

# Multipoint formulas in inverse problems

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This talk is devoted to multipoint formulas for finding leading coefficients in asymptotic expansions arising in potential and scattering theories. In particular, we implement different formulas for finding the Fourier transform of potential from the scattering amplitude at several high energies. We show that the aforementioned approach can be used for essential numerical improvements of classical results including the slowly convergent Born-Faddeev formula for inverse scattering at high energies. The approach of multipoint formulas can be also used for recovering the X-ray transform of potential from boundary values of the scattering wave functions at several high energies. Determination of total charge (electric or gravitational) from several exterior measurements is also considered. In addition, we show that the aforementioned multipoint formulas admit an efficient regularization for the case of random noise. Besides, we present our results on phaseless inverse scattering.

# Abstract multipoint formulas

## Asymptotic expansions

Many functions of scattering theory admit asymptotic expansions  $z(s)$ ,  $s \in (r, +\infty)$  of the form with leading coefficient

$$z(s) = a_1 + \frac{a_2}{s} + \dots + \frac{a_N}{s^{N-1}} + \mathcal{O}(s^{-N}) = \sum_{j=1}^N \frac{a_j}{s^{j-1}} + \mathcal{O}(s^{-N}), \quad (1)$$

see Atkinson 1949, Buslaev 1967, Melrose 1995, Yafaev 2003.

In addition, in some cases, the most important information is contained in  $a_1$  (and/or some next leading terms). In talk we present new results on finding  $a_1$  from  $z(s)$ , given at several sufficiently large  $s$ , with applications to inverse scattering at high energies.

# Reconstruction

So, we consider

$$z(s) = a_1 + \frac{a_2}{s} + \dots = \sum_{j=1}^N \frac{a_j}{s^{j-1}} + \mathcal{O}(s^{-N}). \quad (2)$$

Let  $z = z(s)$  be given at  $n$  points  $s + \tau_1 < s + \tau_2 < \dots < s + \tau_n$ .

The simplest reconstruction of  $a_1$ , which is widely used in direct and inverse scattering, gives

$$a_1 = z(s + \tau_n) + \mathcal{O}(s^{-1}), \quad s \rightarrow +\infty.$$

However, the error of reconstruction has a slow decay as  $\mathcal{O}(s^{-1})$ .

# Reconstruction

In turn, according to (Novikov)  $a_1$  can be found with high accuracy from  $z$  given at  $n$  points  $s + \tau_1 < s + \tau_2 < \dots < s + \tau_n$  via formulas

$$\begin{aligned}
 a_1 &= \sum_{j=1}^n y_j(s, \vec{\tau}) z(s + \tau_j) + \mathcal{O}(s^{-n}), \quad s \rightarrow +\infty, \\
 y_j(s, \vec{\tau}) &= (-1)^n \frac{(s + \tau_j)^{n-1}}{\alpha_j(\vec{\tau}) \beta_{n,j}(\vec{\tau})} \sim s^{n-1}, \\
 \alpha_j(\vec{\tau}) &= \prod_{i=1}^{j-1} (\tau_j - \tau_i), \quad \beta_{n,j}(\vec{\tau}) = \prod_{i=j+1}^n (\tau_i - \tau_j).
 \end{aligned} \tag{3}$$

However, we found that these formulas are unstable to noise.

# Unstability

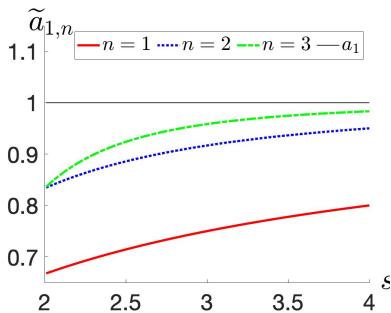
Indeed, if data are given with random noise:

$$z^{\text{noisy}}(s) = z(s) + \varepsilon \mathcal{N}(0, 1)$$

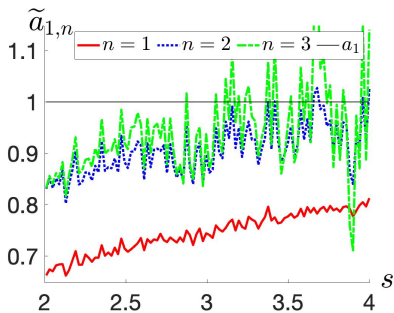
with dispersion  $\varepsilon^2$ , then the dispersion of reconstruction grows rapidly for high  $s$  even for small  $n \geq 2$ :

$$\mathbb{D}(\mathbf{a}_{1,n}) \sim \varepsilon^2 s^{2(n-1)}.$$

# Unstability



(a) Exact data.



(b) Noisy data.

**Figure:** [NSS2023]  $n$ -point reconstructions  $a_{1,n}(s)$  of  $a_1 = 1$  for  $z(s) = s/(s+1)$  with  $\tau_j = j-1$ . (a) Two- and three-point formulas rapidly converge to exact value. (b) Two- and three-point formulas are unstable to noise.

