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ROCK FAILURE

ON ATTENUATION OF PENDULUM-TYPE WAVES IN A BLOCK ROCK MASS

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The data on experimental and theoretical studies of pendulum wave propagation in a system of steel bars separated by rubber interlayers are reported. The one-dimensional mathematical model, describing the given system as a chain of masses jointed by elastic springs and viscous dampers, is proposed. Numerical calculations are made, and approximate analytical estimates are obtained for attenuation rate of maximum velocities and accelerations of the masses under impact loading of the test system. It is proved that the model adequately describes the disturbance behavior in the system of bars with rubber interlayers.

Impact, block medium, pendulum-type waves, interlayers, elasticity, viscosity

When one-dimensional waves generated by impact loading propagate in a block medium, weakly-attenuating, low-frequency waves of a pendulum type arise [1 – 5]. The impact loading tests of systems composed of few blocks (19 – 20 in number), separated by rubber interlayers showed that a wave attenuation rate is mainly determined by damping properties of interlayers [6, 7]. The determination of their attenuation regularities is of practical value. The simultaneous employment of two joint variants for viscous and elastic elements, parallel and successive, is proposed to describe the behavior of interlayers between bars [6]. Below only parallel connection is used in the mathematical model. The said simplification allows obtaining simple analytical expressions to describe the attenuation of one-dimensional pendulum-type waves. To evaluate reliability of the model under consideration, the wave propagation under impact loading of a bar system with rubber interlayers was studied.

Problem Statement. The chain of masses jointed by viscoelastic interlayers (Fig. 1) is considered. The equations of movement of this system are:

$$m\ddot{u}_n = k_2(u_{n-1} - 2u_n + u_{n+1}) + \lambda_2(\dot{u}_{n-1} - 2\dot{u}_n + \dot{u}_{n+1}), \quad n = 1,\ldots,N-1,$$
$$m\ddot{u}_0 = k_2(u_1 - u_0) + \lambda_2(\dot{u}_1 - \dot{u}_0) + Q(t), \quad u_N = 0, \tag{1}$$

where u_n is the displacement of nth mass; k_2 is the rigidity of springs; λ_2 is the viscosity of dampers; m is the mass. The initial conditions are zero. The mass is fixed rigidly at the chain end. At time moment $t = 0$, a semi-sinusoidal load with an amplitude P_0 is applied to a zero mass:

$$Q(t) = P_0 \sin(\omega_* t) H_0(\pi - \omega_* t) H_0(t). \tag{2}$$

Here, ω_* is the impact frequency; H_0 is the Heaviside step function.

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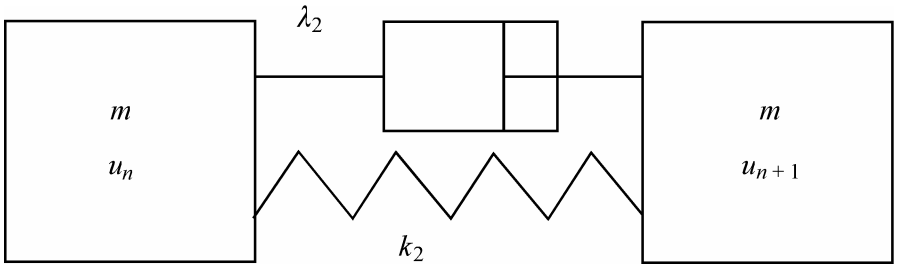

Fig. 1. Viscoelastic model of an interlayer

Asymptotic Estimates. To obtain an analytical solution, it is assumed that the mass chain is infinite. To the system of equations (1) with load (2), Laplace's transformation by time is applied:

$$f^L(p) = \int_0^\infty f(t)e^{-pt}dt\,, \quad f(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} f^L(p)\,e^{pt}dp$$

together with discrete Fourier's transformation by longitudinal coordinate *n:*

$$f^{F_d}(q) = \sum_{n=-\infty}^{n=\infty} f_n e^{iqnl}\,, \quad f_n = \frac{l}{2\pi}\int_{-\pi/l}^{\pi/l} f^{F_d}(q)\,e^{-iqnl}dq\,,$$

where l is the distance between the masses.

