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ROCK FAILURE

ELASTIC WAVE PROPAGATION IN BLOCK MEDIUM UNDER IMPULSE LOADING

N. I. Aleksandrova UDC 539.3

The waveguide properties of a block medium are studied by an example of a discrete-periodical system of spring-connected rods. As a particular case, a chain of masses is considered. It is shown that in the rod system, impulse loading excites two groups of nonstationary waves. The velocities of their propagation are determined. The asymptotic representation is obtained for low-frequency wave at a great period of time since the process onset. The analytical estimations of excitation spectrum are derived.

Block medium, impulse excitation, nonstationary waves, spectral characteristics

Rock masses are primarily characterized by the block-hierarchical structure [1]. General regularities of wave propagation in rocks can be revealed on discrete-periodical models, for example, on a system of rods connected by springs. One of the effects of elastic wave packet propagating in rocks under impulse excitation is formation of pendulum-type waves caused by movement of separate blocks as "perfect solids" [2]. In [2 – 5], main attention is focused on studying the results of laboratory and in situ experiments in block media under impact loads. This paper performs a theoretical analysis of influence exerted by block medium on wave propagation and its spectral characteristics. Periodical system consisting of rods interacting with each other through the connecting springs is considered. In the first approximation, it is a model of block media peculiar to rocks. As a particular case, a chain of masses connected by springs is studied. Dispersion properties and nonstationary processes in this system were analyzed in [6 – 8]. In [9, 10], the results are presented for the experimental and theoretical investigations into spectral characteristics of the dynamic processes in blocks connected by nonlinear springs. Below, general properties of wave propagation processes in chain of masses and system of rods are discussed.

Problem statement. A system of spring-connected elastic rods is considered: ρ is the density; c is the sound velocity; y is the length; F is the cross-section area. The equation of the rod motion has the form:

$$\ddot{u}_j = c^2 u_j'' ; \qquad j = 0,..., J , \tag{1}$$

where u_j is the displacement of the jth rod; prime and point indicate the derivatives with respect to the longitudinal coordinate x and the time t, respectively. At the ends of the rods, the stresses are proportional to the unit elongation of springs:

$$\begin{aligned} FEu_j'\big|_{x=y(j-1/2)} &= -k(u_{j-1}\big|_{x=y(j-1/2)} - u_j\big|_{x=y(j-1/2)}); \quad j = 1,..., J ; \\ FEu_j'\big|_{x=y(j+1/2)} &= -k(u_j\big|_{x=y(j+1/2)} - u_{j+1}\big|_{x=y(j+1/2)}); \quad j = 0,..., J-1 . \end{aligned} \tag{2}$$

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Here, E is Young's modulus; k is the spring rate. All springs are the same. The initial conditions are zero. At $t = 0$ impulse load with the amplitude P_0 is applied to the left end of the rod:

$$Eu_0'|_{x=-y/2} = P_0 Q(t); \qquad Q(t) = \sin(\omega_* t) H_0(\pi - \omega_* t) H_0(t), \tag{3}$$

where H_0 is the Heaviside function, $\omega_* = 1/T_*$, T_* is the action duration.

Analyze a simpler model of this system. The rods are replaced by rigid balls with mass equal to that of the rods $m = \rho y F$, provided the spring rate is the same. For such system, the equations of motion have the form:

$$m\ddot{v}_j = k(v_{j-1} - 2v_j - v_{j+1}); \quad (j = 1,..., J-1); \quad m\ddot{v}_0 = k(v_1 - v_0) - FP_0 Q(t). \tag{4}$$

Here, v_j is the center of mass displacement.