# Regularization

In order to make multipoint formulas appropriate for applications, we propose a regularisation method with a parameter  $r$ . We construct

$$\tilde{a}_{1,n}^r = \sum_{j=1}^n y_j^r(s, \vec{\tau}) z(s_j(s)), \quad (4)$$

where  $y^r = (y_1^r, \dots, y_n^r)$  depends only on  $n$  and  $r$ , and  $r \in [n^{-1/2}, \|(y_{1,n}, \dots, y_{n,n})\|]$ . In particular,  $\|y^r\| = r$ .

We want that the reconstruction  $\tilde{a}_{1,n}^r$  had the following properties:

- the dispersion of reconstruction  $a_{1,n}^r$  from noisy data is bounded by

$$\mathbb{D}(a_{1,n}^r(s, \vec{\tau})) \leq r^2 \varepsilon^2 \quad \text{independently of } s; \quad (5)$$

- for the noiseless function  $z$ , the regularized reconstruction is the best possible under the condition  $\|y^r\| \leq r$ .

# Regularization

Recall that

$$\sum_{j=1}^n y_j z(s + \tau_j) = a_1 + \mathcal{O}(s^{-n}), \quad s \rightarrow +\infty. \quad (6)$$

In fact, the vector  $y = (y_1, \dots, y_n)$  arises as a solution of the system

$$(\gamma_i, y) = \sum_{j=1}^n \frac{y_j}{(s + \tau_j)^{i-1}} = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{for } 1 < i \leq n. \end{cases} \quad (7)$$

Evidently, here  $\gamma_1 = (1, 1, \dots, 1)$ .

# Regularization

Let  $\pi_k$  denote the orthogonal projection of  $\mathbb{R}^n$  on the  $\text{span}(\gamma_1, \dots, \gamma_k)$ , where  $\gamma_j$  are the vectors arising in (7), and  $k = 1, \dots, n$ .

## Lemma

Let  $y$  solve (7). Then:

$$(\gamma_i, \pi_k y) = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{for } 1 < i \leq k, \end{cases} \quad (8)$$

$$\pi_1 y = \frac{\gamma_1}{n}, \quad \pi_n y = y, \quad (9)$$

$$\frac{1}{\sqrt{n}} = \|\pi_1 y\| \leq \|\pi_2 y\| \leq \dots \leq \|\pi_n y\| = \|y\|. \quad (10)$$

## Regularization

We consider a regularisation parameter  $r \in [n^{-1/2}, +\infty)$ , where  $r = +\infty$  corresponds to no regularization, and  $r = n^{-1/2}$  corresponds to the strongest regularisation. Let

$$\mathbb{S}_r^{n-1} := \{x \in \mathbb{R}^n : \|x\| = r\}. \quad (11)$$

Let  $L$  denote the broken line defined by

$$L := \cup_{k=1}^{n-1} [\pi_k y, \pi_{k+1} y], \quad (12)$$

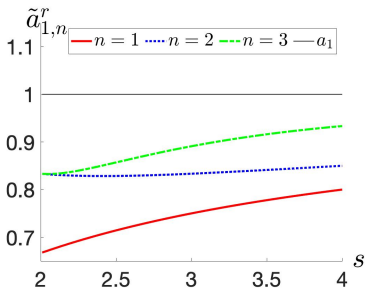
where  $y$  solves (7).

Our regularised  $y^r = (y_1^r, \dots, y_n^r)$  is defined by

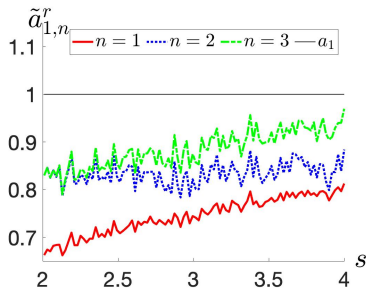
$$y^r := \begin{cases} y, & \text{if } |y| \leq r, \\ L \cap \mathbb{S}_r^{n-1}, & \text{if } |y| > r, \end{cases} \quad (13)$$

where  $y$  solves (7). This definition takes into account formula (10).

# Regularization



(a) Exact data.



(b) Noisy data.

**Figure:** [NSS2023]  $n$ -point regularized reconstructions  $a_{1,n}^r(s)$  of  $a_1 = 1$  for  $z(s) = s/(s+1)$  with  $\tau_j = j-1$ . Regularization parameter  $r = \sqrt{5}$ . (a) Two- and three-point regularized formulas converge to exact value, but not so rapid as in Figure 1(a). (b) Two- and three-point regularized formulas are stable to noise.

