Closures of finite permutation groups (Lectures 3,4)

Ilia Ponomarenko[a] and Andrey Vasil'ev[b]

[a] St.Petersburg Department Steklov Mathematical Institute, St.Petersburg, Russia
[b] Sobolev Institute of Mathematics, Novosibirsk, Russia

The G2A2-Summer School, Novosibirsk
03.08-16.08.2025

Closures of transitive abelian groups

Theorem. A transitive abelian group is k-closed for every $k \geq 2$.

Proof. Let $G \leq \mathrm{Sym}(\Omega)$ be abelian and transitive on Ω.
By Wielandt's criterion it suffices to prove that G is regular
(regular $\implies G_x = 1$ for some $x \in \Omega \implies b(G) = 1$).
Let x, y be arbitrary elements of Ω.
G is transitive $\implies$ there is $g \in G$ such that $y = x^g$.
It follows that $G_y = g^{-1} G_x g$, because $y^h = y \Leftrightarrow (x^g)^h = x^g$.
G is abelian $\implies G_y = G_x$.
Since x, y are arbitrary, it follows that $G_x = 1$, as required.

Example (Churikov). $G = \langle (1,2)(3,4), (12)(5,6) \rangle \simeq C_2 \times C_2$
G has three orbits $\{1,2\}, \{3,4\}, \{5,6\}$ and
$G < G^{(2)} = \langle (1,2), (3,4), (5,6) \rangle \simeq C_2 \times C_2 \times C_2$.

Exercise 1. a) Check the statements in the example;
b) For every k, find an (abelian) group G which is not k-closed.

Closure of a direct product

Theorem. Let $G_1 \leq \mathrm{Sym}(\Delta_1)$, $G_2 \leq \mathrm{Sym}(\Delta_2)$, and let $G_1 \times G_2$ be a permutation group acting on the disjoint union $\Delta_1 \cup \Delta_2$. Then for each $k \geq 1$,

$$(G_1 \times G_2)^{(k)} = G_1^{(k)} \times G_2^{(k)}. \quad (1)$$

Proof. We prove only the $\leq$-part. Let $G = G_1 \times G_2$ and $H = G^{(k)}$. Since $G \approx_k H$, they have the same orbits, in particular, $H \leq \mathrm{Sym}(\Delta_1) \times \mathrm{Sym}(\Delta_2)$. Thus, $h = (h_1, h_2)$, $h_i \in \mathrm{Sym}(\Delta_i)$.

Recall the closure argument from Lecture 2:
$h \in H$ iff for any $x \in \Omega^k$ there is $g = (g_1, g_2) \in G$ s.t. $x^h = x^g$.

Assume that $x = (x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in \Delta_1$. Then $(x_1, \ldots, x_k)^{h_1} = x^h = x^g = (x_1, \ldots, x_k)^{g_1}$. Applying the closure argument once again, we have $h_1 \in G_1^{(k)}$ and, similarly, $h_2 \in G_2^{(k)}$.

Exercise 2. Prove the $\geq$-part.

Closures of abelian groups

Theorem. Let $G \leq \mathrm{Sym}(\Omega)$, $\Omega = \Delta_1 \cup \ldots \cup \Delta_t$, $\Delta_i \in \mathrm{Orb}(G)$, and $G_i = G^{\Delta_i}$ transitive constituents of G. If G is abelian, then $G^{(k)} \leq G_1 \times \cdots \times G_t$, in particular, $G^{(k)}$ is abelian for every $k \geq 2$.

Proof. $G \leq G_1 \times \ldots \times G_t \implies G^{(k)} \leq (G_1 \times \ldots \times G_t)^{(k)}$.

By the previous theorem, $G^{(k)} \leq G_1^{(k)} \times \cdots \times G_t^{(k)}$.

$G_i \leq \mathrm{Sym}(\Delta_i)$ are transitive and abelian $\implies G_i = G_i^{(k)}$ for all i.