Analytical solution of problems. Assume that the discrete system is infinite $(-\infty \le j \le \infty)$ and force (3) is applied to the mass j. Take the Laplace transform of (1) – (4) (p is the parameter):

$$f^L(p) = \int_0^\infty f(t) e^{-pt} dt; \qquad f(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} f^L(p) e^{pt} dp \quad (\alpha > 0)$$

and the Fourier transform of equations (1) with respect to x (q is the parameter):

$$f^F(q) = \int_{-\infty}^{\infty} f(x) e^{iqx} dx; \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^F(q) e^{-iqx} dq.$$

For chain of masses, take the discrete Fourier transform with respect to j:

$$f^{F_d}(q) = \sum_{j=-\infty}^{j=\infty} f_j e^{iqjy}; \qquad f_j = \frac{1}{2\pi} \int_{-\pi/y}^{\pi/y} f^{F_d}(q) e^{-iqjy} dq.$$

Introduce the designations:

$$u_j(X,t)\big|_{X=-y/2} = W_j; \qquad u_j(X,t)\big|_{X=y/2} = V_j,$$

where $X = x - jy$, $-y/2 \le X \le y/2$. Then, from (1), for the jth rod, we have:

$$u_j^L(X) = \left[V_j^L \mathrm{sh}\left(\frac{p}{c}\left(X + \frac{y}{2} \right) \right) + W_j^L \mathrm{sh}\left(\frac{p}{c}\left(\frac{y}{2} - X \right) \right) \right] \Big/ \mathrm{sh}\frac{py}{c}. \tag{5}$$

Thus, for the system of rods, the relation between the continuous and discrete Fourier transforms is obtained:

$$u^{LF} = \int_{-\infty}^{\infty} u^L e^{iqx} dx = \sum_{j=-\infty}^{j=\infty} \int_{-y/2}^{y/2} u_j^L e^{iqX} dX e^{iqjy} = V^{LF_d} A(p,q) + W^{LF_d} B(p,q); \tag{6}$$

$$A(p,q) = -\frac{1}{(q^2 + p^2/c^2)\mathrm{sh}(py/c)} \left[e^{iqy/2} \left(iq\,\mathrm{sh}\frac{py}{c} - \frac{p}{c}\,\mathrm{ch}\frac{py}{c} \right) + e^{-iqy/2}\frac{p}{c} \right],$$

$$B(p,q) = \frac{1}{(q^2 + p^2/c^2)\mathrm{sh}(py/c)} \left[e^{-iqy/2} \left(iq\,\mathrm{sh}\frac{py}{c} + \frac{p}{c}\,\mathrm{ch}\frac{py}{c} \right) - e^{iqy/2}\frac{p}{c} \right].$$

Assume that the load is applied to the middle of zero rod. Then, taking the discrete Fourier transform and the Laplace transform of (2) with boundary conditions (3), derive the set of algebraic equations to determine V^{LF_d} and W^{LF_d}:

$$V^{LF_d}\left(\operatorname{ch}\frac{py}{c}+\frac{\alpha}{p}\operatorname{sh}\frac{py}{c}\right)-W^{LF_d}\left(1+\frac{\alpha}{p}\operatorname{sh}\frac{py}{c}e^{-iqy}\right)=\frac{2cP_0Q^L\operatorname{sh}\dfrac{py}{2c}}{Ep};\tag{7}$$

$$W^{LF_d}\left(1+\frac{\alpha}{p}\operatorname{sh}\frac{py}{c}e^{iqy}\right)-W^{LF_d}\left(\operatorname{ch}\frac{py}{c}+\frac{\alpha}{p}\operatorname{sh}\frac{py}{c}\right)=-\frac{2cP_0Q^L\operatorname{sh}\dfrac{py}{2c}\operatorname{ch}\dfrac{py}{c}}{Ep}.$$