# Examples

- Electrical or gravitational field with potential

$$U(x) = \int_D \frac{\rho(x') dx'}{|x - x'|}, \quad x \in \mathbb{R}^3, \quad (14)$$

where  $D$  is a bounded domain in  $\mathbb{R}^3$ . Then  $sU(s\theta)$  admits multipole expansion of the form (1) with total charge leading coefficient

$$a_1 = \int_D \rho(x) dx. \quad (15)$$

# Applications to inverse scattering

We consider the stationary Schrödinger equation describing a non-relativistic quantum mechanical particle at fixed energy  $E$  interacting with a macroscopic object contained in  $D$ ,

$$-\Delta\psi + v\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0.$$

Solutions describing scattering of planar wave

$$\psi^+(x, k) = \underbrace{e^{ikx}}_{\text{planar wave}} + \underbrace{\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} c(d, |k|) \overbrace{f[v](k, |k| \frac{x}{|x|})}^{\text{scattering amplitude}}}_{\text{leading spherical wave}} + \mathcal{O}\left(\frac{1}{|x|^{(d+1)/2}}\right),$$

$|x| \rightarrow +\infty$  uniformly in  $x/|x|$ . The main scattering functions are:

- $f$  - scattering amplitude (or far-field),
- $\psi^+$  - near-field,
- $|f|^2$  - differential scattering cross section,
- $|\psi^+|^2$  - amplitude of near-field.

We consider direct and inverse scattering problems:

- DSP:  $v \rightarrow \psi^+ \rightarrow f$
- ISP:  $\psi^+, f \rightarrow v$

The main purpose of this talk is to present several recent results in this domain. However, let us first recall some well-known results.

## Direct scattering

Direct problem can be solved via Lippmann-Schwinger integral equation

$$\psi^+(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G^+(x - y, k) v(y) \psi^+(y, k) dy, \quad (16)$$

where  $G^+$  is a Green function for  $\Delta + |k|^2$ , and scattering amplitude can be found from

$$f[v](k, l) = (2\pi)^{-d} \int_D e^{-ily} v(y) \psi^+(y, k) dy. \quad (17)$$

# Born approximation

Note that for small  $v$  LS equation takes the form

$$\psi^+(x, k) \approx e^{ikx}, \quad f(k, l) \approx \hat{v}(k - l), \quad (18)$$

$$\hat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipy} v(y) dy. \quad (19)$$

Note that since  $|k| = |l| = \sqrt{E}$ , then  $k - l \in B_{2\sqrt{E}}$ .

Therefore, in Born approximation, we reconstruct  $\hat{v}(p) \approx f(k - l)$ , for  $B_{2\sqrt{E}}$ .

Then, inverting Fourier transform, we approximately reconstruct  $v$ .

## Old general result

Faddeev 1956:

$$f(k, l) = \hat{v}(k - l) + \mathcal{O}(s^{-1}), \quad s \rightarrow +\infty, \quad (20)$$

where  $|k| = |l| = s$ .

For  $d \geq 2$ , at fixed energy  $E = s^2$ , we reconstruct  $\hat{v}$  up to  $\mathcal{O}(E^{-1/2})$ . But this gives no method for more accurate reconstruction, even for smooth  $v$ .

## Examples with far-field

- (Born, Faddeev, Buslaev, Melrose, Yafaev) For  $v \in C_c^\infty(\mathbb{R}^d)$ , scattering amplitude  $f$  has the expansion

$$f(s) = \hat{v} + \sum_{j=2}^N \frac{a_j}{s^{j-1}} + \mathcal{O}(s^{-N}), \quad \text{as } s \rightarrow +\infty, \quad (21)$$

where  $s = \sqrt{E}$  is the energy level.

Therefore, if the scattering amplitude for **several** energy levels is provided, we hope to reconstruct  $\hat{v}$  very precisely.

- Moreover, (21) has its phaseless analogue:

$$|f(s)|^2 = |\hat{v}|^2 + \sum_{j=2}^N \frac{a_j}{s^{j-1}} + \mathcal{O}(s^{-N}), \quad \text{as } s \rightarrow +\infty. \quad (22)$$

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Here

$$\begin{aligned} f(s) &= f(k_s, l_s), & \hat{v} &= \hat{v}(p), & a_j &= a_j(p, \omega), \\ k_s &= p/2 + (E - p^2/4)^{1/2}\omega, & l_s &= -p/2 + (E - p^2/4)^{1/2}\omega, \\ E &= E(s) = s^2, & p &\in \mathbb{R}^d, & p \cdot \omega &= 0, & \omega &\in \mathbb{S}^{d-1}. \end{aligned}$$

## Example with near- to far-field

- Consider the scattering solutions  $\psi^+$  at fixed energy. Then, according to Atkinson-Wilson type expansion:

$$\psi^+(x, k) = e^{ikx} + \frac{e^{i|k|s}}{s^{(d-1)/2}} \left( f_1 + \sum_{j=2}^N \frac{f_j}{s^{j-1}} + \mathcal{O}\left(\frac{1}{s^N}\right) \right), \quad s \rightarrow +\infty, \quad (23)$$

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where  $s = |x|$ . Here

$$f_j = f_j(k, |k|\hat{x}), \quad x = s\hat{x}, \quad f_1(k, l) = c(d, |k|)f(k, l),$$

$$c(d, |k|) = -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \quad \text{for } \sqrt{-2\pi i} = \sqrt{2\pi} e^{-i\pi/4},$$

where  $\hat{x}, k, l \in \mathbb{R}^d$ , and  $|k|^2 = |l|^2 = E$ ,  $|\hat{x}|^2 = 1$ .

