The Markov and Zariski topologies of a free group

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(Elementary) algebraic sets

Definition (Markov, 1945)

Let G be a group. A subset S of G is called:

(i) *elementary algebraic* if

$$S = \{x \in G : g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \ldots g_n x^{\varepsilon_n} = 1\}$$

for some $n \in \mathbb{N}$, $g_1, g_2, \ldots, g_n \in G$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{-1, 1\}$. (Here 1 is the identity of G.)

(ii) *algebraic* if S is an intersection of finite unions of elementary algebraic sets.

If G is abelian, elementary algebraic sets in G have much simpler form $\{x \in G : mx = g\}$ for some $m \in \mathbb{Z}$ and $g \in G$.

Zariski (verbal) topology of a group

The family of all algebraic subsets of G coincides with the family of closed sets of a unique topology $\mathfrak{Z}_G$ on G called the *verbal topology* (Bryant, 1977) or *Zariski topology* (Baumslag, Myasnikov, Remeslennikov, 1999) of G in connection to the notion in *algebraic geometry*.

Clearly, $\mathfrak{Z}_G$ is the coarsest (=smallest) topology on G in which all elementary algebraic sets are closed.

For every $g \in G$, we have $\{x \in G : g^{-1}x = 1\} = \{g\}$, so the singleton $\{g\}$ is an elementary algebraic subset of G; in particular, $\{g\}$ is $\mathfrak{Z}_G$-closed, and so the Zariski topology $\mathfrak{Z}_G$ is T_1.

Markov topologies of a group

Definition (Markov, 1944)

Let G be a group. A subset S of G is called *unconditionally closed* in G if S is closed in each Hausdorff group topology on G.

The family of all unconditionally closed subsets of G coincides with the family of closed sets of a unique T_1 topology $\mathfrak{M}_G$ on G. This topology was named as the *Markov* topology of G by Dikranjan and Shakhmatov in 2007. Clearly,

$$\mathfrak{M}_G = \bigcap \{\mathcal{T} : \mathcal{T} \text{ is a Hausdorff group topology on } G\}.$$

$\mathfrak{M}_G$ coincides with the infimum (in the lattice of of topologies of G) of all Hausdorff group topologies on G.

Theorem (Markov, 1945)

Let G be an arbitrary group. Then every algebraic subset of G is unconditionally closed; that is, $\mathfrak{Z}_G \subseteq \mathfrak{M}_G$ holds.

Proof: Let $S = \{x \in G : g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \ldots g_n x^{\varepsilon_n} = 1\}$ be an elementary algebraic subset of G. Let $\mathcal{T}$ be an arbitrary Hausdorff group topology on G. Since $\mathcal{T}$ is a group topology on G, the map $f \colon G \to G$ defined by

$$f(x) = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \ldots g_n x^{\varepsilon_n} \quad (x \in G)$$

is $\mathcal{T}$-continuous. Since $\{1\}$ is a $\mathcal{T}$-closed subset of G, $f^{-1}(1) = S$ is a $\mathcal{T}$-closed set. Since this holds for *every* Hausdorff group topology $\mathcal{T}$ on G, S is unconditionally closed.

Markov and Zariski topologies

- $\mathfrak{Z}_G$ is the coarsest (=smallest) topology on G in which all elementary algebraic sets are closed.
- $\mathfrak{M}_G = \bigcap\{\mathcal{T} : \mathcal{T} \text{ is a } T_2 \text{ group topology on } G\}$.
- $\mathfrak{Z}_G \subseteq \mathfrak{M}_G$.

Question (Markov, 1945)

Is every unconditionally closed subset of a group algebraic? Equivalently, does the equality $\mathfrak{Z}_G = \mathfrak{M}_G$ hold for every group G?

If the group G is infinite, the topology $\mathfrak{M}_G$ is discrete if and only if the only Hausdorff group topology on G is discrete. Such groups are called *non-topologizable*.

Question (Markov, 1945)

Does there exist an infinite non-topologizable group? Namely, an infinite group G for which $\mathfrak{M}_G$ is discrete?