For the rod system, the problem is solved from (6) and (7) for the Laplace — Fourier representations:

$$u^{LF}=\frac{P_0Q^LC(p,q)}{ED(p,q)};\quad C(p,q)=\frac{1}{2(q^2+p^2/c^2)\operatorname{ch}(py/c)}\times\tag{8}$$

$$\times\left[e^{iqy/2}\left(\frac{p}{\alpha}\operatorname{sh}\frac{py}{c}+3\operatorname{ch}\frac{py}{c}\right)-e^{-iqy/2}\left(\frac{iqc}{\alpha}\operatorname{sh}^2\frac{py}{c}+\operatorname{ch}\frac{2py}{c}+2+\frac{p}{2\alpha}\operatorname{sh}\frac{2py}{c}\right)+e^{-3iqy/2}\operatorname{ch}\frac{py}{c}-e^{3iqy/2}\right];$$

$$D(p,q)=\cos qy-\operatorname{ch}\frac{py}{c}-\frac{p}{2\alpha}\operatorname{sh}\frac{py}{c};\quad \alpha=\frac{kc}{FE}.\tag{9}$$

The same solution for spring-connected masses:

$$v^{LF_d}=\frac{P_0FQ^L}{mD(p,q)},\qquad D(p,q)=p^2+4\frac{k}{m}\sin^2\frac{qy}{2}.\tag{10}$$

Now, analyze the dispersion properties of two models. For the chain of masses, from the dispersion equation $D(i\omega,q)=0$, where ω is the circular frequency, calculate the phase and group velocities:

$$c_p=\frac{\omega}{q}=\frac{2}{q}\sqrt{\frac{k}{m}}\sin\frac{qy}{2},\qquad c_q=\frac{d\omega}{dq}=y\sqrt{\frac{k}{m}}\cos\frac{qy}{2}.$$

It is obvious that as $q\to 0$ they are equal:

$$c_p\big|_{q\to0}=c_q\big|_{q\to0}=y\sqrt{\frac{k}{m}}=c_1,\tag{11}$$

i.e., the waves propagate without dispersion and form quasi-front.

In the rod system, higher oscillation modes appear due to reflection from the rod ends. The explicit dependences of c_p, c_q on the wave number q fail to obtain from (9). However, on the assumption of infinitesimal $\beta=\alpha y/c$, it is possible to find the explicit approximate formulae for all oscillation modes infinite in number on account of periodicity of trigonometrical functions. For the first and second modes:

$$c_p^{\mathrm{I}}=\frac{2c_1}{qy}\sin\frac{qy}{2},\qquad c_q^{\mathrm{I}}=c_1\cos\frac{qy}{2},\qquad \omega^{\mathrm{I}}=\omega_0\sin\frac{qy}{2},\qquad c_1=y\sqrt{\frac{k}{m(1+\beta)}};\tag{12}$$

$$c_p^{\mathrm{II}}=\frac{1}{q}\left(\omega_2-\frac{4k}{\omega_2 m}\sin^2\frac{qy}{2}\right),\quad c_q^{\mathrm{II}}=c_2\sin qy,\quad \omega^{\mathrm{II}}=qc_p^{\mathrm{II}},\quad c_2=\frac{2\alpha c}{\omega_2};\tag{13}$$

$$\omega_0=2\sqrt{\frac{k}{m(1+\beta/3)}},\quad \omega_1=\frac{\pi c}{y},\quad \omega_2=\frac{\pi c}{y}\left(1+\frac{4\beta}{\pi^2+4\beta}\right).$$

The behavior of dispersion curves for the chain of masses and the first oscillation mode of the rod system coincides qualitatively. There are small quantitative differences, i.e., the maximal c_p and c_q in the rod decrease as compared with the chain of masses by a factor of $\sqrt{1+\beta}$, and the maximal frequency, by a factor of $\sqrt{1+\beta/3}$. The frequency ω_1 corresponds to oscillations with the period equal to the time of double pass of longitudinal wave along the rod. It follows from (12) and (13) that $c_1 > c_2$ for all problem paramers; so, the high-frequency wave never advances the low-frequency one.

