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A NEW EXAMPLE OF A FLEXIBLE POLYHEDRON

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1. Introduction. A surface in the 3-dimensional Euclidean space constituted by a finite set of polygons is said to be a *polyhedron*. The polygons are referred to as *faces* of the polyhedron and the sides of the polygons are referred to as its *edges* . We suppose that exactly two faces are adjacent to every edge.

Shape and size of the faces will be considered to be unchangeable, i. e. the faces will be considered as made from solid plates. On the contrary, suppose we can vary dihedral angles of our polyhedron. We'll call our polyhedron a *flexible* one, if it is possible to change dihedral angles continuously in such a way as to change the spatial shape of the polyhedron.

2. Survey of evolution of theory of flexible polyhedrons. One can formulate the Definition 10 from Book XI of Euclid's Elements [9] as follows: "Equal and similar solid figures are those contained by similar planes equal in multitude and magnitude". Some authors use this fact to prove that Euclid have passed through the notion of flexible polyhedron.

The first rigorous result on flexible polyhedrons was obtained by A. L. Cauchy in 1813. In particular he has proved that each convex polyhedron is not a flexible one [6]. Answering the question whether non convex polyhedron can be flexible, in 1897 R. Bricard have constructed examples of flexible octahedrons (i. e. polyhedrons with 6 vertexes, 12 edges and 8 faces) [5]. All of them have points of self-intersection. The problem on existence of a flexible polyhedron without self-intersection remains open for a long time even though it was interesting for such outstanding mathematicians as Henri Lebesgue [12] and

A. D. Aleksandrov [1]. Many mathematicians were sure that it has the negative answer. Nevertheless in 1976 R. Connelly [7], [8] have obtained the positive answer. Soon K. Steffen has simplified Connelly's example and constructed a flexible polyhedron without self-intersection with only 9 vertexes (only one more then the cube has) (see [3]). In 1994 I. Maksimov has announced that the number 9 can not be replaced by any smaller one [13].

3. Applications of flexible polyhedrons. In the paper [9] R. Connelly discusses applications of flexible polyhedrons to building mechanics. It is based on the intuitively clear reason that each construction made from prefabricated ferro-concrète items can be regarded as polyhedron with rigid items-faces and changeable dihedral angles at joints-edges.

In the articles [4] and [10] applications of flexible octahedrons to stereo chemistry are discussed. The idea is that the carbon skeleton of the cyclohexane molecule may be represented by a spatial hexagon with prescribed sides and angles. Replacing each pair of sides with common vertex by the rigid triangle with the same vertexes we obtain an octahedron. Thus the problem weather the spatial hexagon is rigid or flexible is equivalent to the problem weather the octahedron is flexible.

4. Open problems. By far the main goal is to obtain a criterion for flexibility of polyhedrons, i. e. to obtain a rule which will allows us to conclude after some finite set of operations involving finite number of sizes of our polyhedron whether it is flexible or not. As there are no direct approaches to this problem, we'll discuss the following partial problems:

- Does there exist flexible polyhedron in many dimensional space?
- Is the set of flexible polyhedrons semialgebraic one in the space of all polyhedrons of a prescribed combinatorial type, i. e. is it defined by a finite set of polynomial

equations and inequalities?

- Is it true that each flexible polyhedron preserves volume during the process of bending [9]? (The positive answer for a class of combinatorial one-parameter flexible polyhedrons was announced by I. Kh. Sabitov [14] in 1994. A flexible polyhedron is said to be combinatorial one-parameter if it fails to be flexible after we fix spatial distance between two its vertexes that were not joined by an edge.)
- Which functions except volume can be preserved by all flexible polyhedrons? Can the mean curvature play this role [2]?

Obviously, studying these problems it is useful to have examples of flexible polyhedrons. Connelly's and Steffen's polyhedrons are very elegant, but they are based on the Bricard's octahedrons. For better understanding of the problems it is desirable to have examples based on other ideas.

<u>5. Formulation of the result.</u> In the present report we'll explain a new example of a flexible polyhedron (with self-intersection), that is a piecewise linear realization (but not an immersion) of torus. The Bricard's octahedrons are not used in the construction. Flexibility of the polyhedron is deduced from the purely analytical reason — from the fact that every rational function can be expanded into a sum of proper fractions. We shall verify that, under some relations between parameters of the construction, the polyhedron is flexible. It turns out that precisely with these values of parameters our polyhedron preserves volume and mean curvature during a bending. For more details see [2].

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