## Example with near- to far-field

- Consider the scattering solutions  $a(x, k) = |x|^{(d-1)/2}(|\psi^+(x, k)|^2 - 1)$ .  
Then, for  $d = 3$ ,

$$a(x, k) = \sum_{j=1}^N \frac{e^{i\kappa|x|} f_j + e^{-i\kappa|x|} \bar{f}_j}{|x|^{j-1}} + \mathcal{O}(|x|^{-N}), \quad |x| \rightarrow +\infty, \quad (24)$$

where  $\kappa = \kappa(k, |k| \frac{x}{|x|}) = |k| - \frac{(k, x)}{|x|}$ .

Then there are formulas for reconstruction of the leading coefficient  $f_1(k, |k| \frac{x}{|x|})$  up to  $\mathcal{O}(s^{-n})$  from  $|\psi^+(x, k)|^2$  given at  $m$  points  $x = x_1(s), \dots, x_m(s)$ .

## Examples with near-field

- (Yafaev, Klivanov, Romanov) Consider the scattering solutions  $\psi^+$  of Schrödinger equation at fixed point. For  $v \in C_c^\infty(\mathbb{R}^d)$ , we have that

$$\psi^+(x, k) = e^{ikx} \left( 1 + \sum_{j=1}^{N-1} \frac{b_j(x, \theta)}{s^j} + \mathcal{O}(s^{-N}) \right), \text{ as } s \rightarrow +\infty, \quad (25)$$

$$b_1(x, \theta) = \frac{1}{2i} Dv(x, -\theta), \quad Dv(x, \theta) := \int_0^{+\infty} v(x + \tau\theta) d\tau, \quad (26)$$

where  $x \in \mathbb{R}^d$ ,  $s = |k| = \sqrt{E}$  is energy level,  $\theta = k/|k|$  ( $x$  and  $\theta$  are fixed). Note that  $Dv$  is known as the divergent beam transform of  $v$ , and is very popular in tomography.

# Phase retrieval problem

Phase retrieval problem consists in finding a function  $v : \mathbb{R}^d \rightarrow \mathbb{C}$  from the magnitude  $|\widehat{v}|$  of its Fourier transform

$$\widehat{v}(p) = \mathcal{F}v(p) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ip \cdot x} v(x) dx, \quad (27)$$

given for  $p$  in some subset of  $\mathbb{R}^d$ , e.g.,  $B_R := \{x \in \mathbb{R}^d : |x| \leq R\}$ .

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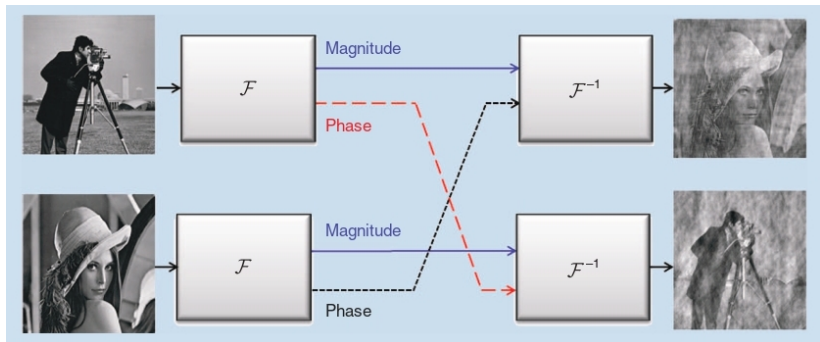
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given for  $p$  in some subset of  $\mathbb{R}^d$ , e.g.,  $B_R := \{x \in \mathbb{R}^d : |x| \leq R\}$ .

This problem is ill-posed: in particular, there is nonuniqueness of different types:

$$(\mathcal{F}v_y)(p) = e^{ipy} \mathcal{F}v(p), \quad v_y(x) = v(x - y). \quad (28)$$

To compensate for the missing phase information  $\widehat{v}/|\widehat{v}|$ , one either assumes a-priori information on  $v$  or additional data. Such inversions of the Fourier transform from phaseless data are much more complicated than the inversion of the Fourier transform from phased data.



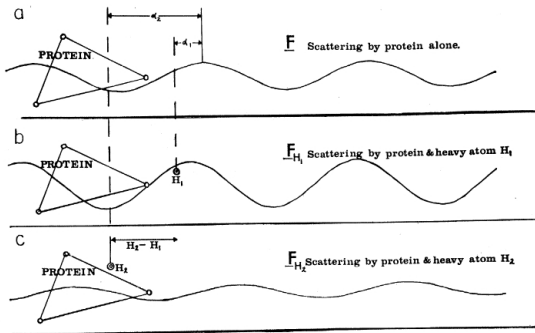
**Figure:** [Shechtman et al, 2015] The importance of Fourier phase. Two images, a cameraman and Lenna, are Fourier transformed. After swapping their phases, they are inverse Fourier transformed. The result clearly demonstrates the importance of phase information for image recovery.