System (1) yields the solutions in images:

$$u^{LF_d} = \frac{Q^L}{mD(p,q)}, \quad D(p,q) = p^2 + \frac{4}{m}\sin^2\left(\frac{ql}{2}\right)(\lambda_2 p + k_2)\,, \tag{3}$$

where $D(p,q)$ is the dispersion system operator.

Evaluate frequencies ω of progressing waves in the system, depending on their length, from the dispersion equation $D(i\omega,q) = 0$:

$$\omega^{I,II} = 2i\sin^2\left(\frac{ql}{2}\right)\frac{\lambda_2}{m} \pm \omega_0 \sin\left(\frac{ql}{2}\right)\sqrt{1 - \frac{\lambda_2^2}{mk_2}\sin^2\left(\frac{ql}{2}\right)}\,, \quad \omega_0 = 2\sqrt{\frac{k_2}{m}}\,. \tag{4}$$

As the phase and group velocities of the first mode (sing "–" in formula (4)) of the dispersion curve when $q \to 0$ are equal in value:

$$c_f^I\big|_{q\to 0} = c_g^I\big|_{q\to 0} = c_1\,, \quad c_1 = l\sqrt{\frac{k_2}{m}}\,, \tag{5}$$

long waves move with no dispersion and form a quasi-front [8]. Velocity of the long waves c_1 is the same as for $\lambda_2 = 0$.

By employing the reconversion in the ray neighborhood $x = ct$ ($x = nl$) ($c = c_1 + c'$, $c' \to 0$) [8], an asymptotic form of long-wave disturbances ($|q| < \varepsilon$) is obtained as $t \to \infty$ in the quasi-front neighborhood $x = c_1 t$:

$$\ddot{u}_n = \frac{2P_0 c_1 l}{m\omega_*(\gamma t)^{2/3}} \frac{1}{\pi} \int_0^\infty \sin(æz + z^3/3) e^{-\mu z^2} z dz,$$

$$\dot{u}_n = \frac{c_1}{l}\Delta u_n = \frac{2P_0 l}{\omega_* m(\gamma t)^{1/3}} \frac{1}{\pi} \int_0^\infty \cos(æz + z^3/3) e^{-\mu z^2} dz, \tag{6}$$

$$\gamma = \frac{c_1 l^2}{8}\left(1 + \frac{3\lambda_2^2}{mk_2}\right), \quad æ = \frac{nl - c_1 t}{(\gamma t)^{1/3}}, \quad \mu = \frac{\alpha t}{(\gamma t)^{2/3}}, \quad \alpha = \frac{\lambda_2 l^2}{\lambda m}.$$

From formulas (6) it follows that at $\lambda_2 = 0$, there is no exponential attenuation ($\mu = 0$) and formulas are transformed into:

$$\ddot{u}_n = \frac{2P_0 c_1 l}{m\omega_*(\gamma t)^{2/3}} \frac{dAi(æ)}{dæ}, \quad \dot{u}_n = \frac{2P_0 l}{\omega_* m(\gamma t)^{1/3}} Ai(æ),$$
$$\gamma = \frac{c_1 l^2}{8}, \quad Ai(æ) = \frac{1}{\pi}\int_0^\infty \cos(æz + z^3/3) dz. \tag{7}$$

Here, $Ai(æ)$ is Airy's function [9]. It is explicit that the range of velocities and difference of displacements reduces proportionally to $t^{-1/3}$ with time, the maximum acceleration amplitude reduces proportionally to $t^{-2/3}$. The maximum amplitude of velocities and displacement difference is gained at the first semi-wave and equals:

$$\max_t \dot{u}_n \approx \frac{2.14\ P_0}{m\omega_*} n^{-1/3}, \quad n \to \infty. \tag{8}$$