Figure 1 shows the graphs of the dependence $\omega(q)$ obtained from the dispersion equation for the first two oscillation modes ($\beta = 0, 2$). The dashed lines indicate the approximate values of ω_0, ω_1, and ω_2. It is evident that they are consistent with numerical results for small β to a high accuracy. If the dispersion equation is rewritten as $\cos qy = K(\omega)$, the frequency ranges with $|K(\omega)| \le 1$ coincide with the zones where undamped waves exist. For the frequencies satisfying the inequality $|K(\omega)| > 1$, the dispersion equation has complex solutions to determine damped waves. For the rod system, such zones are infinitely large in number, the first one is $\omega_0 < \omega < \omega_1$. For the chain of masses, such zone is an interval $\omega > 2\sqrt{k/m}$.

Let us perform an analysis for propagation of low-frequency disturbance. For this purpose, inversions of (8) and (10) are used in the ray $x = ct$ $(c = c_1 + c',\ c' \to 0)$ [6]. In the problem on excitation of chain of masses by load (3), the asymptotic of long-wave disturbance near the quasi-front $x = c_1 t$ at $t \to \infty$:

$$v_{j+1} - v_j = \frac{P_0 F c_1}{k\pi\omega_*(\gamma t)^{1/3}} Ai(æ), \quad Ai(æ) = \int_0^\infty \cos(æz + z^3)dz; \tag{14}$$

$$æ = \frac{(j+1)y - c_1 t}{(\gamma t)^{1/3}}, \quad \gamma = \frac{c_1 y^2}{24}, \quad c_1 = y\sqrt{\frac{k}{m}}.$$

Here, *Ai* is the Airy function [11]. It is apparent that as the time increases, the oscillation amplitude decreases proportionally to $t^{1/3}$. For the rod system, the asymptotic of low-frequency longitudinal wave is analogous to the other coefficients:

$$\frac{\partial u}{\partial x} = \frac{P_0 c_1}{E\pi\omega_*(\gamma t)^{1/3}} Ai(æ), \quad æ = \frac{x - c_1 t}{(\gamma t)^{1/3}}, \quad \gamma = \frac{c_1 y^2}{24(1+\beta)^2}, \quad c_1 = y\sqrt{\frac{k}{m(1+\beta)}}. \tag{15}$$

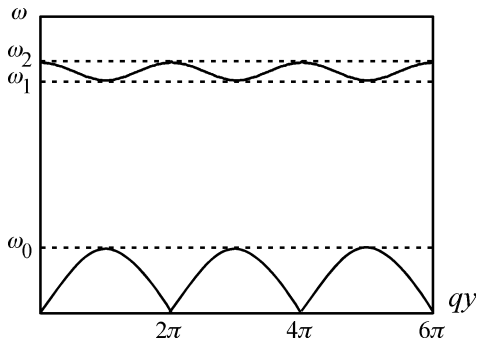

Fig. 1

Spectral analysis. Study spectral characteristics of wave process for semi-infinite chain of masses. Apply the Fourier transform:

$$f^{\tilde{L}}(\omega)=\int_0^{\infty} f(t)e^{-i\omega t}dt$$

with respect to t of set (4). The solution has the form of $v_j^{\tilde{L}}=Ve^{-i\lambda j}$, where V is obtained from the boundary conditions. As a result we have:

$$v_{j+1}^{\tilde{L}}-v_j^{\tilde{L}}=\frac{P_0F}{k}Q^{\tilde{L}}e^{-i\lambda(j+1)},\quad \lambda=2\arcsin\mu,\quad \mu=\frac{\omega}{\omega_0},\quad \omega_0=2\sqrt{\frac{k}{m}}.$$

Whence it follows that spectrum of spring elongation is determined by the formula:

$$\left|v_{j+1}^{\tilde{L}}-v_j^{\tilde{L}}\right|=\frac{P_0F}{k}\left|Q^{\tilde{L}}\right|\begin{cases}1, & \mu\le 1,\\ e^{2(j+1)\ln(\mu-\sqrt{\mu^2-1})}, & \mu\ge 1,\end{cases} \tag{16}$$

where according to (3):

$$\left|Q^{\tilde{L}}\right|=\frac{2\omega_*\left|\cos(\pi\omega/2\omega_*)\right|}{\left|\omega^2-\omega_*^2\right|}.$$

