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ROCK FAILURE

EFFECT OF BOREHOLE CHARGE STRUCTURE ON THE PARAMETERS OF A FAILURE ZONE IN ROCKS UNDER BLASTING

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The calculation scheme of rock failure under explosion of a borehole charge is presented with account for density of borehole filling with an explosive, availability of air and inert intervals. According to the calculated data, the introduction of air intervals diminishes the specific explosive consumption by 9–27% depending on the explosive power intensity. The similar effect is gained by introducing an inert rod into a borehole charge.

Failure, explosion, rocks, borehole charge, air interval, charge structure

Studies of explosion effects on solid media were conducted since the establishment of the Siberian Branch of the Russian Academy of Sciences, they were supported mainly by the personal interest of Academician M. A. Lavrent'ev to explosion problems. Investigations concerned different aspects: cumulative action, development of technological schemes for welding by explosion, explosive metal processing, compacting of powder materials. M. A. Lavrent'ev paid much attention to the practical application of explosion action in the field of the national economy. Thus, models describing explosion effect in soils and rocks, were developed at the laboratory headed by V. M. Kuznetsov, Institute of Hydrodynamics, SB RAS. The fluid and fluid-solid models [1, 2] were proposed to calculate an excavating explosion, later their development was greatly advanced [3]. Based on these models, the method of the pinpoint explosion of soils and rocks was worked out, it provided theoretically 100% pinpoint throwing out of rocks [4–6]. M. A. Lavrent'ev initiated studies of the cord-like charge to be applied to blasting operations during channel construction, ice breaking and creation of vertical cylindrical cavities in plastic soils [7].

The practical schemes for underground blasting fragmentation were developed at the Institute of Mining, SB RAS. It was under the supervision of N. G. Dubynin, the new technologies were developed for underground iron ore mining in the Gornaya Shoria, by the level-and-chamber method with forced ore drawing ,[8, 9]. According to the said technology it is required to failure rock in a large portion of a block up to 80 m high by a single explosion. Moreover, the fragmentation should have of such quality that guarantees a continuous flow of the failed rock drawing. The explosion schemes ensuring the required fragmentation quality were developed and commercially introduced [10]. The blasting scheme modernization is in advance. Thus, it was proposed to use the bundle pattern of borehole charges to provide the required quality of rock fragmentation [9]. Lately, large-diameter charges were introduced in order to reduce blasting costs [11, 12].

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Along with the development and introduction of new blasting schemes, the appreciable advance has been gained in the theory of rock failure by blasting. In [13, 14] the fundamentals of the zone model for blasting of fractured rocks are worded. The calculation models, describing the development of the monolith brittle rock failure under borehole charge blasting, were elaborated at the Institute of Mining, SB RAS [15 – 19]. The effects of rock pressure, explosive and rock properties, stemming, gas penetration into the failing rock and gas emission from a borehole were considered and taken into account. The said models allow that the time dependences are derived for development of explosion cavities and failure — fragmentation zone and radial fractures characterized by radii a, b, and L_N, where N is the number of the radial zone fractures crossing the circle with radius L_N (Fig. 1).

The data obtained permit to calculate dimensions of a failure zone for a single borehole charge, where the required fragmentation quality is gained with no oversize produced. Reasoning from the above, it is possible to calculate a blasting array pitch for a large-scale explosion under the first approximation for the array pitch of $1.5 - 2\,L_N$, followed by the correction for blasting near the free surface. This is valid for estimation of the other important blasting parameter, delay time in large-scale short-delay blasting. The delay time in use should be longer than the time of the failure zone development.

Thus, the proposed calculation schemes are applicable in designing the drilling-and-blasting operations and optimal borehole structures, in particular.

In the calculation scheme of rock failure under explosion of a cylindrical charge infinite in length, the case is studied when the blasting fragmentation wave has a lower velocity as compared to the velocity of an elastic wave (weak explosion) [16]. Two stages can be singled out in the fragmentation process developing under blasting.

At the first stage the rock fails in the shear-fragmentation wave having the velocity higher as compared to the maximum velocity of fracture growth, $V_{\max}$. Along with the said, the deformation domain contains two zones: the shear fragmentation zone with $a(t) \le r \le b(t)$ and the elasticity zone with $r \ge b(t)$, where $a(t)$ is the radius of explosion cavity, $b(t)$ is the position of the fragmentation front, r is a current radius.

At the second stage the lowering fragmentation-front velocity and fulfillment of inequality $b \le V_{\max}$ give rise to a radial fractured zone within the domain $b(t) \le r \le l(t)$, where $l(t)$ is the radius of this zone.