Shelah's consistent answer to Markov's 2nd problem

Question (Markov, 1945)

Does there exist an infinite non-topologizable group? Namely, a group G for which $\mathfrak{M}_G$ is discrete?

Theorem (Shelah, late 1970s – revised 1980)

Under CH there exists a non-topologizable Kurosh group (= with every proper subgroup at most countable) $\mathcal{S}$ of cardinality ω_1.

Definition

A group G is said to be *n-Shelah* for $n \in \mathbb{N}$ if $X^n = G$ for each subset $X \subseteq G$ of cardinality $|G|$. G is said to be *Shelah* if it is n-Shelah for some $n \in \mathbb{N}$.

Shelah's group $\mathcal{S}$ is 6640-Shelah.

Hesse's example

Hesse, a young mathematician working in the theory of κ-boundedness (or κ-*gebunden*) as inspired by the works of Podewski on algebraic subsets. He obtained the following result via a modification of Shelah's example:

Example (Hesse (1979), PhD thesis in German)

Let λ, κ be cardinals satisfying $cf(\lambda) > \kappa = cf(\kappa)$. Under ZFC, there exists a group G such that:

- $|G| = \lambda$,
- Every Hausdorff semigroup topology on G is discrete (in particular, $\mathfrak{M}_G$ is discrete, and
- G is not κ-gebunden.

Hesse's main focus was obtaining the third condition above, however, he was not aware that this condition also implies that $\mathfrak{Z}_G$ is non-discrete. Namely, $\mathfrak{Z}_G \neq \mathfrak{M}_G$ for Hesse's group.

Hesse's example

Question (Markov, 1945)

Is every unconditionally closed subset of a group algebraic?
Equivalently, does the equality $\mathfrak{Z}_G = \mathfrak{M}_G$ hold for every group G?

Question (Markov, 1945)

Does there exist an infinite non-topologizable group? Namely, a group G for which $\mathfrak{M}_G$ is discrete?

Example (Hesse (1979), PhD thesis in German)

There exists an infinite group G for which the topology $\mathfrak{M}_G$ is discrete and $\mathfrak{Z}_G$ is non-discrete. Therefore, $\mathfrak{M}_G \neq \mathfrak{Z}_G$.

This result remains published only in Hesse's original thesis, and remains only in German language. No known reprints of this thesis exist, beyond a partial fragment published by Shelah in a re-print of his original paper.

Beyond Hesse's example

Theorem (Shelah, late 1970s – revised 1980)

Under CH there exists a non-topologizable 6640-Shelah group $\mathcal{S}$ of cardinality ω_1.

Theorem (Ol'shanskij, late 1980)

There exists a countable non-topologizable group (with discrete Zariski topology).

Theorem (Sipacheva, 2007)

Under CH, there exists a non-topologizable 10000-Shelah group $\mathcal{M}$ of cardinality ω_1 for which $\mathfrak{Z}_G \neq \mathfrak{M}_G$ holds.

Theorem (Poór and Rinot, preprint 2023)

In ZFC, there exists a non-topologizable 10120-Shelah group of cardinality λ^+ (for any cardinal λ) for which $\mathfrak{Z}_G \neq \mathfrak{M}_G$ holds.

Groups with coinciding Markov and Zariski topology

Question (Markov, 1945)

Is every unconditionally closed subset of a group algebraic?
Equivalently, does the equality $\mathfrak{Z}_G = \mathfrak{M}_G$ hold for every group G?

A continuation of Markov's original problem is the following natural question:

Definition

$\mathcal{MZ}$ denotes the class of groups for which the Markov and Zariski topologies coincide.

Question

Can one describe groups contained in the class $\mathcal{MZ}$?

Markov topologies of a group

Definition

A topological group G is said to be *precompact* if it is isomorphic to a subgroup of some compact group K.

Dikranjan and Shakhmatov defined the precompact Markov topology $\mathfrak{P}_G$ of G as the infimum of all Hausdorff precompact group topologies on G. Namely,

$$\mathfrak{P}_G = \bigcap \{\mathcal{T} : \mathcal{T} \text{ is a precompact Hausdorff group topology on } G\}.$$

Therefore, for each group G the following inclusion holds:

$$\mathfrak{M}_G \subseteq \mathfrak{P}_G.$$

Definition

$\mathcal{PM}$ denotes the class of groups for which the Markov and precompact Markov topologies coincide.