The maximum acceleration amplitude nearby the quasi-front tends to fall with the growth of the mass number as $n^{-2/3}$:

$$\max_t \ddot{u}_n = \frac{2.07 P_0 c_1}{m\omega_* l} n^{-2/3}, \quad n \to \infty. \tag{9}$$

Viscosity contributes to additional attenuation. In the case of $\lambda_2 \neq 0$ as $n \to \infty$, formulae (6) are simplified:

$$\ddot{u}_n = \frac{4P_0 c_1}{\omega_* l \lambda_2 t} \frac{1}{\pi} \int_0^\infty \sin(\tilde{æ}z) e^{-z^2} z dz = \frac{P_0 c_1}{\omega_* l \lambda_2 \sqrt{\pi}} \frac{\tilde{æ}}{t} \exp\left(-\frac{\tilde{æ}^2}{4}\right),$$
$$\dot{u}_n = \frac{2P_0 l}{\omega_* m\sqrt{\alpha t}} \frac{1}{\pi} \int_0^\infty \cos(\tilde{æ}z) e^{-z^2} dz = \frac{P_0 l}{\omega_* m\sqrt{\pi\alpha t}} \exp\left(-\frac{\tilde{æ}^2}{4}\right), \quad \tilde{æ} = \frac{nl - c_1 t}{\sqrt{\alpha t}}. \tag{10}$$

The maximum amplitudes reduce proportionally to $n^{-1/2}$ for the mass velocities and to n^{-1} for the accelerations:

$$\max_t \dot{u}_n \to \frac{P_0}{\omega_*}\sqrt{\frac{2c_1}{\pi m l \lambda_2}} n^{-1/2}, \quad n \to \infty,$$
$$\max_t \ddot{u}_n \to \frac{P_0 k_2}{\omega_* m \lambda_2}\sqrt{\frac{2}{\pi e}}\, n^{-1}, \quad n \to \infty. \tag{11}$$

At $\lambda_2 = 0$ and $\omega_* = \omega_0$, Laplace's and Fourier's reconversion in neighborhood of a specific point $q = \pi / l$, $p = i\omega_0$ results in the asymptotic formula:

$$\ddot{u}_n \to \frac{P_0(-1)^{n+1}}{m}\sqrt{\frac{2\pi}{\omega_0 t}}\cos\left(\frac{2n}{\omega_0 t}\right)\sin(\omega_0 t)H_0(c_1 t - nl). \tag{12}$$

Hence, it is apparent that the maximum of acceleration is proportional to $n^{-1/2}$:

$$\max_t \ddot{u}_n \to \frac{P_0\sqrt{\pi}}{m}n^{-1/2}, \quad n \to \infty. \tag{13}$$

Thus, from formulas (9) and (13) it follows that at $\lambda_2 = 0$, the acceleration amplitude at the wave front $x = c_1 t$ falls quicker as compared to that beyond the front.

To determine the limits of their applicability, the values obtained were compared with the solutions of system of equations (1) by the finite difference method according to the explicit scheme of "cross" type. In Fig. 2 the graphs of the functions $f(n) = \ln(\max_t \dot{u}_n / \max_t \dot{u}_{n-1}) / \ln(n/(n-1))$, specifying the velocity attenuation rate nearby the *n*th mass are given. The velocities $\dot{u}_n$ were calculated by the finite difference method for different values of the viscosity parameter $\lambda_2 = 0$; 0.1; 0.5; 1 N·s/m^2. The rest parameters of the problem were assumed as: $k_2 = 4.4$ kg/m^2, $m = 0.3822$ kg, $l = 0.1$ m, $\omega_* = 3.14$ ms^{-1}. The dotted lines in Fig. 2 correspond to the limit values of $f(n)$, evaluated from formulas (8) and (11), $f(n) = -1/3$ and $f(n) = -1/2$, respectively. It is obvious that at $\lambda_2 = 0$ the rate of the maximum velocity amplitude fall converges to $-1/3$ with the growth of n. Given $\lambda_2 \neq 0$, the function $f(n)$ tends to $-1/2$ with the growth of *n*, and the higher is λ_2, the quicker $f(n)$ reaches its limit value. These results coincide with theoretical estimates (8) and (11).

