda^k$. Consider

$$\lambda P_n(\lambda) = \sum_{m=0}^{\infty} \gamma_{nm} P_m(\lambda). \tag{59}$$

Using the orthogonal relation, we can calculate that λ is a 3-diagonal matrix given by

$$\gamma = \begin{pmatrix} \gamma_{0,0} & 1 & & & \\ \gamma_{1,0} & \gamma_{1,1} & 1 & & \\ & \gamma_{2,1} & \gamma_{2,2} & 1 & \\ & & \cdots & \cdots & \cdots \end{pmatrix}, \tag{60}$$

satisfying the equation

$$\frac{\partial}{\partial(-t_q)} \gamma = \left[\gamma_q^+, \gamma\right], \quad q \geq 1. \tag{61}$$

Here, γ_q^+ represents the upper triangular part of γ_q.

Toda Lattice Hierarchy

Consider the following linear shift operator

$$L := \Lambda + v_n + w_n \Lambda^{-1}, \tag{62}$$

where

$$\Lambda : f_n \mapsto f_{n+1} \tag{63}$$

is the shift operator, and v_n, w_n depend on time $\mathbf{s} = (s_0, s_1, s_2, ...)$. Then, the Toda lattice hierarchy can be given by the following Lax representation

$$\frac{\partial L}{\partial s_j} = \left[\left(L^{j+1} \right)_+ , L \right], \quad j \geq 0, \tag{64}$$

where, for a difference operator of the form $P = \sum_{i \in \mathbb{Z}} P_i \Lambda^i$, P_+ is defined as $P_+ := \sum_{i \geq 0} P_i \Lambda^i$. Letting $t_i = -s_{i-1}$ in equation (21), we can see that $v_n = \gamma_{n,n}(\mathbf{s})$, $w_n = \gamma_{n,n-1}(\mathbf{s})$, provides a set of solutions for the Toda lattice hierarchy (24).

Lemma (Dubrovin, Yang)

For any solution $v_n = v_n(\mathbf{s}), w_n = w_n(\mathbf{s})$ of the Toda lattice hierarchy, there exists a function $\tau(\mathbf{s})$ satisfying the following equations:

$$\sum_{i,j\geq 0} \frac{1}{\lambda^{i+2}\mu^{j+2}} \frac{\partial^2 \log \tau_n(\mathbf{s})}{\partial s_i \partial s_j} \tag{65}$$

$$= \frac{\operatorname{tr} R_n(\mathbf{s},\lambda) R_n(\mathbf{s},\mu) - 1}{(\lambda-\mu)^2} \tag{66}$$

$$\sum_{i\geq 0} \frac{1}{\lambda^{i+2}} \frac{\partial}{\partial s_i} \log \frac{\tau_{n+1}(\mathbf{s})}{\tau_n(\mathbf{s})} \tag{67}$$

$$= [R_{n+1}(\mathbf{s},\lambda)]_{21} \tag{68}$$

$$\frac{\tau_{n+1}(\mathbf{s})\tau_{n-1}(\mathbf{s})}{\tau_n^2(\mathbf{s})} = w_n. \tag{69}$$

Here, $R_n(\lambda)$ is the pre-resolvent matrix of the difference operator L. The function $\tau_n(\mathbf{s})$ is uniquely determined, up to a factor of the form:

$$\tau_n(\mathbf{s}) \mapsto e^{a_0 + a_1 n + \sum_{j\geq 0} b_j s_j} \tau_n(\mathbf{s}), \tag{70}$$

Weighted Hermitian Matrix Models as τ-Functions of Toda Lattice Hierarchy

By verifying the three conditions in the lemma, we obtain:

Theorem(Y.,Zhou)

For

$$\tilde{Z}_n^w(\mathbf{t}) := C_n^w Z_n^w(\mathbf{t})$$

under the variable substitution

$$t_i = s_{i-1}, \quad i > 0,$$

then $\tilde{Z}_n^w$ is a Dubrovin-Zhang type τ-function of the Toda lattice hierarchy with solutions $v_n = \gamma_{n,n}(\mathbf{s})$ and $w_n = \gamma_{n,n-1}(\mathbf{s})$.