Examples of a-priori informations include (approximate) knowledge of  $\text{supp } v$ , constraints like  $|v| = 1$  or  $v \geq 0$ , and knowledge of  $v$  on part of the domain.

### Problem 1

- 1 Reconstruct a function  $v$  from  $|\hat{v} + \hat{w}|^2$  on  $B_R$  for some known function  $w$  under the a-priori assumption that  $\text{supp } v$  and  $\text{supp } w$  are compact and sufficiently separated.
- 2 Reconstruct  $v$  from  $|\hat{v}|^2$  and  $|\hat{v} + \hat{w}_j|^2$ ,  $j = 1, \dots, n$ , on  $B_R$  for some appropriate known functions  $w_1, \dots, w_n$  separated from  $v$ .

Problem 1.2 was considered, in particular, by Novikov, Agaltsov, Hohage, 2014-19. In addition, related considerations go back, at least, to [M. Perutz (Cambridge), 1963] on X-ray analysis of hemoglobin (Nobel Prize for Chemistry 1962):



## Reconstruction formulas

Let  $v, w \in L_{1,loc}(\mathbb{R}^d)$ ,  $w \neq 0$ ,  $\text{supp } v \subset D$ ,  $\text{supp } w \subset \Omega$ , where  $D, \Omega$  are open convex bounded domains in  $\mathbb{R}^d$ .

Let  $\mathcal{F}v(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx$ .

### Phase Retrieval Theorem [NS2021]

Let  $\text{dist}(D, \Omega) > \text{diam } D$ . Then  $|\mathcal{F}(v + w)|^2$  and  $w$  uniquely determine  $v$  via inverse Fourier transform by the formulas

$$\mathcal{F}v(p) = (\overline{\mathcal{F}w(p)})^{-1} \mathcal{F}q(p), \quad (29)$$

$$q(x) = \chi_{D-\Omega}(x) \left( u(x) - (2\pi)^{-d} \int_{y \in \Omega} w(x+y) \overline{w(y)} dy \right), \quad (30)$$

$$u(x) = \mathcal{F}^{-1}(|\mathcal{F}(v + w)|^2)(x). \quad (31)$$

### Similar Phase Retrieval Theorem [NS2021]

Let  $\text{dist}(D, \Omega) > 0$ . Then  $|\mathcal{F}v|^2$ ,  $|\mathcal{F}(v + w)|^2$  and  $w$  uniquely determine  $v$ .

# Reconstruction formulas

## Some remarks

- Formula (2) includes division, and denominator can vanish at some points. We use Tikhonov regularization to avoid zeros:

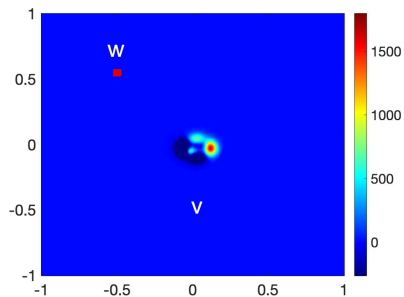
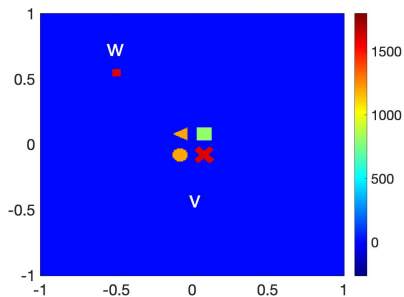
$$(\overline{\mathcal{F}w(p)})^{-1} \mathcal{F}q(p) \approx \frac{\mathcal{F}q(p)\mathcal{F}w(p)}{|\mathcal{F}w(p)|^2 + \varepsilon}.$$

- We exclude the case  $w \equiv 0$ , since  $\mathcal{F}w(p) \equiv 0, \forall p$ .
- We suppose that we approximately know the domains  $D$  and  $\Omega$ .
- The operation  $\mathcal{F}^{-1}$  is the most difficult for computations, especially in the case when data  $|\mathcal{F}(v+w)|^2$  are provided on non-uniform grid.

Our phase retrieval formulas have similar theoretical and numerical properties as standard recovering of  $v$  from  $\mathcal{F}v$  !

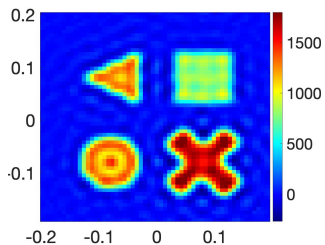
This includes the case of approximate recovering of  $v$  from  $|\mathcal{F}(v+w)|^2$  restricted to a ball  $B_{2\sqrt{E}}$ , including the form of error estimate as  $E \rightarrow +\infty$ .  
Related analysis can be found in the works of the thesis.