In Fig. 3 the graphs $\max_t \dot{u}_n$, obtained by the finite difference method (thick curves) and from analytical formulas (8), (11) (thin curves) at different values of λ_2 ($\lambda_2 = 0$; 0.1; 0.5; 1 Ns/m^2) are presented. The rest parameters of the problem are the same as the abovementioned ones. The comparison of the numerical and analytical solutions shows that at $\lambda_2 = 0$ simple estimate (8) describes the velocity maximums with an error less than 5 %, starting from $n = 200$; while at $\lambda_2 \neq 0$, the higher is the viscosity factor, the closer to the action start the difference between numerical and analytical results (11) tends to reduce.

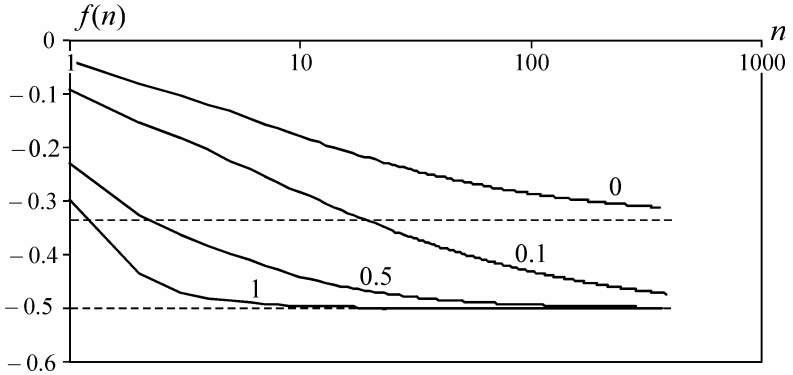

Fig. 2. Rate of mass velocity attenuation depending on their number at different values of the parameter λ_2

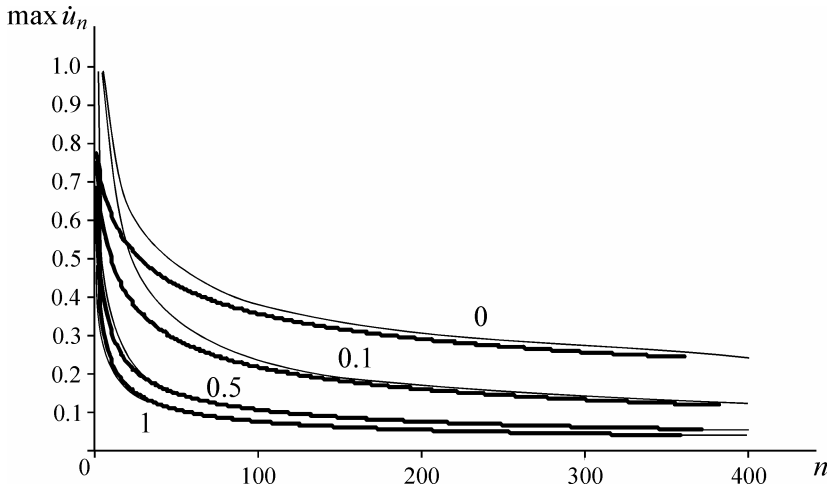

Fig. 3. Dependence of maximum velocity amplitude on mass number at different λ_2

Figure 4 demonstrates the graphs of the functions $q(n) = \ln(\max\limits_t \ddot{u}_n / \max\limits_t \ddot{u}_{n-1}) / \ln(n/(n-1))$, specifying the rate of acceleration attenuation, obtained numerically from equations (1), (2) for $\lambda_2 = 0$; 0.001; 0.01; 0.1; 0.2; 0.5 N·s/m^2. The rest parameters of the problem are the same as in Fig. 2. The dotted lines correspond to the limit values $q(n) = -1/2$, $-2/3$, -1, calculated from formulas (9), (11), (13). It is explicit that at $\lambda_2 = 0$ the attenuation rate for the acceleration maximums converges to -0.5 with the growth of n, and at $\lambda_2 \neq 0$ it tends to -1 and fits completely analytical estimates (11), (13).