The case of discrete Zariski topology

Question

Can one describe groups contained in the class $\mathcal{MZ}$ (alt. $\mathcal{PM}$)?

Recall that

$$\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G.$$

If $\mathfrak{Z}_G$ is discrete for a group G, then $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$. The theory of $\mathfrak{Z}$-discrete groups connects back to the theory of Podewski. Major advances in this case were achieved by Olshanski'j:

Example (Olshanski'j, late 1980)

There exists a countable group G with discrete Zariski topology.

Further work in this direction was achieved by Klyachko-Trofimov, Trofimov, and Morris-Obraztsov.

Positive solutions to Markov's problem

Question

Can one describe groups contained in the class $\mathcal{MZ}$ (alt. $\mathcal{PM}$)?

Theorem (Markov, 1946)

All countable groups belong to $\mathcal{MZ}$.

Markov attributed to Perel'man a solution for Abelian groups.

Theorem (Sipacheva, 2006)

All Abelian groups belong to $\mathcal{MZ}$.

Theorem (Dikranjan-Shakhmatov, 2006)

All Abelian groups G belong to $\mathcal{MZ}$ and $\mathcal{PM}$, i.e,
$\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$.

Markov's problem for non-abelian groups

Question

Can one describe groups contained in the class $\mathcal{MZ}$ (alt. $\mathcal{PM}$)?

Dikranjan and Shakhmatov asked in 2007 whether permutation groups and free groups belong to the class $\mathcal{MZ}$.

Theorem (Banakh, Guran and Protasov, 2012)

All permutation groups belong to the class $\mathcal{MZ}$.

By a result of Gaughan, the Markov topology of a permutation group coincides with the topology of pointwise convergence (which is not precompact).

Corollary (conseq. of Gaughan, 1966)

If G is an infinite permutation group, then $\mathfrak{M}_G$ is non-discrete but $\mathfrak{P}_G$ is discrete; in particular, $\mathfrak{M}_G \neq \mathfrak{P}_G$ and so $G \notin \mathcal{PM}$.

The Zariski reflection principle

A complete characterization of the groups belonging to the class $\mathcal{MZ}$ was achieved by Dikranjan and Shakhmatov in 2008.

For a set X we shall denote by $[X]^{\leq\omega}$ the set of all (at most) countable subsets of X.

Definition

A family $\mathcal{C} \subseteq [X]^{\leq\omega}$ is called:

(i) *closed in* $[X]^{\leq\omega}$ if $\bigcup\{C_n : n \in \mathbb{N}\} \in \mathcal{C}$ whenever $\{C_n : n \in \mathbb{N}\} \subseteq \mathcal{C}$ and $C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n \subseteq C_{n+1} \subseteq \ldots,$

(ii) *unbounded in* $[X]^{\leq\omega}$ if for every $Y \in [X]^{\leq\omega}$ there exists $C \in \mathcal{C}$ with $Y \subseteq C$,

(iii) a *club in* $[X]^{\leq\omega}$ if $\mathcal{C}$ is both closed and unbounded in $[X]^{\leq\omega}$.

Intuitively, clubs in $[X]^{\leq\omega}$ are "big" subsets of $[X]^{\leq\omega}$.

“Standard” club in $[G]^{\leq\omega}$

Let G be a group. We denote by $\mathcal{S}(G)$ the family of all (at most) countable subgroups of G; that is,

$$\mathcal{S}(G) = \{H \in [G]^{\leq\omega} : H \text{ is a subgroup of } G\}.$$

Example

For every group G, $\mathcal{S}(G)$ is a club in $[G]^{\leq\omega}$.

Proof: $\mathcal{S}(G)$ is closed in $[G]^{\leq\omega}$. Assume that $\{H_n : n \in \mathbb{N}\} \subseteq \mathcal{S}(G)$ and $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n \subseteq H_{n+1} \subseteq \ldots$. Then each H_n is an at most countable subgroup of G, and so $\bigcup\{H_n : n \in \mathbb{N}\}$ is an at most countable subgroup of G. Therefore, $\bigcup\{H_n : n \in \mathbb{N}\} \in \mathcal{S}(G)$.