Generalized Hermitian Matrix Models as τ-Functions of KP Hierarchy

Orthogonal polynomials $P_n(\lambda)$ can be expressed as determinants:

$$P_n(\lambda) = \det_{n\times n}[\lambda - \gamma]. \tag{71}$$

Here, $\det_{n\times n}[A]$ means taking the nth order leading principal minor of the infinite-order matrix A, and the 0th order principal minor is defined as 1.
Define $P_m^*(\lambda)$ as

$$P_m^*(\lambda) := \lambda^{-m} \det_{m\times m}\left[\left(1 - \frac{\gamma}{\lambda}\right)^{-1}\right], \quad m \geq 0 \tag{72}$$

Define

$$\hat{W}_n^*(\lambda, t) := \lambda^{-n} P_n(\lambda, t), \quad \hat{W}_n\left(\lambda, t'\right) := \lambda^n P_n^*(\lambda, t) \tag{73}$$

The following bilinear relation can be proven:

$$\oint d\lambda \hat{W}_n^*(\lambda, t) \hat{W}_n\left(\lambda, t'\right) e^{V\left(\lambda, t'-t\right)} = 0 \tag{74}$$

Generalized Hermitian Matrix Models as τ-Functions of KP Hierarchy

From the bilinear relation, we know that $\hat{W}_n(\lambda,t)e^{V(\lambda,t)}$ is a wave function of the KP hierarchy, and $\hat{W}_n^*(\lambda,t)e^{-V(\lambda,t)}$ is its adjoint wave function. Thus, there exists a τ-function for KP, denoted as $\tau_n(t)$, such that

$$\hat{W}_n^*(\lambda,t) = \frac{\tau_n\left(t_\ell + \ell^{-1}\lambda^{-\ell}\right)}{\tau_n\left(t_\ell\right)}, \quad \hat{W}_n(\lambda,t) = \frac{\tau_n\left(t_\ell - \ell^{-1}\lambda^{-\ell}\right)}{\tau_n\left(t_\ell\right)}. \tag{75}$$

This leads to

$$P_n(\lambda,t) = \lambda^n \frac{\tau_n\left(t_\ell + \ell^{-1}\lambda^{-\ell}\right)}{\tau_n\left(t_\ell\right)}, \quad P_n^*(\lambda,t) = \lambda^{-n} \frac{\tau_n\left(t_\ell - \ell^{-1}\lambda^{-\ell}\right)}{\tau_n\left(t_\ell\right)} \tag{76}$$

Finally, by expressing $\tilde{Z}_n^w(\mathbf{t}) = C_n^w Z_n^w(\mathbf{t}) = h_0 h_1 \cdots h_{n-1}$ in the form of a Wronskian determinant,

$$\tilde{Z}_n^w(t) = W\left(h_0, h_0', \cdots, h_0^{(n-1)}\right), \tag{77}$$

it can be proven that $\tilde{Z}_n^w$ exactly corresponds to the τ_n satisfying equation (35). In other words, Z_n^w is the τ-function of the KP hierarchy.

Virasoro Constraints of Generalized Hermitian Matrix Models

Theorem(Y.,Zhou)

Assume $\frac{w'(y)}{w(y)} \in \mathbb{R}[y]y^{-r}$, where $r \in \mathbb{Z}_{\geq 0}$. Let $\left(\frac{w'(y)}{w(y)}\right)_{[y^k]}$ denote the coefficient of y^k in $\frac{w'(y)}{w(y)}$. Suppose $\left(\frac{w'(y)}{w(y)}\right)_{[y^{-r}]} \neq 0$ and for any $q \geq r-1$, the condition $\left(w(y)y^{q+1}\right)|_{\partial D} = 0$ holds, where D is the support of $w(y)$. Then, the partition function $Z_N^w(\mathbf{t})$ satisfies