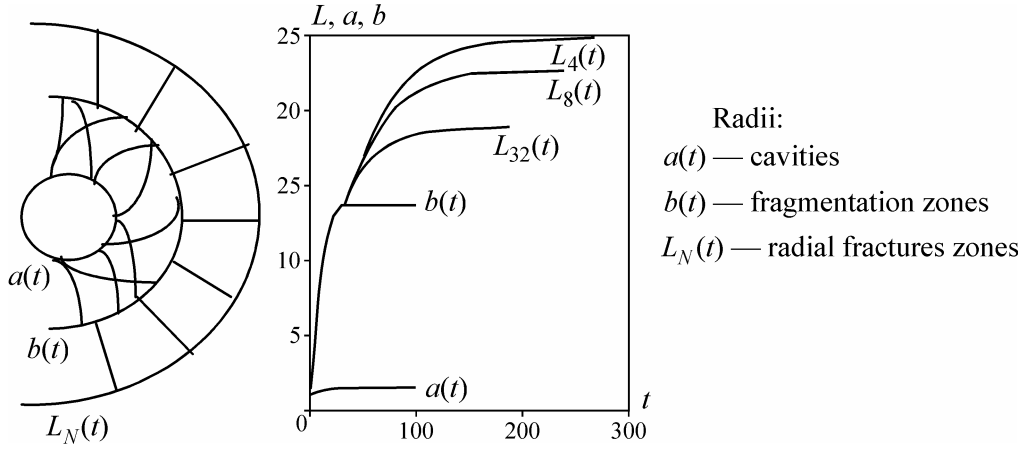

Fig. 1. Zone model of rock fracture by blasting

In the fragmented medium zone, movement is described by the equation:

$$\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial t}\right)=\frac{\partial \sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}, \tag{1}$$

where ρ is the medium density; v is the radial velocity; σ_r, σ_θ are the stress tensor components in the cylindrical coordinates (r, θ, z).

It is assumed that the effect of medium density variations in the fragmentation zone in equation (1) is small and the density ρ is equal to the initial value. From the condition of incompressibility in the fragmentation zone we have:

$$v(r,t)=\dot{a}\frac{a}{r}, \tag{2}$$

where $\dot{a}$ is the velocity of variation for the explosion cavity radius. The similar relation is obtained with account for the dilatancy effect under shear deformation of rocks in [14].

It is assumed that in the shear fragmentation zone when the explosion cavity is expanding, the stress tensor components satisfy the Mohr – Coulomb plasticity condition [20]:

$$(1+\alpha)\sigma_\theta-\sigma_r-Y=0, \tag{3}$$

$$Y=\frac{2C\cos\varphi}{1-\sin\varphi}, \quad \alpha=\frac{2\sin\varphi}{1-\sin\varphi},$$

where C is the cohesion; φ is the friction angle. Under compression of the cavity, the plasticity condition is:

$$(1+\alpha)\sigma_r-\sigma_\theta-Y=0,$$

which can be reduced to (3) with new α_1, Y_1:

$$(1+\alpha_1)\sigma_\theta-\sigma_r-Y_1=0,\ \alpha_1=-\alpha/(1+\alpha),\ Y_1=-Y/(1+\alpha).$$

At the boundary of the cavity $r=a(t)$, the radial stress within the fragmentation zone is determined by gas pressure in the cavity:

$$\sigma_r(a+0)=-p(\hat{\rho}), \tag{4}$$

that is in the adiabatic curve for detonation products of the Jones – Miller type [20], relating the gas pressure p and the density $\hat{\rho}$ of gases produced by detonation.

At the boundary of the fragmentation front $r=b(t)$, the stresses and velocities are assumed continuous:

$$\sigma_r(b-0)=\sigma_r(b+0), \tag{5}$$

$$\dot{u}_b=\dot{u}(b+0)=v(b-0)=\dot{a}\frac{a}{b}. \tag{6}$$

From (1) – (3) integrated by r, the follows the general solution to the radial stress in the fragmentation zone, it depends on the unknown functions $a(t)$, $F(t)$:

$$\sigma_r=\frac{Y}{\alpha}+\rho\left[\frac{(\dot{a}a)'_t}{g}-\frac{(\dot{a}a)^2r^{-2}}{g-2}\right]+\frac{F(t)}{r^g}, \quad g=\frac{\alpha}{1+\alpha}. \tag{7}$$