$\mathcal{S}(G)$ is unbounded in $[G]^{\leq\omega}$. Let $Y \in [G]^{\leq\omega}$ be arbitrary. Then Y is at most countable subset of G, so the subgroup H of G algebraically generated by Y is at most countable as well. Now $Y \subseteq H \in \mathcal{S}(G)$.

Markov Reflection principle

For a subset A of a group G,

$$\mathcal{R}_A(G) = \{H \in \mathcal{S}(G) : A \cap H \text{ is } \mathfrak{Z}_H\text{-closed } (=\mathfrak{M}_H\text{-closed}) \text{ in } H\}.$$

Theorem (Zariski Reflection P. (Dikranjan-Shakhmatov, 2008))

Let G be a group. Then for every $\mathfrak{Z}_G$-closed subset A of G, the family $\mathcal{R}_A(G)$ contains a club $\mathcal{C}$ in $[G]^{\leq\omega}$.

Theorem (Markov Reflection P. (Dikranjan-Shakhmatov, 2008))

For every group G, the following conditions are equivalent:

(i) *for every $\mathfrak{M}_G$-closed subset A of G, the family $\mathcal{R}_A(G)$ contains a club $\mathcal{C}$ in $[G]^{\leq\omega}$.*

(ii) $\mathfrak{Z}_G = \mathfrak{M}_G$ *holds* ($= G \in \mathcal{MZ}$);

In order to apply Reflection Principle successfully, we need "good" sufficient conditions for a family $\mathcal{R}_A(G)$ to contain a club.

Definition (Dikranjan-Shakhmatov, 2008)

A subgroup H of G is *Hausdorff embedded in* G if every Hausdorff group topology $\mathcal{T}$ on H can be extended to a Hausdorff group topology $\mathcal{T}'$ on G (i.e, $(H, \mathcal{T})$ becomes a subspace of $(G, \mathcal{T}')$).

Lemma (folklore)

Each subgroup H of an abelian group G is Hausdorff embedded.

For a group G define

$$\mathcal{H}(G) = \{H \in \mathcal{S}(G) : H \text{ is Hausdorff embedded in } G\}.$$

Corollary (Dikranjan-Shakhmatov, 2008)

If G is a group such that $\mathcal{H}(G)$ contains a club $\mathcal{C}$ in $[G]^{\leq\omega}$, then $\mathfrak{Z}_G = \mathfrak{M}_G$ holds ($= G \in \mathcal{MZ}$). In particular, $G \in \mathcal{MZ}$ for every Abelian group G.

Theorem (Shakhmatov-Y.)

Let X be a set and $F(X)$ be the free (non-commutative) group over a set X. Define

$$\mathcal{C} = \{F(Z) : Z \in [X]^{\leq\omega}\}.$$

Then:

(i) *$\mathcal{C}$ is a club in $[F(X)]^{\leq\omega}$, and*

(ii) *$\mathcal{C} \subseteq \mathcal{H}(F(X))$.*

Corollary (Dikranjan-Shakhmatov, 2008)

If G is a group such that $\mathcal{H}(G)$ contains a club $\mathcal{C}$ in $[G]^{\leq\omega}$, then $\mathfrak{Z}_G = \mathfrak{M}_G$ holds (=Markov's problem has a positive answer for G).

We have proved that $\mathcal{H}(F(X))$ contains the club $\mathcal{C} = \{F(Z) : Z \in [X]^{\leq\omega}\}$. By the above corollary $\mathfrak{Z}_{F(X)} = \mathfrak{M}_{F(X)}$, so Markov's problem has a positive answer for $F(X)$. □

Markov's problem for free groups.

We confirm Markov's conjecture for all free (non-commutative) groups.

Theorem (Shakhmatov-Y., 2022)

All free groups belong to $\mathcal{MZ}$.

This result is the first non-Abelian application of the reflection principle of Dikranjan and Shakhmatov.