$$L_n Z_N^w(\mathbf{t}) = 0, \quad n \geq r-1 \tag{78}$$

where L_n is given by the following operators

$$L_{-1} = \sum_{k=2}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}} + N t_1 + \sum_k \left(\frac{w'(y)}{w(y)}\right)_{[y^k]} \frac{\partial}{\partial t_k}, \tag{79}$$

$$L_0 = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + N^2 + \sum_k \left(\frac{w'(y)}{w(y)}\right)_{[y^k]} \frac{\partial}{\partial t_{k+1}}, \tag{80}$$

$$L_m = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + 2N \frac{\partial}{\partial t_m} + \sum_{l=1}^{m-1} \frac{\partial}{\partial t_l} \frac{\partial}{\partial t_{m-l}} + \sum_k \left(\frac{w'(y)}{w(y)}\right)_{[y^k]} \frac{\partial}{\partial t_{k+m+1}}, \quad m \geq 1, \tag{81}$$

Generalized Hermitian Matrix Model's Virasoro Constraints (Continued)

Virasoro Constraints of Generalized Hermitian Matrix Models

Satisfying Virasoro commutation relations

$$[L_n, L_m] = (n-m)L_{n+m}. \tag{82}$$

Remarks:

- When $\frac{w'(y)}{w(y)}$ contains negative powers of y, suppose the lowest order term with a nonzero coefficient is y^{-r} (where $r > 0$), then the expressions for $\{L_n, n \geq -1\}$ will be undefined for the first r terms. In other words, the effective operators are $\{L_n, n \geq r-1\}$.
- Condition: For any $q \geq r-1$,
$$\left(w(y)y^{q+1}\right)|_{\partial D} = 0 \tag{83}$$
is imposed to ensure that the boundary terms from integration by parts in the derivation are zero.
- It is understood that if $\frac{\partial}{\partial t_0}$ appears, its meaning is "multiply by N."

Probability Distribution of the Acharya-Swamy Model

The probability distribution associated with the Acharya-Swamy model is given by

$$\mu^{\mathsf{AS}}(\alpha, x) = w(\alpha, x; y)dy \tag{84}$$

where

$$w(\alpha, x; y) = \frac{1}{\alpha\Gamma(\frac{x}{\alpha})}\left(\frac{y}{\alpha}\right)^{\frac{x}{\alpha}-1} e^{-\frac{y}{\alpha}}\chi_{(0,\infty)}(y), \quad x > 0, \alpha > 0; \tag{85}$$

$$w(\alpha, x; y) = \frac{-1}{\alpha\Gamma(\frac{x}{\alpha})}\left(\frac{y}{\alpha}\right)^{\frac{x}{\alpha}-1} e^{-\frac{y}{\alpha}}\chi_{(-\infty,0)}(y), \quad x < 0, \alpha < 0; \tag{86}$$

$$w(\alpha, x; y) = \delta(y - x), \quad x \in \mathbb{R}, \quad \alpha = 0. \tag{87}$$

Further Discussion on the Acharya-Swamy Model

The probability measure associated with Acharya-Swamy depends on two parameters, (α, x).

- x is the first moment of the considered probability distribution.
- α is introduced in the Acharya-Swamy model as an interpolation parameter between the Bose-Einstein and Fermi-Dirac models.

The cluster series for Acharya-Swamy is given by

$$w_\alpha^{\mathsf{AS}}(X) = \frac{X}{1-\alpha X},$$

As an interpolation, it satisfies:

$$w_0^{\mathsf{AS}} = w^{\mathsf{B}}, \quad w_1^{\mathsf{AS}} = w^{\mathsf{BE}}, \quad w_{-1}^{\mathsf{AS}} = w^{\mathsf{FD}}. \tag{88}$$