In Fig. 5 the graphs are presented for the acceleration oscillograms for a mass with the number $n = 100$, calculated by the finite difference method (thick curves) and by asymptotic formula (6) (thin curves) for $\lambda_2 = 0$; 0.01; 0.1 N·s/m^2. The rest parameters have the same values as in Fig. 2. The data comparison shows that at $\lambda_2 = 0.1$ N·s/m^2 the asymptotics and numerical experimental data coincide with an accuracy of the graph construction error. As λ_2 lowers the quantitative ratio of the asymptotics and numerical data is obtainable only within the quasi-front range.

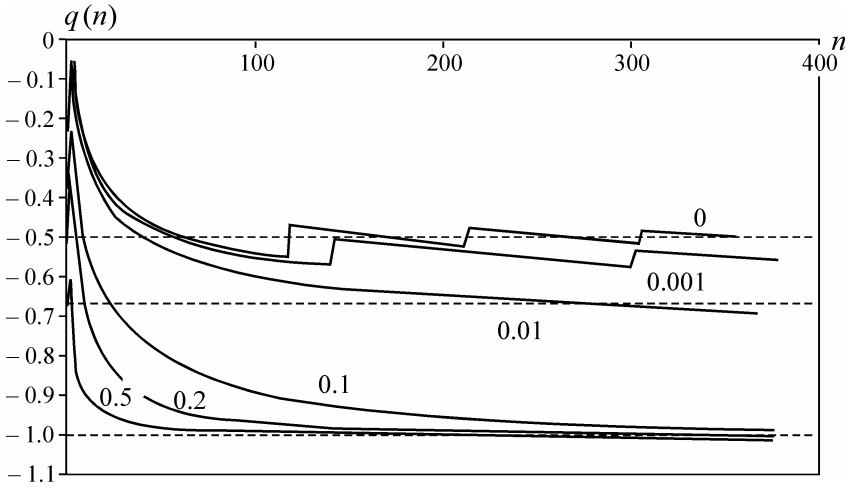

Fig. 4. Attenuation rate for acceleration maximums depending on their number at different λ_2