Substituting (7) into boundary conditions (4), (5) gives a system of ordinary differential equations of the first order, relative to $a(t)$ and $V(t) = \dot{a}(t)$:

$$\frac{da}{dt} = V, \quad \frac{dV}{dt} = \frac{p(\hat{\rho}) - K_3 - V^2(K_1 - K_2)}{aK_1},$$

$$K_1 = \frac{\rho}{g}\left[\left(\frac{b}{a}\right)^g - 1\right], \quad K_2 = \frac{\rho}{g-2}\left[\left(\frac{b}{a}\right)^{g-2} - 1\right], \tag{8}$$

$$K_3 = \left(\frac{Y}{\alpha} - q\right)\left(\frac{b}{a}\right)^g - \frac{Y}{\alpha}, \quad q = \sigma_r(b+0).$$

In the elasticity zone, the medium movement is described by the wave equation with respect to the radial displacement *u*, additional to the initial one:

$$\frac{\partial^2 u}{\partial t^2} = c_p^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right), \tag{9}$$

where c_p is the velocity of longitudinal waves in the elastic medium.

In the elastic zone, at the boundary with the fragmentation front $r = b(t)$, the strength condition, assumed as the Mohr – Coulomb condition, is fulfilled [15]:

$$(1+\alpha_2)\sigma_\theta - \sigma_r - Y_2 = 0, \quad \alpha_2 = \frac{\sigma_c}{\sigma_t} - 1, \quad Y_2 = \sigma_c, \tag{10}$$

where σ_c, σ_t are uniaxial compressive and tensile strengths.

Using the expression for stresses in an elastic medium in terms of displacements, the failure condition is obtained from (10) at $r = b(t)$ in the form:

$$f = A\frac{\partial u}{\partial r} - B\frac{u}{r} + \frac{Y_2 + \alpha_2 P}{\rho c_p^2} = 0, \tag{11}$$

$$A = (1+\alpha_2)\varepsilon + 1, \quad B = 1 + \varepsilon + \alpha_2, \quad \varepsilon = -\frac{\nu}{1-\nu},$$

where *P* is the rock pressure; ν is Poisson's ratio. Value *b* is calculated as a radius at which condition (11) is valid.

At the first stage when $\dot{b} > V_{\max}$, we have to find *a*(*t*), *b*(*t*), and the displacement *u*(*r*, *t*) in the elastic zone with $r > b(t)$ from equations (9), (8) and (11) under the following initial conditions:

$$t = 0: \quad a = b = a_0, \quad V = \frac{p_0 - P}{\rho c_p^2}, \quad u = 0, \quad \dot{u} = \delta(r - a_0)V,$$

where δ is the delta function.

At first, after blasting, the cavity tends to expand with no medium failure up to a moment t_1 when failure condition (11) is fulfilled on the cavity surface. The movement parameters $a(t_1)$, $b(t_1)$, $V(t_1)$ and $u(t_1, r)$ are used as initial ones to solve (8), (9) with boundary conditions (6), (11). At t_2 when the failure velocity gets lower than the maximum velocity $V_{\max}$ of fracture growth, the radial fractures zone arises between the zones of fragmentation and elasticity. The results of the first stage are used as the initial data for the second stage.

It is shown in [16] that the problem of the radial fracture development can be solved in a quasi-static statement, namely, the statics is assumed available in the elasticity zone and the fractured zone. The error of the maximum size estimation of a fractured zone as compared to the dynamic solution is, at least, within 5 %.

The velocity of the fracture front movement is calculated according to [16]:

$$\dot{l} = \begin{cases} 0 \quad , & \gamma \le \tilde{\gamma}_0, \\ \dfrac{V_{\max}}{c_0} \cdot \dfrac{1 - e^{1-\sqrt{\gamma/\tilde{\gamma}_0}}}{1 - e^{1-\sqrt{\tilde{\gamma}_1/\tilde{\gamma}_0}}}, & \tilde{\gamma}_0 < \gamma < \tilde{\gamma}_1, \\ \dfrac{V_{\max}}{c_0}, & \gamma \ge \tilde{\gamma}_1, \end{cases} \tag{12}$$

where $\tilde{\gamma}_0$, $\tilde{\gamma}_1$ is the specific surface energy of fracturing at the start of movement and at the beginning of bifurcation, γ is the current value of energy spent for formation of a unit of fractured surface, $c_0 = \sqrt{E/\rho}$, where E is Young's modulus for rock.