## Numerical reconstructions of [HNS2022]

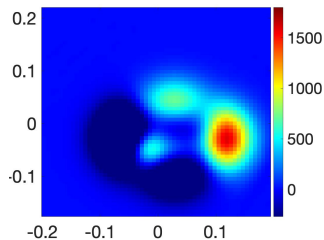
Our  $v + w$ :

# Numerical reconstructions

Reconstructions  $v_E$  from  $|\mathcal{F}(v+w)|^2$  on  $\mathcal{P}_N \cap B_{2\sqrt{E}}$ .



$L_2$  error: 0.2816 (0.0292)



$L_2$  error: 0.005

$\mathcal{F}, \mathcal{F}^{-1}$  via FFT

In the framework of phaseless inverse scattering, at fixed frequency  $\kappa$  the phaseless Fourier transform  $|\mathcal{F}(v + w)|^2(p)$  arises on discrete family of discrete Ewald circles for  $d = 2$  (non-uniform grid  $\mathcal{P}_{E, M_1, M_2}$ ).

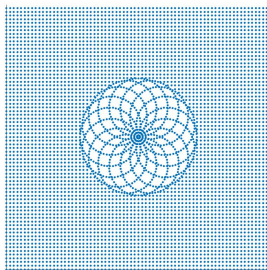


Figure: Non-uniform grid  $\mathcal{P}_{E, M_1, M_2}$

## Numerical reconstructions of [HNS2022]

- Recall that, in our formulas,  $\hat{u} = |\mathcal{F}(v + w)|^2$ .
- It was difficult to realize  $\mathcal{F}^{-1}$  from  $\hat{u}$  defined on non-uniform grid  $\mathcal{P}_{E, M_1, M_2}$ .
- For non-uniform grid we can not use FFT for  $\mathcal{F}^{-1}$ .
- Inversion  $\mathcal{F}^{-1}$  realised directly via CG method for the normal equation

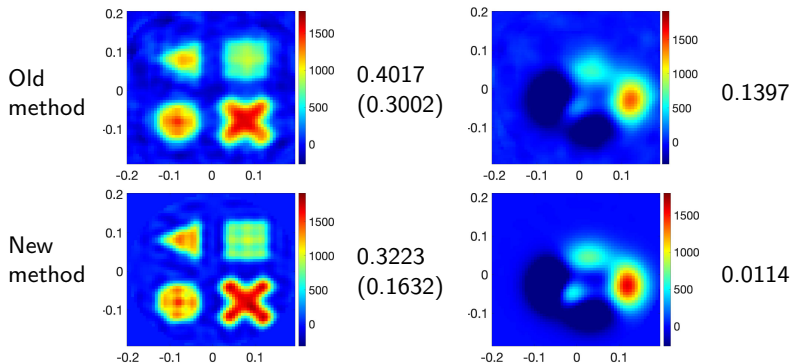
$$\mathcal{F}^* \mathcal{F} u = \mathcal{F}^* \hat{u}$$

did not work properly. Interpolations of  $\hat{u}$  to uniform grid did not help.

- We overcome this numerical problem using the theoretical result that  $\text{supp } u \subset D - \Omega$ . We replace normal equation by

$$\chi_{D-\Omega} \mathcal{F}^* \mathcal{F} \chi_{D-\Omega} u = \chi_{D-\Omega} \mathcal{F}^* \hat{u}.$$

## Numerical reconstructions



Reconstruction from  $|\mathcal{F}(v + w)|^2$  given on non-uniform grid in  $B_R$ .

# Applications to phaseless inverse scattering

We consider the stationary Schrödinger equation

$$-\Delta\psi + v\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \quad (32)$$

$$v \in L^\infty(\mathbb{R}^d), \quad \text{supp } v \subset D \text{ (open bounded domain)}. \quad (33)$$

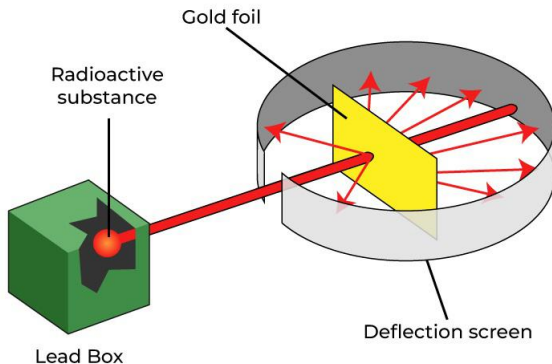
Solutions describing scattering of planar wave

$$\psi^+(x, k) = \underbrace{e^{ikx}}_{\text{planar wave}} + \underbrace{\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} c(d, |k|) \overbrace{f[v](k, |k| \frac{x}{|x|})}^{\text{scattering amplitude}}}_{\text{leading spherical wave}} + \mathcal{O}\left(\frac{1}{|x|^{(d+1)/2}}\right), \quad (34)$$

$|x| \rightarrow +\infty$  uniformly in  $x/|x|$ .