Description of the Zariski topology of free groups

Let G be a group and $a \in G$ be its element. The set

$$C_G(a) = \{x \in G : xa = ax\} = \{x \in G : xax^{-1}a^{-1} = 1\}$$

is called the *centralizer of a in G*.

The following explicit description of the Zariski topology of free groups is deduced from results of Bryant (1977), Guba (1986), and Chiswell and Remeslennikov (2000):

Theorem (Dikranjan–Toller (2010))

Zariski closed subsets of a free group G are finite unions of singletons or sets of the form $gC_G(a)h$, where $a, g, h \in G$ and $C_G(a)$ is the centralizer of a.

Corollary (Shakhmatov–Y.)

The unconditionally closed subsets of a free group G are finite unions of singletons or sets of the form $gC_G(a)h$, where $a, g, h \in G$ and $C_G(a)$ is the centralizer of a.

Definition

A topological space is *Noetherian* if it satisfies the descending chain condition for closed sets. Kaplansky denotes them as *Z-groups*.

Remark

The following are equivalent for a group G with a Noetherian T_1 topology:

1. *G is Hausdorff,*
2. *G is finite,*
3. *G is a topological group.*

Theorem (Bryant, 1977)

The Markov-Zariski topology of an Abelian group is Noetherian.

Theorem (Dikranjan–Toller, 2012)

The Zariski topology of a free group is Noetherian.

Theorem (Shakhmatov-Y.)

$\mathfrak{M}_G \neq \mathfrak{P}_G$ (i.e, $G \notin \mathcal{PM}$) for every free group G of rank at least 2. Moreover, $\mathfrak{P}_G$ is non-discrete.

The restriction on rank of the free group G in this theorem is essential, as $\mathfrak{M}_{\mathbb{Z}} = \mathfrak{P}_{\mathbb{Z}}$ for the free group $\mathbb{Z}$ of rank 1 (Dikranjan-Shakhmatov, 2010).

The Bohr topology of a group

A topological group is *precompact* if it is a subgroup of some compact group. The finest precompact group topology on a group G is called its *Bohr topology*. We shall use the symbol $\mathfrak{B}_G$ for denoting the Bohr topology of a group G. The $\mathfrak{B}_G$-closure of the identity 1 of G is called the *von Neumann kernel* of G; it is the kernel of a natural homomorphism of G into its Bohr compactification bG.

Theorem (folklore)

For every group G, the following conditions are equivalent:

(i) *The Bohr topology $\mathfrak{B}_G$ of G is Hausdorff;*

(ii) *G admits a precompact Hausdorff group topology;*

(iii) *the discrete group G is maximally almost periodic in the sense of von Neumann.*

All discrete abelian and free groups are maximally almost periodic.

Precompact Markov topology and Bohr topology

Recall that the precompact Markov topology $\mathfrak{P}_G$ of G is the infimum of all Hausdorff precompact group topologies on G. i.e,

$$\mathfrak{P}_G = \bigcap\{\mathcal{T} : \mathcal{T} \text{ is a precompact Hausdorff group topology on } G\}.$$

Lemma (Shakhmatov-Y.)

For an arbitrary group G, the following conditions are equivalent:

(i) *The Bohr topology $\mathfrak{B}_G$ of G is Hausdorff,*

(ii) $\mathfrak{P}_G \subseteq \mathfrak{B}_G$.

Corollary (Shakhmatov-Y.)

For an infinite group G, the following conditions are equivalent:

(i) *G is maximally almost periodic in its discrete topology,*

(ii) *$\mathfrak{P}_G$ is non-discrete.*

Idea of proof

Lemma (Shakhmatov-Y.)

Let H be a subgroup of a group G. Then every $\mathfrak{B}_H$-compact (i.e, Bohr-compact) subset of H is $\mathfrak{P}_G$-closed.

Proof.