According to [16], the value of γ and the stress at boundary of the fragmentation zone $q = \sigma_r(b+0)$ are calculated from formulas:

$$\gamma = \frac{\pi l\,(1-\nu^2)}{2NE}\left(2P + q\frac{b}{l}\right)^2, \quad q = -\frac{\dfrac{u_b}{b}\dfrac{E}{1+\nu} + \dfrac{2Pl(1-\nu)}{b}}{1+(1-\nu)\ln(l/b)}, \tag{13}$$

$$u_b = u_{b2} + \frac{a^2 - a_2^2}{2b}, \tag{14}$$

where a_2 u_{b2} are the radius of cavity and the displacement at the failure boundary at the time moment t_2, respectively. As in the fractured zone $\sigma_\theta = 0$, Mohr – Coulomb criterion (10) for medium failure is $\sigma_r(b+0) + Y_2 = 0$, fragmentation in the core zone of radial fractures occurs under fulfillment of the condition $r = b(t)$ at the boundary:

$$f = E\frac{\partial u}{\partial r} + [Y_2(1-\nu) - P(1-2\nu)](1+\nu) = 0. \tag{15}$$

At $\sigma_r(b+0) = -Y_2$ the fragmentation front is immovable (b = const).

The described scheme is supplemented with calculations of packing and movement of borehole charge stemming, account for redistribution of gas pressure in explosion cavity and its further emission from a borehole in the case of the stemming outburst [17]. The studies also concerned the effect of explosive gas penetration into the failing medium on the failure dynamics [19].

To provide practical application of the proposed calculation approach for the parameters of the borehole charge pattern under large-scale blasting, the software in DELFI system with serviceable interface for input of rock and charge-design parameters has been developed to calculate the failure parameters for blasting of borehole charges under different conditions.

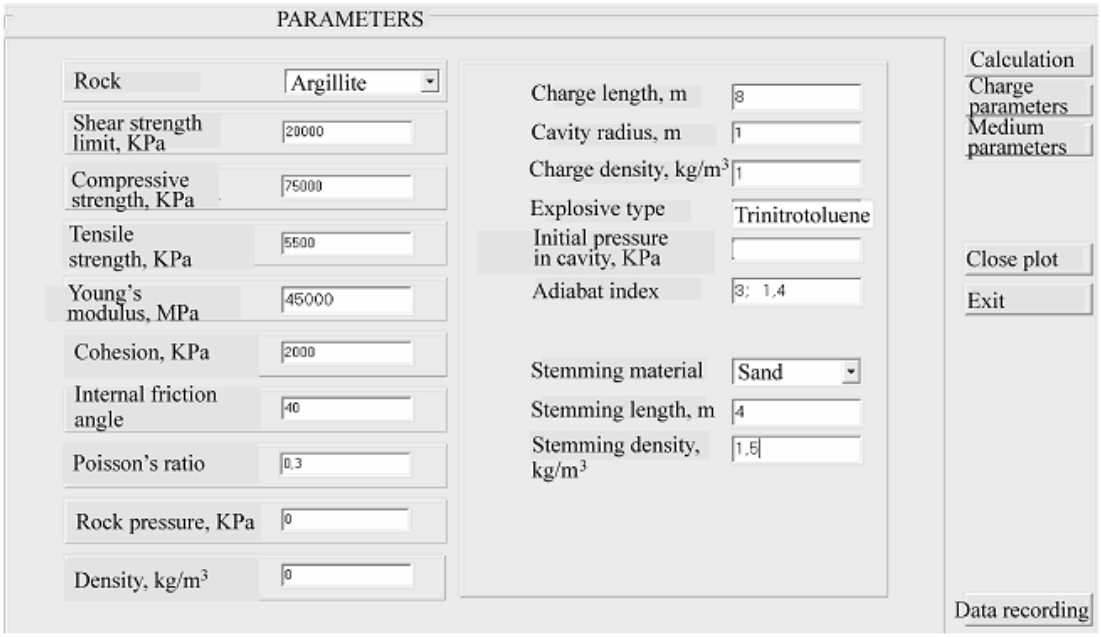

Fig. 2. Data input window in software for calculation of rock mass failure by borehole charge blasting

The software was employed to estimate the effect of charge structure on efficiency of the borehole charge blasting. Different charge structures: complete/incomplete filling with an explosive with/without air- and inert-intervals, are applied to break rocks by borehole blasting. To study the effect of a charge structure, the calculations were performed for failure zones created by blasting charges with a preset mass, placed into a borehole with uniformly distributed air- and inert-intervals. Depending on the degree of a borehole filling, the average initial gas pressure after detonation and the borehole filling with it were calculated. Evolution of the gas pressure was described by the detonation adiabat. The charge structure efficiency was estimated by the specific explosive consumption, equal to the ratio of the linear explosive weight to the linear rock volume crushed down to a prescribed size.