$\sigma[v] = |f[v]|^2$  is differential scattering cross section.

In quantum mechanics complex values of  $\psi^+$ ,  $f$  have no direct physical sense, whereas  $|\psi^+|^2$ ,  $|f|^2$  have direct probabilistic interpretations and can be measured; [M. Born, 1926] (Nobel Prize for Physics, 1954).



**Figure:**  $|f(k, l)|^2$  is differential scattering cross section, describing probability density of scattering of particle with initial impulse  $k$  into direction  $l/|l| \neq k/|k|$ .

# Inverse scattering phaseless problem

## Problem 2

Reconstruct  $v$  from differential scattering cross section  $|f[v + w]|^2 = \sigma[v + w]$ , and background  $w$ .

## The previous result

In [N14], [AHN19] the function  $v$  was reconstructed from  $\sigma[v]$ ,  $\sigma[v + w_1]$ ,  $\sigma[v + w_2]$ , and  $w_1, w_2$ .

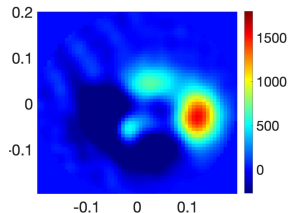
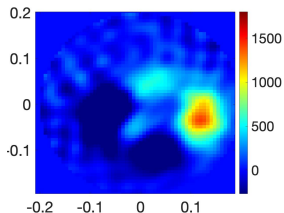
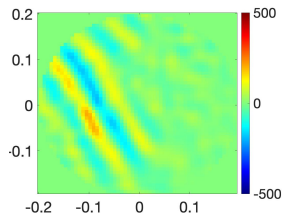
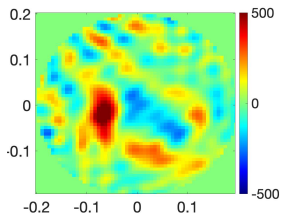
Real  
partImaginary  
part(a) First iteration  $u_E^1$ (b) 6th iteration  $u_E^6$ 

Figure: Reconstructions  $u_E^J$  of smooth real  $v$ . Relative errors on  $\mathcal{X}_N$ :

$$\mathfrak{E}(u_E^1, \underline{v}) = 0.4112; \quad \mathfrak{E}(u_E^6, \underline{v}) = 0.1977.$$

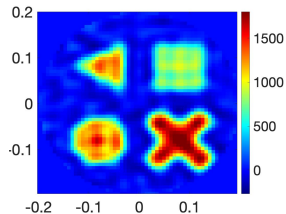
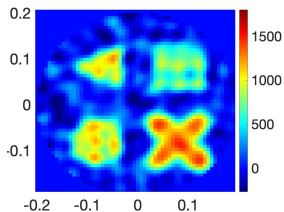
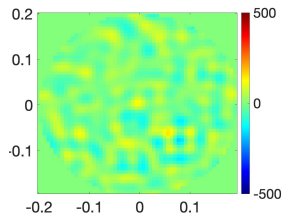
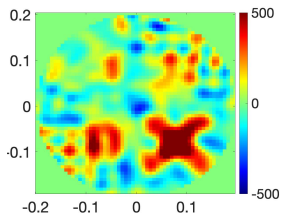
Real  
partImaginary  
part(a) First iteration  $u_E^1$ (b) 10th iteration  $u_E^{10}$ 

Figure: Reconstructions  $u_E^J$  of non-smooth real  $v$ . Relative errors on  $\mathcal{X}_N$ :  
 $\mathfrak{E}(u_E^1, \underline{v}) = 0.5140(0.4466)$ ;  $\mathfrak{E}(u_E^{10}, \underline{v}) = 0.3396(0.1873)$ .

## Remarks

- We give iterative method for Problem 2. Both first approximation and iterative step are new.
- We obtain error estimates and stability results for our method.
- We reconstruct complex-valued potential  $v$  on  $\mathbb{R}^d$  from one real-valued function  $\sigma[v + w]$  defined on  $d$ -dimensional set, where  $w$  is a priori known and sufficiently separated from  $v$ .
- The examples of our reconstruction from  $\sigma[v + w]$  include Poisson noise with  $N = 3 \cdot 10^7$  particles. This corresponds to the level of Poisson noise in real experiments in X-ray imaging. In this case the particles are X-ray photons measured by detectors.
- We found examples where our method converges to correct solution, whereas the conventional Newton method minimizing the corresponding discrepancy does not. This effect arises already in reconstruction from phaseless Fourier transform.

## Ideas for Problem 2

Differential cross section is related with phaseless Fourier transform via the following Born type formula with background  $w$

$$\sigma[v + w](k, l) = |\mathcal{F}(v + w)(p)|^2 + \mathcal{O}(E^{-1/2}), \quad (35)$$

for  $p = k - l$ ,  $k^2 = l^2 = E$ ,  $E \rightarrow +\infty$ ; see [B1926], [F1956], [N2014].