Example. $G = \langle (1,2) \cdots (2t-1, 2t) \rangle = G^{(2)} \neq$
$\neq \langle (1,2) \rangle \times \cdots \times \langle (2t-1, 2t) \rangle = G_1 \times \cdots \times G_t$.

The explicit criterion for abelian group to be k-closed is unknown, see [Churikov–Praeger, 2021] and [Churikov–Ponom., 2022].

Theorem. For a fixed k, the k-closure of an abelian group $G \leq \mathrm{Sym}(\Omega)$ can be found in time polynomial in n, where $n = |\Omega|$.

Proof. Apply the Babai-Luks algorithm for $H = G_1 \times \cdots \times G_t$.

Systems of blocks

Let $G \leq \mathrm{Sym}(\Omega)$ and $\Delta \subseteq \Omega$.

- $G_{(\Delta)} = \{x \in G \mid \delta^x = \delta \text{ for all } \delta \in \Delta\}$ is pointwise stabilizer
- $G_{\{\Delta\}} = \{x \in G \mid \Delta^x = \Delta\}$ is setwise stabilizer of Δ in G.

A subset Δ of Ω is called a block for G if for each $x \in G$ either $\Delta^x = \Delta$ or $\Delta^x \cap \Delta = \varnothing$.

Every one-element subset of Ω and Ω itself are blocks for G, they are called trivial blocks. A block called minimal, if it is nontrivial and does not include any other nontrivial block.

Lemma. Let $G \leq \mathrm{Sym}(\Omega)$ be transitive, Δ a block for G. Then

1. Δ^x is a block for every $x \in G$.
2. the blocks from $\Sigma = \{\Delta^x \mid x \in G\}$ form a partition of Ω called the system of blocks for G, containing Δ.
3. $|\Sigma| = |G : G_{\{\Delta\}}|$ and $|\Delta| = |G_{\{\Delta\}} : G_\alpha|$, where $\alpha \in \Delta$.

Inclusion to a wreath product

Let $G \leq \mathrm{Sym}(\Omega)$ be transitive, Σ a system of blocks for G, $\Delta \in \Sigma$.

Define $\rho : G_{\{\Delta\}} \to \mathrm{Sym}(\Delta)$ by $x \mapsto x^{\Delta}$ with $\delta^{x^{\Delta}} := \delta^{x}$. Then ρ is a homomorphism, $\ker \rho = G_{(\Delta)}$, $G^{\Delta} := (G_{\{\Delta\}})^{\rho} \simeq G_{\{\Delta\}}/G_{(\Delta)}$.

Define $\sigma : G \to \mathrm{Sym}(\Sigma)$ by $x \mapsto x^{\Sigma}$ with $\Delta^{x^{\Sigma}} := \Delta^{x}$. Then σ is a homomorphism, $G^{\Sigma} := G^{\sigma} \simeq G/G_{\Sigma}$, where

$$G_{\Sigma} = \ker \sigma = \bigcap_{\Delta \in \Sigma} G_{\{\Delta\}} \simeq \underbrace{G^{\Delta} \times \cdots \times G^{\Delta}}_{|\Sigma| \text{ times}}.$$

Theorem. There is a bijection between Ω to $\Delta \times \Sigma$ which induces an embedding of G into the (imprimitive) wreath product $G^{\Delta} \wr G^{\Sigma}$ of G^{Δ} and G^{Σ}, that is the semidirect product $G_{\Sigma} \rtimes G^{\Sigma}$.

Here the action of G^{Σ} on $G_{\Sigma} = G^{\Delta} \times \cdots \times G^{\Delta}$ is given by $(x_1, \ldots, x_s)^{g} = (x_{1\sigma(g^{-1})}, \ldots, x_{s\sigma(g^{-1})})$, where $s = |\Sigma|$.

Primitivity

Let $G \leq \mathrm{Sym}(\Omega)$ be transitive, $|\Omega| \geq 2$. A group G is imprimitive if it has a nontrivial system of blocks, and primitive otherwise.