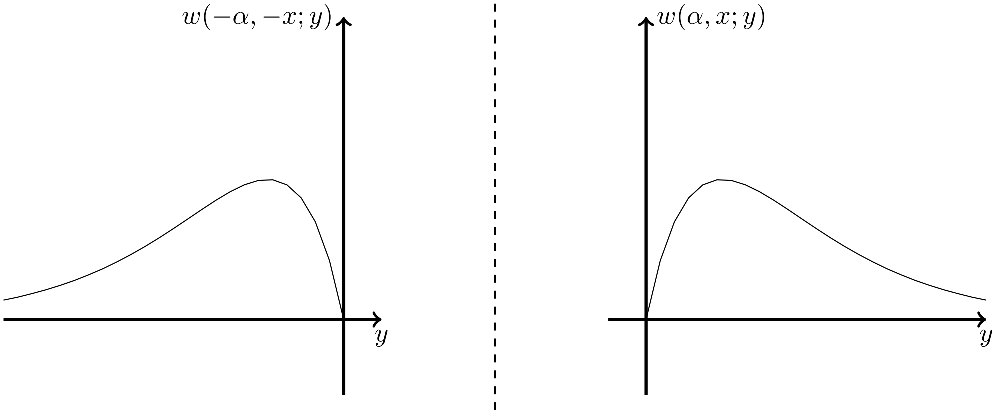
It is also a one-parameter subgroup of $\mathcal{S}$:

$$\begin{aligned} \left(w_\alpha^{\mathsf{AS}}\right)^{-1} &= w_{-\alpha}^{\mathsf{AS}}, \quad \forall \alpha \in \mathbb{R} \\ w_{\alpha_1}^{\mathsf{AS}} \circ w_{\alpha_2}^{\mathsf{AS}} &= w_{\alpha_1+\alpha_2}^{\mathsf{AS}}, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R} \end{aligned} \tag{89}$$

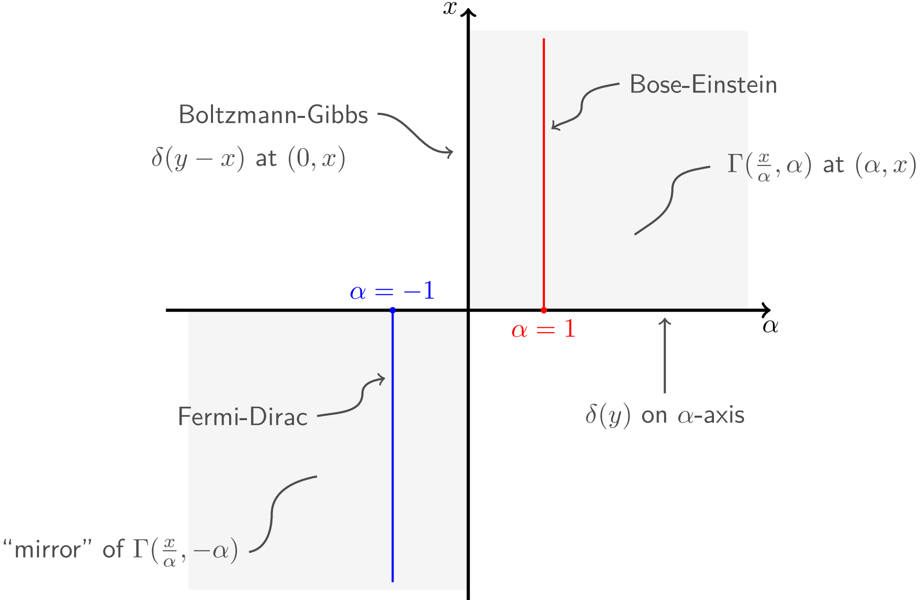
"Mirror Symmetry"

On the (α, x) parameter plane, the probability distributions located at (α, x) and $(-\alpha, -x)$ are symmetric about the y-axis:

$$w(\alpha, x; y) = w(-\alpha, -x; -y). \tag{90}$$



Phase Transition Near the Boundary of Parameter Space



A Subfamily of Acharya-Swamy: Laguerre Unitary Ensemble

Let's set the Acharya-Swamy parameters to

$$\alpha = 1, \quad x = a + 1,$$

Then $Z_N^{\mathsf{AS}}(t)$ gives the partition function for the Laguerre Unitary Ensemble (LUE):

Definition (LUE)

$$Z_N^{\mathsf{LUE}}(\mathbf{t}, a) = \frac{1}{C_N^{LUE} N!} \int_{\mathbb{R}^N_{\geq 0}} \prod_{i=1}^{N} \lambda_i^a e^{-\lambda_i} d\lambda_i \exp\left(\sum_{i=1}^{N} \sum_{k=1}^{\infty} t_k \lambda_i^k \right) \prod_{i>j=1}^{N} (\lambda_i - \lambda_j)^2. \tag{91}$$

In other words, LUE corresponds to the Bose-Einstein model.