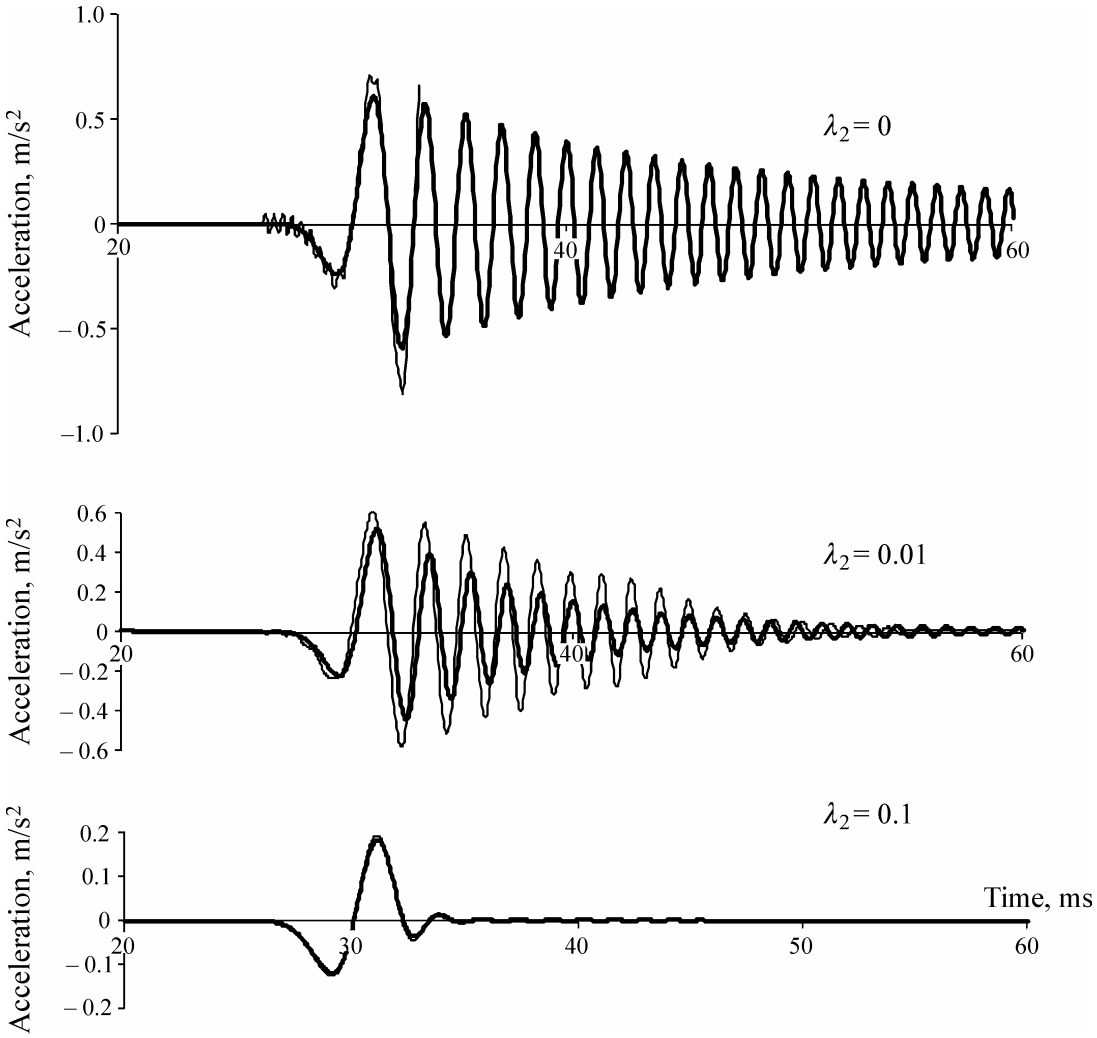

Fig. 5. Acceleration oscillograms for masses with number $n = 100$ at different values of the parameter λ_2

EXPERIMENTAL MODELING OF WAVE PROPAGATION IN COMPOUND BAR SYSTEM

The tests were performed in a bar system described in [6]. Bars separated by rubber interlayers, were placed into a vertically mounted pipe with longitudinal cuts provided for cables, recording the sensor movements. Accelerometers KD-91 built longitudinally in the bars, were employed as sensors. The accelerometers were distributed along the length of the top section of the system in $1-4$ and 11 bars. The impact was performed on the free end surface of the top bar, the impact intensity was registered by an accelerometer mounted on an impactor. The length of the system (2 m) allowed recording the bar oscillations before the wave arrival as a result of their reflection from the bottom end of the assembly.

The acceleration oscillodrams obtained in the test (thin curves) and the data calculated from equations (1), (2) (thick curves) at $k_2 = 5.5$ kg/m^2, $\lambda_2 = 0.7$ N·s/m^2, m = 0.3822 kg, l = 0.1 m, $\omega_* = 31.4$ ms^{-1} are presented in Fig. 6. The dotted curves correspond to the asymptotics (*b*). Starting from $n = 11$, a good compliance of the asymptotics and numerical data is gained. It is apparent that in the given tests, at the above mentioned values of rigidity and viscosity, the mathematical model satisfactorily describes the behavior of the bar system with the rubber interlayers under the impulsive excitation.