Corollary. Every transitive permutation group G is permutation isomorphic to a subgroup of an iterated wreath product $G_1 \wr \cdots \wr G_t$, where $G_i = G^{\Delta_i}$ are primitive groups isomorphic to some sections of a group G.

Recall that H is a section of an (abstract) group G if there is a subgroups K and L of G such that $K \trianglelefteq L$ and $L/K \simeq H$.

Lemma (Criterion of primitivity). A transitive permutation group G is primitive if and only if a point stabilizer is a maximal subgroup of G.

Corollary. Every primitive p-group is a regular group of order p.

Exercise 3. Prove the lemma and its corollary.

Closure of an imprimitive wreath product

[Kalužnin–Klin, 1976]: Let $G \leq \mathrm{Sym}(\Omega)$ be transitive group with a nontrivial system of blocks Σ and $\Delta \in \Sigma$. Then for each $k \geq 2$,

$$G^{(k)} \leq (G^{\Delta} \wr G^{\Sigma})^{(k)} = (G^{\Delta})^{(k)} \wr (G^{\Sigma})^{(k)}. \qquad (2)$$

Proof. The inclusion follows from the above arguments. The equality can be derived applying the closure argument.

Corollary 1. For $k \geq 2$, the k-closure of a transitive group G can be embedded into an iterated wreath product of primitive sections of G.

Corollary 2. For every prime p and integer $k \geq 2$, the k-closure of a p-group is a p-group.

Closures of nilpotent groups

Theorem. Let $G_1 \leq \mathrm{Sym}(\Delta_1)$, $G_2 \leq \mathrm{Sym}(\Delta_2)$, and let $G_1 \times G_2$ be a permutation group acting on the Cartesian product $\Delta_1 \times \Delta_2$. Then for each $k \geq 2$,

$$(G_1 \times G_2)^{(k)} = G_1^{(k)} \times G_2^{(k)}. \qquad (3)$$

Recall that a finite group G is nilpotent $\iff$ G is the direct product of its Sylow subgroups.

[Churikov, 2021]: Let $G \leq \mathrm{Sym}(\Omega)$ be a nilpotent group. Then for each $k \geq 2$, $G^{(k)}$ is the direct product of the k-closures of the Sylow subgroups of G, in particular, $G^{(k)}$ is nilpotent.

[Ponom., 1994]: For a fixed k, the k-closure of an nilpotent group $G \leq \mathrm{Sym}(\Omega)$ can be found in time polynomial in n, where $n = |\Omega|$.

Solvable groups

A group G is solvable (supersolvable, respectively) if it has a series

$$G = G_0 \geq G_1 \geq \ldots \geq G_{n-1} \geq G_n = 1,$$

where $G_i \trianglelefteq G_{i-1}$ ($G_i \trianglelefteq G$ respectively) and G_{i-1}/G_i is cyclic for every $i = 1, \ldots, n$.

The Babai-Luks argument works inside solvable groups. The problem is that there are solvable, supersolvable and even metacyclic groups such that their 2-closures include large nonabelian composition factors.

Exercise 4. If $G \leq \mathrm{Sym}(\Omega)$ is k-transitive, then $G^{(k)} = \mathrm{Sym}(\Omega)$.

Example. For a field $\mathbb{F}_p$, where p is a prime, $G = \mathrm{AGL}_1(p) = \{x \mapsto ax + b, a \in \mathbb{F}_p^\times, b, x \in \mathbb{F}\} \simeq \mathbb{F}^+ \rtimes \mathbb{F}^\times$ is 2-transitive, so $G^{(2)} = \mathrm{Sym}(p)$.

Closures of groups with restricted composition factors

[O'Brien–Ponom.–V.–Vdovin, 2022]: If G is a solvable group, then $G^{(k)}$ is solvable for $k \geq 3$.