Virasoro Constraints for Acharya-Swamy

Calculating $\frac{w'(y)}{w(y)}$ with $w(y) = \frac{1}{\alpha\Gamma(\frac{x}{\alpha})}\left(\frac{y}{\alpha}\right)^{\frac{x}{\alpha}-1} e^{-\frac{y}{\alpha}}\chi_{(0,\infty)}(y), x > 0, \alpha > 0$, we obtain

$$\frac{w'(y)}{w(y)} = (\frac{x}{\alpha} - 1)y^{-1} - \frac{1}{\alpha}, \tag{92}$$

The Virasoro constraints for the Acharya-Swamy model's partition function $Z_N^{AS}(t)$ are given by $L_m^{AS} Z_N^{AS}(t) = 0, m \geq 0$, where L_m^{AS} is defined as

$$L_0^{AS} = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_k} - \frac{1}{\alpha}\frac{\partial}{\partial t_1} + N(N + \frac{x}{\alpha} - 1), \tag{93}$$

$$L_1^{AS} = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+1}} + (2N + \frac{x}{\alpha} - 1)\frac{\partial}{\partial t_1} - \frac{1}{\alpha}\frac{\partial}{\partial t_2}, \tag{94}$$

$$L_m^{AS} = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+m}} + (2N + \frac{x}{\alpha} - 1)\frac{\partial}{\partial t_m} - \frac{1}{\alpha}\frac{\partial}{\partial t_{m+1}} + \sum_{l=1}^{m-1}\frac{\partial}{\partial t_l}\frac{\partial}{\partial t_{m-l}}, m \geq 2. \tag{95}$$

Acharya-Swamy and Grothendieck's Dessins d'Enfants

Grothendieck's theory of dessins d'enfants considers the weighted counting problem of dessins. Let (C, f) be a Belyi pair of genus g and degree d, where $\Gamma = f^{-1}[0,1]$ is the corresponding dessin. Denote $k = |f^{-1}(0)|$, $l = |f^{-1}(1)|$, and $m = |f^{-1}(\infty)|$, such that $2g-2 = d-(k+l+m)$. The poles of f are labeled by an ordered sequence of positive integers $\mu = (\mu_1, ..., \mu_m)$, and $d = \sum_{i=1}^{m} \mu_i$. The type of the dessin Γ is denoted by (k, l, μ).
The set of all dessins of type (k, l, μ) is denoted as $\mathcal{D}_{k,l;\mu}$. The weighted count of all dessins of type (k, l, μ) is defined as

$$N_{k,l}(\mu) = N_{k,l}(\mu_1, \ldots, \mu_m) = \sum_{\Gamma \in \mathcal{D}_{k,l,\mu}} \frac{1}{|\mathrm{Aut}_b \Gamma|}, \tag{96}$$

The free energy of this theory is defined as the generating function for the count of all connected bicolor ribbon graphs:

$$F(s, u, v, p_1, p_2, \ldots) = \sum_{k,l,m \geq 1} \frac{1}{m!} \sum_{\mu \in \mathbb{Z}_+^m} N_{k,l}(\mu) s^d u^k v^l p_{\mu_1} \ldots p_{\mu_m}. \tag{97}$$