The calculations were performed by the three-zone model for the fragmentation effect of an elongated charge blasting with consideration for the fragmented medium movement nearby the charge, development of a radial-fractured zone and dynamics of an external elastic zone. The description of the radial fractures based on the brittle failure theory is the specific feature of the above calculation scheme as compared to the before developed ones.

The pressure of detonation products in the failure process was determined by a two-section adiabat, similar to that used for the calculation of pressure under trotyl blasting (Jones-Miller adiabat) [20]. Under high gas pressure and density $\hat{\rho}$, the pressure p tends to lower with diminishing density starting from the initial value $\hat{\rho}_0$ according to the adiabat law with the index $\gamma_1 = 3$; when the density is lower than $\hat{\rho}_* = 400$ kg/m^3, the index lowers to $\gamma_2 = 1.27$. Within the whole density range:

$$p(\rho) = \begin{cases} p_0(\hat{\rho}/\hat{\rho}_0)^{\gamma_1}, & \hat{\rho} \geq \hat{\rho}_*, \\ p_0(\hat{\rho}_*/\hat{\rho}_0)^{\gamma_1+\gamma_2}, & \hat{\rho} \leq \hat{\rho}_*. \end{cases}$$

The initial gas pressure p_0 was evaluated based on the term that the gas work under its adiabatic expansion from the initial pressure to the atmospheric pressure p_{00} amounts to the blasting heat of the used explosive. Under this assumption p_0 is found in the dimensionless form $x = p_0 / p_{00}$ like the solution of the following equations:

when $\hat{\rho}_0 > \hat{\rho}_*$

$$\frac{\hat{\rho}_0 Q\eta}{p_{00}} - x\left[\frac{1}{\gamma_1 - 1} + \frac{(\gamma_1 - \gamma_2)}{(\gamma_1 - 1)(\gamma_2 - 1)}\left[\frac{\hat{\rho}_*}{\hat{\rho}_0}\right]^{\gamma_1 - 1}\right] + \frac{x^{1/\gamma_2}}{\gamma_2 - 1}\left[\frac{\hat{\rho}_*}{\hat{\rho}_0}\right]^{\frac{\gamma_1 - \gamma_2}{\gamma_1}} = 0;$$

when $\hat{\rho}_0 \le \hat{\rho}_*$

$$\frac{\hat{\rho}_0 Q(\gamma_2 - 1)\eta}{p_{00}} - x + x^{1/\gamma_2} = 0.$$

In these equations, Q is the specific heat of blasting explosive, η is the coefficient of power losses, the calculation of which is not assumed in the theory, for example, losses of seismic wave energy when in use are quasi-static explosion models. In [20] it is assumed that $\eta = 0.333$.

The calculations were performed for an elongated borehole charge with no account for gas emission through a borehole mouth. The borehole radius a_0, the inert rod radius a_1, introduced to describe the distributed volume of inert intervals in a charge, the charge weight m per unit length and the specific blasting heat Q were varied.

The initial and current gas densities for detonation products were assessed by m and the explosion cavity radius:

$$\hat{\rho}_0 = \frac{m}{\pi(a_0^2 - a_1^2)}, \quad \hat{\rho} = \frac{m}{\pi(a^2 - a_1^2)}.$$

The rest parameters of the problem, characterizing deformation and strength properties of rocks, had averaged values for rocks of the categories V – VIII by M. M. Protod'yakonov.

The calculations proved the known fact that introduction of an air interval into a charge reduces the specific explosive consumption. In Fig. 3*a*, *b* the curves for the specific explosive consumption for two borehole sizes with radii of 0.11 and 0.20 m without inert intervals ($a_1 = 0$). The charge weights and the specific explosive heat were varied (curves *1 – 3*, corresponding to Q = 3, 4, 5 MJ/kg). It is explicit that there are optimal charge weights, providing the minimum specific explosive consumption. This effect is most pronounced for explosives with high explosive heat. The reduction in the specific consumption of explosives with Q = 3, 4, 5 MJ/kg in the optimal variant is in average 9, 16 and 27 % as compared to a charge with no air interval.

The quantitative data on the ratio of the specific consumptions on blasting of borehole charges at their complete and optimal filling are summarized in Table 1 for different borehole radii and explosives.