Using the same line of reasoning for the rod system, find the strain spectrum in the middle of the rods:

$$G(\omega,\omega_*)=\left|\frac{\partial u_j^{\tilde{L}}}{\partial x}\right|_{X=0}=\frac{P_0\left|Q^{\tilde{L}}\right|R}{ES}\begin{cases}1,\ 0\le\omega\le\omega_0\vee\omega_1\le\omega\le\omega_2 & (|K|\le 1);\\ e^{2j\ln(\sqrt{-\mu}-\sqrt{-\nu})},\ \omega_0\le\omega\le\omega_1 & (K\le -1);\\ e^{2j\ln(-\sqrt{\mu}+\sqrt{\nu})},\ \omega_2\le\omega\le\omega_3 & (K\ge 1);\end{cases} \tag{17}$$

$$R=\left[(1+K)(1+L)+\frac{2(K-L)^2}{1+2K-L}\right]^{1/2},\quad S=[L(2K+L)+1]^{1/2},$$

$$K=\cos\frac{\omega y}{c}-\frac{\omega}{2\alpha}\sin\frac{\omega y}{c},\quad L=\cos\frac{\omega y}{c}-\frac{\omega}{\alpha}\sin\frac{\omega y}{c},$$

$$\mu=\frac{K-1}{2},\quad \nu=\frac{K+1}{2}.$$

Apparently the intervals $\omega_0\le\omega\le\omega_1$ and $\omega_2\le\omega\le\omega_3$ correspond to the waves propagating with damping. Consequently, the validity is proven for the relation $G(\omega_1,\omega_1)/G(0,\omega_1)=\pi^2/(8\beta)$ showing that spectrum amplitude on the frequency $\omega=\omega_1$ at $\beta<\pi^2/8$ exceeds the amplitude on $\omega=0$ under load (3) with ω_1.

Numerical results. Analytical estimations (14) and (15) were yielded for infinitely great time and low-frequency waves. To determine the limits of applicability of (14), (15) and find the solutions for all wavelengths, numerical calculations were conducted, and analytical estimations were compared with numerical results. Sets of equations (1) with zero initial conditions and boundary conditions (2), (3) were solved by the finite-difference method using the explicit “cross”-type scheme. Set (4) was also solved by the finite-difference method.

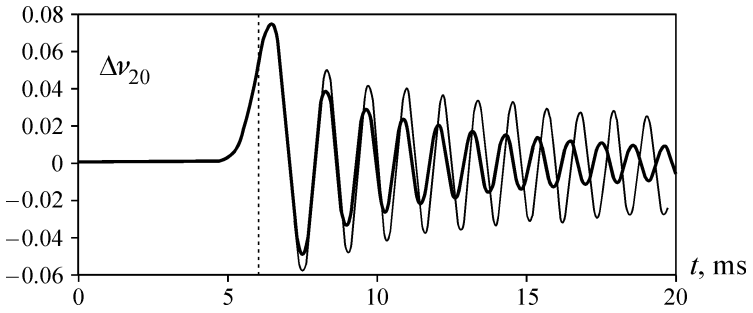

Fig. 2

Figure 2 demonstrates the graphs of unit elongations of the 20th spring $\Delta v_{20} = (v_{20} - v_{19})k / P_0 F$ calculated for the chain of masses (the fine and solid lines correspond to analytical (14) and finite-difference solutions, respectively). The dashed line indicates quasi-front $x = c_1 t$ (11). The problem parameters are as follows: $m = 4.875$ kg, $k = 48.75 \cdot 10^6$ H/m, $\omega_* = 32640$ s^{-1}, and $J = 50$. By the result analysis, (14) describes qualitatively and quantitatively propagation of long-wave disturbances near the quasi-front in the chain of masses even at $j \geq 5$. In [2], these low-frequency waves are called the waves of pendulum type.