Acharya-Swamy and Grothendieck's Dessins d'Enfants

Zograf proved that the partition function $Z^D = e^{F(s,u,v,p_1,p_2,\ldots)}$ of Grothendieck's dessins d'enfants is a τ-function of the KP hierarchy with three parameters. Kazarian and Zograf provided the Virasoro constraints satisfied by Z^D:

$$\begin{aligned} L_n =& -\frac{n+1}{s}\frac{\partial}{\partial p_{n+1}} + (u+v)n\frac{\partial}{\partial p_n} + \sum_{j=1}^{\infty} p_j(n+j)\frac{\partial}{\partial p_{n+j}} \\ &+ \lambda^2 \sum_{i+j=n} ij\frac{\partial^2}{\partial p_i \partial p_j} + \delta_{n,0}uv\lambda^{-2}, \quad n \geq 0. \end{aligned} \tag{98}$$

For the Acharya-Swamy model, consider the variable substitutions:

$$t_k = \frac{p_k}{k}, \quad \alpha = s, \quad N = u, \quad N + \frac{x}{\alpha} - 1 = v.$$

This allows us to identify the Virasoro constraints of Acharya-Swamy with those of Grothendieck's dessins d'enfants.

Acharya-Swamy and Modified GUE with Even Coupling Constants

The partition function of the Hermitian matrix model with even coupling constants is defined as

$$Z^N_{\text{even}} = \frac{\int_{\mathbb{M}N} dM \exp\left(\frac{1}{gs}\operatorname{tr}\left(-\frac{1}{2}M^2 + \sum_{n=1}^{\infty}\frac{g_{2n}}{2n}M^{2n}\right)\right)}{\int_{\mathbb{M}N} dM \exp\left(-\frac{1}{2gs}\operatorname{tr} M^2\right)}, \tag{99}$$

Dubrovin, Liu Siqi, Yang Di, and Zhang Youjin introduced the modified partition function Z^E, defined by the equation

$$\log Z^N_{\text{even}} = \left(\Lambda^{1/2} + \Lambda^{-1/2}\right)\log Z^E, \tag{100}$$

where $\Lambda = e^{g_s \partial_t}$, $t = Ng_s$ is the t'Hooft coupling constant, and it was proved that Z^E is uniquely determined by the following Virasoro constraints:

$$L^E_n Z^E = 0, n \geq 0, \tag{101}$$

Acharya-Swamy and Modified GUE with Even Coupling Constants

Here, $\{L_n^E\}_{n\geq 0}$ are given by

$$\begin{aligned} L_0^E =& \sum_{k\geqslant 1} k\left(g_{2k}-\delta_{k,1}\right)\frac{\partial}{\partial g_{2k}}+\frac{t^2}{4g_s^2}-\frac{1}{16}, \\ L_n^E =& 2nt\frac{\partial}{\partial g_{2n}}+\sum_{k\geqslant 1}(n+k)\left(g_{2k}-\delta_{k,1}\right)\frac{\partial}{\partial g_{2n+2k}}+g_s^2\sum_{k=1}^{n-1}2k(2n-2k)\frac{\partial^2}{\partial g_{2k}\partial g_{2n-2k}} n\geq 1. \end{aligned} \tag{102}$$

Letting $t_k=\frac{g_{2k}}{2kg_s}$, $\alpha=1$, $x=2t+1$, and $N=-t\pm\sqrt{t^2+\frac{t^2}{4g_s^2}-\frac{1}{16}}$, we can equate the Virasoro constraints of Acharya-Swamy with the modified Gaussian matrix model with even coupling constants.

Classification of known cases via Virasoro Constraints

Virasoro Constraints for Generalized Hermitian Matrix Models

Suppose $\frac{w'(y)}{w(y)} \in \mathbb{R}[y]y^{-r}$, where $r \in \mathbb{Z}_{\geq 0}$. Let $\left(\frac{w'(y)}{w(y)}\right)_{[y^k]}$ be the coefficient of y^k in $\frac{w'(y)}{w(y)}$. Assume that $\left(\frac{w'(y)}{w(y)}\right)_{[y^{-r}]} \neq 0$ and for any $q \geq r-1$, the condition $\left(w(\lambda)\lambda^{q+1}\right)|_{\partial D} = 0$ holds, where D is the support of $w(y)$. Then, the partition function $Z_N^w(\mathbf{t})$ satisfies $L_n Z_N^w(\mathbf{t}) = 0, n \geq r-1$, where L_n is given by the following operators:

$$L_{-1} = \sum_{k=2}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}} + N t_1 + \sum_k \left(\frac{w'(y)}{w(y)}\right)_{[y^k]} \frac{\partial}{\partial t_k}, \tag{103}$$

$$L_0 = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + N^2 + \sum_k \left(\frac{w'(y)}{w(y)}\right)_{[y^k]} \frac{\partial}{\partial t_{k+1}}, \tag{104}$$

$$L_m = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + 2N \frac{\partial}{\partial t_m} + \sum_{l=1}^{m-1} \frac{\partial}{\partial t_l} \frac{\partial}{\partial t_{m-l}} + \sum_k \left(\frac{w'(y)}{w(y)}\right)_{[y^k]} \frac{\partial}{\partial t_{k+m+1}}, m \geq 1, \tag{105}$$

$r = 0$: GUE

If $\frac{w'(y)}{w(y)} = \sum_{k\geq 0} a_k y^k \in \mathbb{R}[y]$. then $w(y) = C\exp\left(\sum_{k\geq 1} \frac{a_{k-1}}{k} y^k\right)$, where C is some constant. In this case, let

$$\tilde{t}_n = t_n + \frac{a_{n-1}}{n}, n \geq 0, \tag{106}$$

we obtain

$$L_{-1} = \sum_{k=2}^{\infty} k\tilde{t}_k \frac{\partial}{\partial t_{k-1}} + N\tilde{t}_1, \tag{107}$$

$$L_0 = \sum_{k=1}^{\infty} k\tilde{t}_k \frac{\partial}{\partial t_k} + N^2, \tag{108}$$

$$L_1 = \sum_{k=1}^{\infty} k\tilde{t}_k \frac{\partial}{\partial t_{k+1}} + 2N\frac{\partial}{\partial t_1} \tag{109}$$

$$L_m = \sum_{k=1}^{\infty} k\tilde{t}_k \frac{\partial}{\partial t_{k+m}} + 2N\frac{\partial}{\partial t_m} + \sum_{l=1}^{m-1} \frac{\partial}{\partial t_l}\frac{\partial}{\partial t_{m-l}}, m \geq 2, \tag{110}$$

$r = 1$: Acharya-Swamy/Grothendieck's dessins d'enfants/LUE, JUE

If $a_{-1} \neq 0$, then $\frac{\partial}{\partial t_{-1}}$ will appear in the formula of L_{-1} above and be dropped. After the same shift $\tilde{t}_n = t_n + \frac{a_{n-1}}{n}, n \geq 0$, the Virasoro constraints are

$$L_0 = \sum_{k=1}^{\infty} k\tilde{t}_k \frac{\partial}{\partial t_k} + N(N + a_{-1}), \tag{111}$$

$$L_1 = \sum_{k=1}^{\infty} k\tilde{t}_k \frac{\partial}{\partial t_{k+1}} + (2N + a_{-1})\frac{\partial}{\partial t_1} \tag{112}$$

$$L_m = \sum_{k=1}^{\infty} k\tilde{t}_k \frac{\partial}{\partial t_{k+m}} + (2N + a_{-1})\frac{\partial}{\partial t_m} + \sum_{l=1}^{m-1} \frac{\partial}{\partial t_l}\frac{\partial}{\partial t_{m-l}}, m \geq 2. \tag{113}$$

Definition (LUE)

$$Z_N^{\mathsf{LUE}}(\mathbf{t},a)=\frac{1}{C_N^{LUE}N!}\int_{\mathbb{R}^N_{\geq 0}}\prod_{i=1}^{N}\lambda_i^a e^{-\lambda_i}d\lambda_i\exp\left(\sum_{i=1}^{N}\sum_{k=1}^{\infty}t_k\lambda_i^k\right)\prod_{i>j=1}^{N}(\lambda_i-\lambda_j)^2. \tag{114}$$