Introduction of inert intervals can contribute to the reduction in the specific consumption; it is explicit in Fig. 4, where the specific explosive consumption is cited for the borehole charge 0.20 m in radius, and 48 kg of explosives per 1 m length is provided (196 kg consumption for complete filling).

It is obvious from the graphs that the introduction of the inert rod with the radius a_1 can diminish the specific consumption. The reduction is maximal for explosives with lower exploding heat.

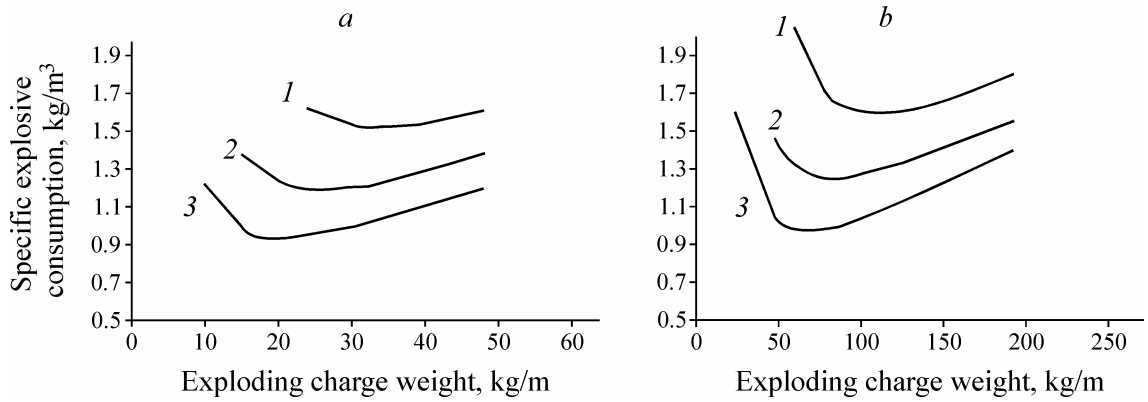

Fig. 3. Dependence of the specific explosive consumption during breakage by borehole charges with air intervals on the explosive weight per unit length: *a*, *b* — borehole diameters of 0.22 and 0.4 m, respectively; *1 – 3* refer to explosives with specific explosion heat of 3, 4, 5 MJ/kg

It is essential to note that with this charge-design optimization, boreholes appeared to be charged by half or one third, thus, resulting in the increased drilling costs. That is the reasonable substantiation why the rational charge design should be established with the help of the developed calculation model, taking into account the costs of drilling and explosives.

TABLE 1

Borehole radius, m	Specific explosion heat, MJ/kg	Linear explosive mass, kg		Optimal mode of borehole filling with explosive, %	Specific consumption for filling, kg/m^3		Reduction in consumption for optimal filling, %
		maximum	optimal		maximum	optimal	
	3	12	8	67	1.7	1.53	10
0.05	4	12	6	50	1.47	1.2	18
	5	12	5	42	1.32	0.94	29
	3	48	32	67	1.61	1.52	6
0.11	4	48	24	50	1.38	1.19	14
	5	48	20	42	1.2	0.93	22
	3	192	112	58	1.7	1.52	11
0.20	4	192	70	36	1.47	1.2	17
	5	192	60	31	1.33	0.94	29

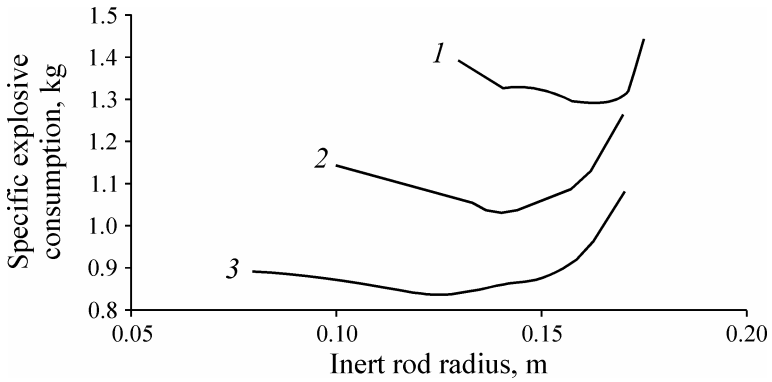

Fig. 4. Effect of inert rod size in a borehole 0.2 m in diameter on the specific explosive consumption. The charge weight in the borehole is 48 kg/m, curves *1 – 3* are described in Fig. 3

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