From this estimate and Phase retrieval theorem [NS2021], we obtain

## Phaseless inverse scattering theorem, [NS2021]

If  $\text{dist}(D, \Omega) > \text{diam } D$ , then  $\sigma[v + w], w$  uniquely determine  $v$ .

## Error estimates

## Error estimate theorem, [NS2021]

Let  $\underbrace{v \in W^{m,1}(\mathbb{R}^d)}_{m \text{ times smooth}}$ ,  $\underbrace{|\hat{w}(p)| \geq c(1 + |p|)^{-\beta}}_{\text{far from zeros}}$ ,  $m, \beta > d$ . Then

$$|v(x) - u_{\text{Born}}(x, E)| = \mathcal{O}(E^{-\frac{1}{2} \frac{m-d}{m+\beta}}). \quad (36)$$

Note that even for infinitely smooth  $v$  the convergence rate does not exceed  $\mathcal{O}(E^{-1/2})$ .

The approximation  $u_{\text{Born}}(x, E)$  of [NS2021] is drastically improved in [HNS2022] via an iterative method going back to [N2015] for the phased case.

In order to obtain iterative improvements, we use the following

### Phaseless iterative lemma, [HNS2022]

Let  $u_j$  be some approximation to  $v$  such that

$$|u_j(x, E) - v(x)| \leq bE^{-\alpha}, \quad x \in D, \quad (37)$$

$$u_j(x, E) = v(x), \quad \text{else.} \quad (38)$$

Then, as  $E \rightarrow +\infty$ ,

$$|\sigma[v + w] - \sigma[u_j + w] + (|\hat{u}_j + \hat{w}|^2 - |\hat{v} + \hat{w}|^2)| = \mathcal{O}(E^{-\alpha - \frac{1}{2}}). \quad (39)$$

This estimate gives us the following improvement:

$$|\hat{u}_{j+1} + \hat{w}|^2 = |\hat{u}_j + \hat{w}|^2 + (\sigma[v + w] - \sigma[u_j + w]). \quad (40)$$

## Error estimates for iterative reconstruction

## Iterative phaseless inverse scattering theorem, [HNS2022]

Let  $v \in \underbrace{W^{m,1}(\mathbb{R}^d)}_{m \text{ times smooth}}$ ,  $\underbrace{|\hat{w}(p)| \geq c(1 + |p|)^{-\beta}}_{\text{far from zeros}}$ ,  $m, \beta > d$ .

Let  $u_j(x, E)$  be  $j$ -iteration. Then

$$|v(x) - u_j(x, E)| = \mathcal{O}(E^{-\alpha_j}), \quad \alpha_j = \frac{1}{2} \frac{m - d}{\beta + d} \left( 1 - \left( \frac{m - d}{m + \beta} \right)^j \right).$$

## Remarks.

- $\alpha_j \rightarrow \frac{j}{2}$ , as  $m \rightarrow +\infty$ .
- The result is better than in [AHN2019] with  $2\beta$  in place of  $\beta$ .

# Stability results

## Approximate Lipschitz stability theorem

Let  $v_1, v_2 \in L^\infty(D)$ ,  $\underbrace{v_1 - v_2 \in W^{m,1}(\mathbb{R}^d)}_{m \text{ times smooth}}$ ,

$w \in L^\infty(\Omega)$ ,  $\underbrace{|\hat{w}(p)| \geq c(1 + |p|)^{-\beta}}_{\text{far from zeros}}$ ,  $m, \beta > d$ .

Then, for some  $C_1, C_2, \varepsilon \in (0, 1/2)$

$$\|v_1 - v_2\|_{L^\infty(D)} \leq C_1 E^{\frac{1}{2} - \varepsilon} \|\sigma[v_1 + w] - \sigma[v_2 + w]\|_{C(\Gamma_E^-)} + C_2 E^{-(\frac{1}{2} - \varepsilon) \frac{m-d}{\beta+d}}.$$

# Conclusions

In this talk we present contributions to studies on phase retrieval, phaseless inverse scattering, and multipoint formulas in inverse problems:

- 1 We give the first numerical implementation of the method of multipoint formulas in inverse problems, including an efficient regularization of these formulas for the noisy case. By this method we also obtain new theoretical results on polychromatic inverse scattering problems (from far-field data, phaseless far-field data, and from near-field data).
- 2 We give phase retrieval formulas from a single phaseless Fourier transform  $|\mathcal{F}(v + w)|^2$  with appropriate background  $w$ . Proceeding from these formulas we obtain different theoretical and numerical results on the phase retrieval problem for the classical Fourier transform.
- 3 We propose an iterative algorithm for inverse scattering from a single differential scattering cross section with appropriate background scatterer. This includes error estimates, stability estimates, and numerical implementation.

Thank you for attention!