Figure 3 depicts the graphs of the disturbance spectrum $\left| \Delta v_{10}^{\tilde{L}} \right|$ (the fine and solid lines represent analytical (16) and numerical solutions, respectively). The problem parameters are similar to those in Fig. 2. For the chain of masses, estimation (16) coincides with numerical calculations of $\left| \Delta v_{10}^{\tilde{L}} \right|$. The differences are mainly associated with the fact that analytical solution is obtained for the semi-infinite time interval.

For the case of resonance excitation ($\omega_* = \omega_1$) by load (3), the character of disturbance propagation along the rod system is illustrated in Fig. 4. Here, the graphs are presented for the strains $u'_j E / P_0$ ($j = 5$, 10, and 20 in Fig. 4*a*, *b*, and *c*, respectively) and the velocities $\dot{u}E / P_0$ ($j = 5$ in Fig. 4*d*) in the middle of the rods ($X = 0$). The dashed lines stand for location of the quasi-fronts $x = c_1 t$, $x = c_2 t$. The parameters of the problem are $\rho = 2600$ kg/m^3, $y = 0.5$ m, $F = 0.00375$ m, $k = 48.75 \cdot 10^6$ H/m, $c = 5\,000$ m/s, $J = 50$, $m = 4.875$ kg, and $\beta = 0.1$. For the chain of masses, the finite-difference solution ($m = 4.875$ kg, $k = 48.75 \cdot 10^6$ H/m) is shown by the fine line in Fig. 4*c*. Asymptotic (15) of long-wave disturbances is not illustrated, since it describes the amplitude and shape of the head pick in the same way as in the chain of masses. In Fig 5, the strain graphs are depicted at $\beta = 0.4$ ($k = 195 \cdot 10^6$ H/m); at that, $j = 5$ in Fig. 5*a* and $j = 10$ in Fig. 5*b*, the rest parameters are the same as in Fig. 4.