Definition (JUE)

$$Z_N^{\mathsf{JUE}}(\mathbf{t},a,b)=\frac{1}{C_N^{\mathsf{JUE}}N!}\int_{[0,1]^N}\prod_{i=1}^{N}\lambda_i^a(1-\lambda_i)^b d\lambda_i\exp\left(\sum_{i=1}^{N}\sum_{k=1}^{\infty}t_k\lambda_i^k\right)\prod_{i>j=1}^{N}(\lambda_i-\lambda_j)^2. \tag{115}$$

$$w^{\mathsf{LUE}}(y)=y^a e^{-x}\chi([0,\infty)),\qquad \frac{w^{\mathsf{LUE}}(y)'}{w^{\mathsf{LUE}}(y)}=\frac{a}{y}-1$$

$$w^{\mathsf{JUE}}(y)=y^a(1-y)^b\chi([0,1]),\qquad \frac{w^{\mathsf{JUE}}(y)'}{w^{\mathsf{JUE}}(y)}=\frac{a}{y}-\frac{b}{1-y}=\frac{a}{y}-b(1+y+y^2+\cdots)$$

$r = 2$: Bessel Model

The probability density function corresponding to the Bessel model is given by
$w_A(y) = \frac{x}{\sqrt{4A\pi y^3}} \exp\left(-\frac{(y-x)^2}{4Ay}\right) \chi_{(0,\infty)}(y)$, where $x > 0$.

$$\frac{w'_A(y)}{w_A(y)} = \frac{x^2}{4Ay^2} - \frac{3}{2y} - \frac{1}{4A},$$

The Virasoro constraints for Z_N^{Bes} are given by the following operators

$$L_1^{\mathrm{Bes}} = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+1}} + 2N \frac{\partial}{\partial t_1} + \frac{x^2}{4A} N - \frac{3}{2} \frac{\partial}{\partial t_1} - \frac{1}{4A} \frac{\partial}{\partial t_2} \tag{116}$$

$$L_m^{\mathrm{Bes}} = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + 2N \frac{\partial}{\partial t_m} + \sum_{l=1}^{m-1} \frac{\partial}{\partial t_l} \frac{\partial}{\partial t_{m-l}} + \frac{x^2}{4A} \frac{\partial}{\partial t_{m-1}} - \frac{3}{2} \frac{\partial}{\partial t_m} - \frac{1}{4A} \frac{\partial}{\partial t_{m+1}}, \quad m \geq 2. \tag{117}$$

Theorem(Y.,Zhou)

For the partition function of the Bessel model type, the Bogoliubov transformation is given by $Z^{\text{Bes}} = \exp\left(\sum_{m,n\geq 0} A_{m,n}\psi_{-m-\frac{1}{2}}\psi^*_{-n-\frac{1}{2}}\right)|0>$, where the coefficients $A_{m,n}$ satisfy the following recursive relations

$$\sum_{j=0}^{1} A_{1-j,j} = -\frac{2N+a_{-1}}{a_0}A_{0,0} - \frac{Na_{-2}}{a_0} \tag{118}$$

$$\begin{aligned}\sum_{j=0}^{n} A_{n-j,j} = &-\frac{2N+a_{-1}}{a_0}\sum_{j=0}^{n-1} A_{n-1-j,j} - \frac{a_{-2}}{a_0}\sum_{j=0}^{n-2} A_{n-2-j,j} \\ &-\frac{1}{a_0}\sum_{j=0}^{n-1}(n-1-2j)\cdot A_{n-1-j,j}, n \geq 2.\end{aligned} \tag{119}$$

$$\begin{aligned}A_{m,n+2} = &A_{m+2,n} - A_{m,1}A_{0,n} - A_{m,0}A_{1,n} + \frac{2N+a_{-1}}{a_0}\left(-A_{m,0}A_{0,n} + A_{m+1,n} - A_{m,n+1}\right) \\ &+\frac{1}{a_0}\left((m+1)A_{m+1,n} + (n+1)A_{m,n+1}\right).\end{aligned} \tag{120}$$

Thank you very much for your attentions!