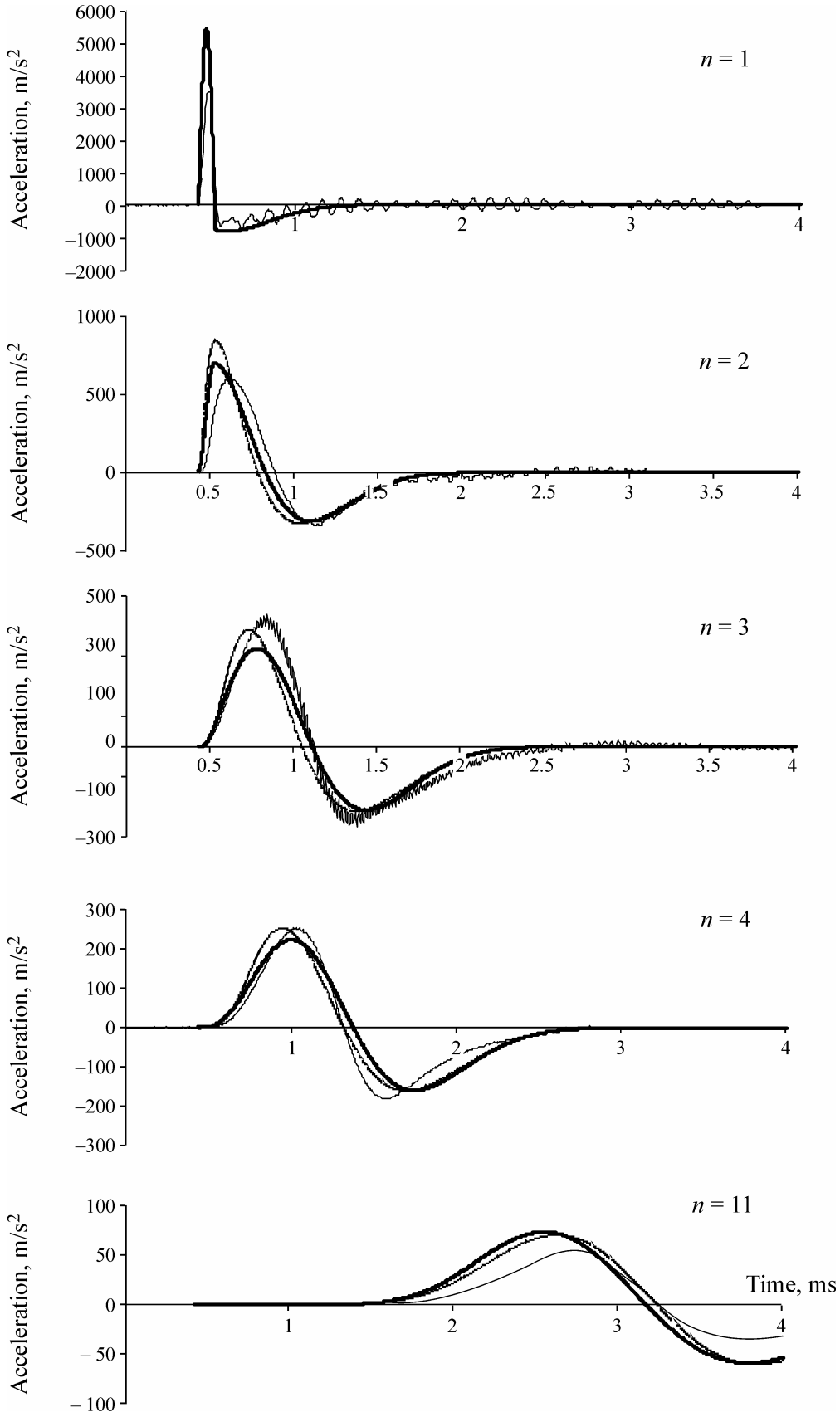

Fig. 6. Comparison of the test and theoretical data for acceleration

CONCLUSIONS

It is demonstrated by the example of a system of steel bars connected by rubber interlayers, that the model of the chain of masses jointed by viscoelastic springs adequately describes propagation of low-frequency pendulum-type waves in a one-dimensional block medium, the scope of their applicability is identified.

The numerical and analytical studies allowed establishing the asymptotic regularities for the attenuation of one-dimensional pendulum-type waves generated by the impact action in the block medium with the viscoelastic interlayers.

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