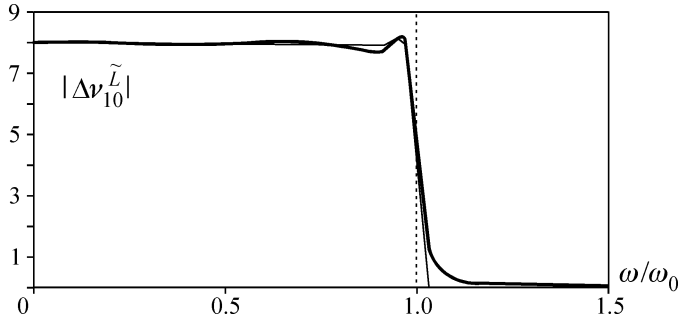

Fig. 3

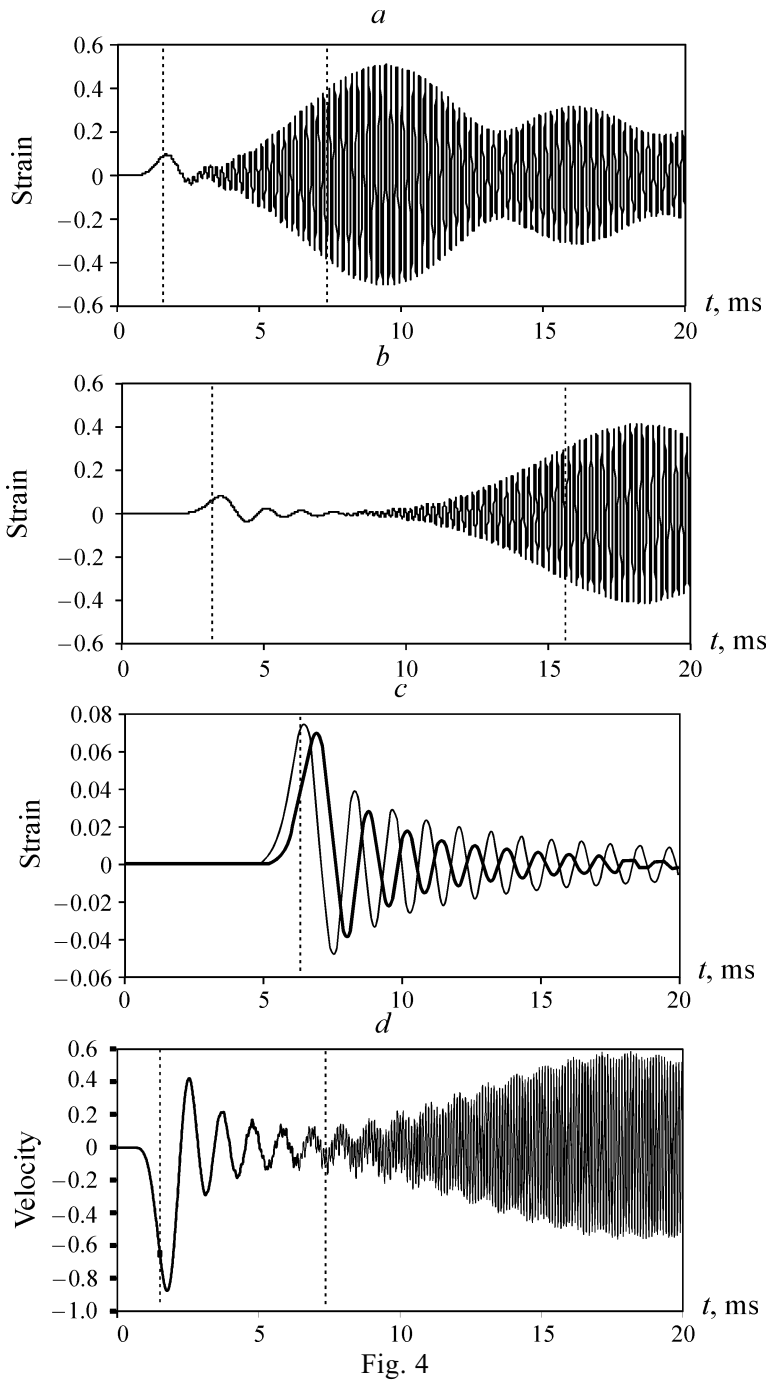

Fig. 4

By a comparison of Fig. 4*a* and *d*, in low-frequency wave, the relation $\dot{u} = -cu'$ is satisfied for the velocities and strains, while in high-frequency waves, the behavior of envelopes differs greatly. From Figs. 4 and 5, it is seen that long-wave disturbances propagating with the velocity c_1 are followed by high-frequency disturbances with the frequency ω_1. The velocity of the wave envelope motion is close to the maximum group velocity of the second mode c_2, and the maximal amplitude exceeds that of the long-wave disturbances if $\omega_* = \omega_1$. An increase in β rises the velocities c_1 and c_2 as well as the envelope frequency (Figs. 4, 5). The velocity of pendulum-type waves is greater in the chain of masses than in the rod system at the equal rigidities of springs and masses of balls and rods in both systems, which is proved by Fig. 4*c*.

