2/12/09

THE AMERICAN MATHEMATICAL MONTHLY

Volume 116, Number 9 November 2009

Mike Paterson, Yuval Peres, Mikkel Thorup, Peter Winkler, Uri Zwick	Maximum Overhang	763
Persi Diaconis, Jason Fulman	Carries, Shuffling, and an Amazing Matrix	788
Luis Verde-Star	Rational Functions	804
NOTES		
Theo de Jong	Lagrange Multipliers and the Fundamental Theorem of Algebra	828
D. M. Einstein, C. C. Heckman, T. S. Norfolk	On Sara's Dove Bar Habit	831
Bakir Farhi	An Identity Involving the Least Common Multiple of Binomial Coefficients and Its Application	836
Valerio De Angelis	Stirling's Series Revisited	839
PROBLEMS AND SOLUTIONS		844
REVIEWS		
Victor J. Katz	*Mathematical Masterpieces: Further Chronicles by the Explorers*. By Arthur Knoebel, Reinhard Laubenbacher, Jerry Lodder, and David Pengelley	852
EDITOR'S ENDNOTES		857

AN OFFICIAL PUBLICATION OF THE MATHEMATICAL ASSOCIATION OF AMERICA

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mario Benedicty, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before March 31, 2010. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11460. *Proposed by Cosmin Pohoaţă, Tudor Vianu National College of Informatics, Bucharest, Romania.* Given a triangle of area S with sides of lengths a, b, and c, and positive numbers x, y, and z, show that

$$a^2+b^2+c^2 \geq 4\sqrt{3}S + \frac{2}{x+y+z}\left(a^2\frac{x^2-yz}{x}+b^2\frac{y^2-zx}{y}+c^2\frac{z^2-xy}{z}\right).$$

11461. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.* Let a, b, and c be the lengths of the sides opposite vertices A, B, and C of an acute triangle. Let H be the orthocenter. Let d_a be the distance from H to side BC, and similarly for d_b and d_c. Show that

$$\frac{1}{d_a+d_b+d_c} \geq \frac{2}{3}\left(\frac{3}{abc}\left(\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}\right)\right)^{1/4}.$$

11462. *Proposed by Nadezhda Alexandrova, Institute of Mining, Novosibirsk, Russia.* Find

$$\lim_{\alpha\to 0^+} \log\alpha + \int_{x=0}^{\pi} \frac{dx}{\sqrt{\sin^2 x + i\alpha}}.$$

11463. *Proposed by Xiang Qian Chang, Massachusetts College of Pharmacy and Health Sciences, Boston, MA.* Let A be a positive-definite $n \times n$ Hermitian matrix with minimum eigenvalue λ and maximum eigenvalue Λ. Show that

$$\left(\frac{n}{\operatorname{tr}((A+\lambda I)^{-1})}-\lambda\right)^n \leq \det(A) \leq \left(\frac{n}{\operatorname{tr}((A+\Lambda I)^{-1})}-\Lambda\right)^n.$$

doi:10.4169/000298909X474927

11464. *Proposed by David Beckwith, Sag Harbor, NY.* Let $a(n)$ be the number of ways to place n identical balls into a sequence of urns $U_1, U_2, \ldots$ in such a way that U_1 receives at least one ball, and while any balls remain, each successive urn receives at least as many balls as in all the previous urns combined. Let $b(n)$ denote the number of partitions of n into powers of 2, with repeated powers allowed. (Thus, $a(6) = 6$ because the placements are 114, 123, 15, 24, 33, and 6, while $b(6) = 6$ because the partitions are 111111, 11112, 1122, 114, 222, and 24.) Prove that $a(n) = b(n)$ for all $n \in \mathbb{N}$.

11465. *Proposed by Pantelimon George Popescu, Polytechnic University of Bucharest, Bucharest, Romania, and José Luis Díaz-Barrero, Polytechnic University of Catalonia, Barcelona, Spain.* Consider three simple closed curves in the plane, of lengths p_1, p_2, and p_3, enclosing areas A_1,A_2, and A_3, respectively. Show that if $p_3 = p_1 + p_2$ and $A_3 = A_1 + A_2$ then $8\pi A_3 \leq p_3^2$.

11466. *Proposed by Tian Yongge, Central University of Finance and Economics, Beijing, China.* For a real symmetric $n \times n$ matrix A, let $r(A)$, $i_+(A)$, and $i_-(A)$ denote the rank, the number of positive eigenvalues, and the number of negative eigenvalues of A, respectively. Let $s(A) = i_+(A) - i_-(A)$. Show that if P and Q are symmetric $n \times n$ matrices, $P^2 = P$, and $Q^2 = Q$, then $i_+(P - Q) = r(P + Q) - r(Q)$, $i_-(P - Q) = r(P + Q) - r(P)$, and $s(P - Q) = r(P) - r(Q)$.

SOLUTIONS

A Weighted Sample of the Riemann Zeta Function

11331 [2007, 926]. *Proposed by Richard Bagby, New Mexico State University, Las Cruces, NM.* Show that if k is a positive integer, then

$$\int_{t=0}^{\infty} \left(\frac{\log(1+t)}{t}\right)^{k+1} dt = (k+1)\sum_{j=1}^{k} a_j \zeta(j+1),$$

where ζ denotes the Riemann zeta function and a_j is the coefficient of x^j in $x\prod_{n=1}^{k-1}(1-nx)$.

Composite solution by David Beckwith, Sag Harbor, NY, and Douglas B. Tyler, Raytheon, Torrance, CA. The change of variable $t = e^y - 1$ transforms the integral to

$$I = \int_0^{\infty} \frac{(e^{-y})^k}{(1-e^{-y})^{k+1}} y^{k+1} dy,$$

and the convergent expansion

$$\frac{u^k}{(1-u)^{k+1}} = \sum_{m=1}^{\infty} \binom{m}{k} u^m, \quad \text{for } |u| < 1,$$

contains only positive terms when $u = e^{-y}$ with y real. This justifies the interchange