Let K be a $\mathfrak{B}_H$-compact subset of H. Let $\mathcal{T}$ be a precompact Hausdorff group topology on G. Then its restriction $\mathcal{T} \restriction_H$ to H is a precompact group topology on H. Since the Bohr topology of H is the finest precompact group topology on H, $\mathcal{T} \restriction_H$ is coarser than $\mathfrak{B}_H$. Since K is a $\mathfrak{B}_H$-compact subset of H, it is also compact in the weaker topology $\mathcal{T} \restriction_H$, and thus also in the topology $\mathcal{T}$ on G. Since $\mathcal{T}$ is Hausdorff, K is $\mathcal{T}$-closed.

We have proved that K is closed in every precompact Hausdorff group topology $\mathcal{T}$ on G. Therefore, K is $\mathfrak{P}_G$-closed by the definition of $\mathfrak{P}_G$. □

Theorem (Shakhmatov-Y.)

Let H be a subgroup of a group G, K be a $\mathfrak{B}_H$-compact subset of H and $\iota_K : (K, \mathfrak{B}_H \restriction_K) \to (K, \mathfrak{P}_G \restriction_K)$ be the identity map of K. Then:

(i) *ι_K is a closed map;*

(ii) *if $\mathfrak{B}_G$ is Hausdorff, then ι_K is a homeomorphism.*

Corollary (Shakhmatov-Y.)

Let G be a group. If $\mathfrak{P}_G$ is non-discrete, then $\mathfrak{B}_G$ and $\mathfrak{P}_G$ induce the same (subspace) topology on every Bohr compact (=$\mathfrak{B}_G$-compact) subset of G.

Corollary (Shakhmatov-Y.)

If G is a non-commutative free group, then $(G, \mathfrak{P}_G)$ contains a countably infinite compact Hausdorff subspace with a single non-isolated point.

$\mathfrak{P}_G \neq \mathfrak{M}_G$ for non Abelian free groups

Corollary (Shakhmatov-Y.)

If G is a non-commutative free group, then $(G, \mathfrak{P}_G)$ contains a countably infinite compact Hausdorff subspace with a single non-isolated point.

Theorem (Shakhmatov-Y.)

$\mathfrak{M}_G \neq \mathfrak{P}_G$ (i.e, $G \notin \mathcal{PM}$) for every free group G of rank at least 2. Moreover, $\mathfrak{P}_G$ is non-discrete.

Proof. Let G be a non-commutative free group. Then the space $(G, \mathfrak{Z}_G)$ is Noetherian; that is, each strictly decreasing sequence of its closed subsets stabilises (Dikranjan-Toller, 2012). Therefore, all Hausdorff subspaces of $(G, \mathfrak{Z}_G)$ are finite (indeed, a topological space is Noetherian if and only if all its subspaces are compact). On the other hand, $(G, \mathfrak{P}_G)$ contains a countably infinite Hausdorff subspace by the previous corollary. Therefore, $\mathfrak{Z}_G \neq \mathfrak{P}_G$.

Open questions about the class $\mathcal{MZ}$

Question (Dikranjan-Shakhmatov, 2008)

Let G be a group having an abelian normal subgroup H such that the quotient G/H is also abelian. Does G belong to the class $\mathcal{MZ}$? This question is open even for all nilpotent groups.

The Zariski topology of the Heisenberg group has been fully described by Dikranjan and Toller in 2012. This would be a natural candidate for attacking the above question. Recall that

$$\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G$$

If one considers a field K of characteristic 0, the corresponding Heisenberg group H_K can be made to be $\mathfrak{P}_G$-discrete and not $\mathfrak{Z}_G$-discrete.

Question (Dikranjan–Toller, 2012)

Does there exist a group G for which $\mathfrak{Z}_G \neq \mathfrak{M}_G \neq \mathfrak{P}_G$?

Open questions about the class $\mathcal{MZ}$

Theorem (essentially Banakh, Guran, Protasov (2012))

Every group is a subgroup of some group in the class $\mathcal{MZ}$.

Theorem (Shakhmatov-Y.)

Every group is a quotient of some group from the class $\mathcal{MZ}$.

Question (Shakhmatov-Y.)

(i) *Can every group be represented as a normal subgroup of a group from the class $\mathcal{MZ}$?*

(ii) *Does every normal subgroup of a group in the class $\mathcal{MZ}$ belong to the class $\mathcal{MZ}$?*

Selected references

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