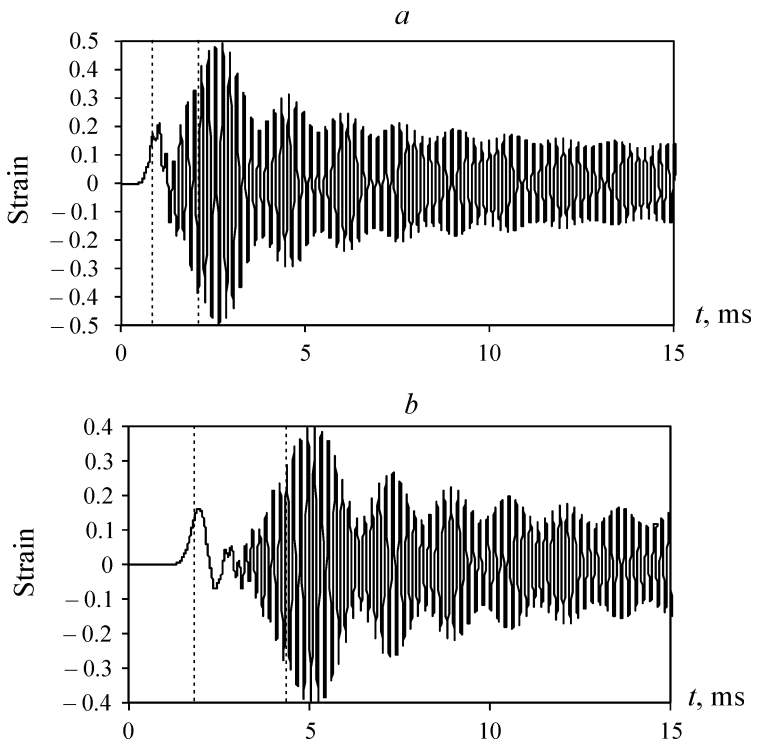

Fig. 5

According to calculations, as the frequency ω_* diminishes, the energy is redistributed. The amplitude of low-frequency wave increases and that one of high-frequency wave decreases. In the internal cross-sections of the rod, wave reflection from the rod ends influences greatly the deformation; at the same time, these reflections leave practically unaffected the unit elongation of springs connecting rods.

The spectrum of strains presented in Fig. 4*b* is shown in Fig. 6 (solid line). The fine line corresponds to analytical estimation (17) on the interval $\omega \le \omega_0$; the dashed lines are the values of the frequencies ω_0, ω_1, and ω_2. Unlike the chain of masses, the narrow peak arisen conforms to the zone $\omega_1 \le \omega \le \omega_2$. The peak height is less than the analytical value of $G(0,\omega_1)\pi^2/(8\beta)$ by a factor of 1.5. This is caused by the errors of numerical solution.

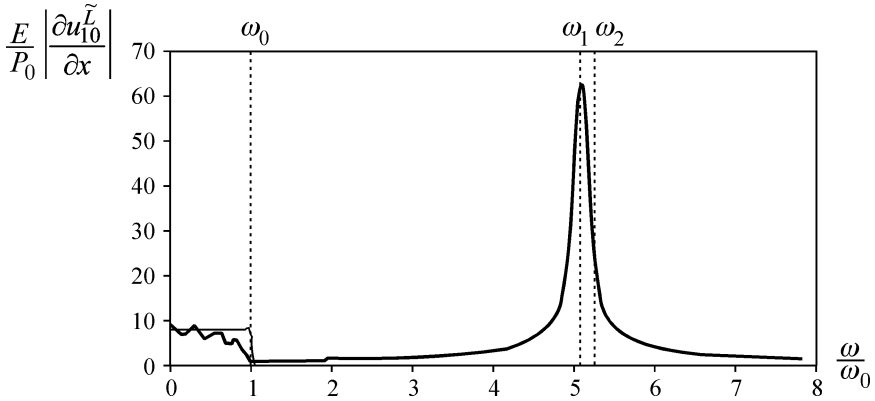

Fig. 6

Conclusions. Two groups of nonstationary waves originate in the block medium under impact excitation. The low-frequency wave moves with the velocity c_1. Its amplitude decreases as $t^{1/3}c$ with increase in time. The high-frequency wave packet propagates with the velocity c_2. Its base frequency equals natural frequency of a separate block.

To describe the low-frequency wave processes, the block medium can be replaced by the equivalent discrete system of concentrated masses elastically interacting with each other.

Thus, the structure and mechanical parameters of the block medium can be determined on the basis of experimental estimations of frequency and propagation velocity of wave groups considered above.

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