[Ponom.–Skresanov–V., 2025]: If G is an $\mathrm{Alt}(d)$-free group with $d \geq 25$, then $G^{(k)}$ is $\mathrm{Alt}(d)$-free group for $k \geq 4$.

(Abstract) group is called $\mathrm{Alt}(d)$-free, $d \geq 5$, if it does not contain section isomorphic to the alternating group of degree d.

$\mathrm{Alt}(d)$-free groups are 'good enough' for the Babai–Luks algorithm, as it follows from

[Babai–Cameron–Palfy, 1982]: The order of a primitive $\mathrm{Alt}(d)$-free group of degree n is at most $n^{f(d)}$.

Wreath products in product action

As a direct product of groups can act naturally in two ways, a wreath product $H \wr L$ of groups $H \leq \mathrm{Sym}(\Delta)$ and $L \leq \mathrm{Sym}(\Sigma)$ can act not only on $\Delta \times \Sigma = \bigsqcup_{\sigma \in \Sigma} \Delta_\sigma$ but on $\Delta^\Sigma = \prod_{\sigma \in \Sigma} \Delta_\sigma$.

Let $H \uparrow L$ denote the permutation group induced by the action of $H \wr L$ on Δ^Γ, this action is called a product action. As an abstract group $H \uparrow L$ is isomorphic to $H^{|\Sigma|} \rtimes L$, where $H^{|\Sigma|}$ acts on Δ^Σ componentwise and L permutes the coordinates.

The complete analog of the Kalužnin–Klin theorem for closures of imprimitive wreath product does not hold for wreath product in product action:

Example. $(\mathrm{Sym}(2) \uparrow \mathrm{Alt}(3))^{(2)} = \mathrm{Sym}(2) \uparrow \mathrm{Sym}(3) \not\leq$
$\not\leq \mathrm{Sym}(2) \uparrow \mathrm{Alt}(3) = \mathrm{Sym}(2)^{(2)} \uparrow \mathrm{Alt}(3)^{(2)}$.

Particular results for this case: [Praeger–Saxl, 1992], [Evdokimov-Ponom., 2001], [O'Brien–Ponom.–V.–Vdovin, 2022].

Closure with respect to partitions

Let $\Omega^{[m]}$ be the set of all ordered partitions Π of Ω with $|\Pi| \leq m$.

For $G \leq \mathrm{Sym}(\Omega)$, we denote by $G^{[m]}$ the largest permutation group on Ω having the same orbits as G in its induced action on $\Omega^{[m]}$.

$$\mathrm{Sym}(\Omega) = G^{[1]} \geq G^{[2]} \geq \ldots \geq G^{[m]} \geq \ldots \geq G^{[|\Omega|]} = G.$$

It is again a closure operator and the closure argument holds.

$G^{[m+1]} \leq G^{(m)}$ (but not necessarily $G^{[m]} \leq G^{(m)}$).

Indeed, given $\alpha = (\alpha_1, \ldots, \alpha_m) \in \Omega^m$, take $\Pi = (\Pi_1, \ldots, \Pi_{m+1})$, where $\Pi_i = \{\alpha_i\}$, $i = 1, \ldots, m$, and $\Pi_{m+1} = \Omega \setminus \{\alpha_1, \ldots, \alpha_m\}$, and apply the closure arguments: if $f \in G^{[m+1]}$, then there is $g \in G$ with $\Pi^g = \Pi^f$, so $\alpha^f = \alpha^g$ and $f \in G^{(m)}$, as required.

Closures of wreath products in product action

[Ponom.–V., 2021]: Let H and L be permutation groups and $k \geq 2$ an integer. Then

$$(H \uparrow L)^{(k)} = H^{(k)} \uparrow L^{[m]},$$

where $m = \min\{m_k, d\}$ with $m_k = |\operatorname{Orb}_k(H)|$ and $d = \deg L$.

Corollary. In the same notation, $(H \uparrow L)^{(k)} \leq H^{(k)} \uparrow L^{(k)}$, unless $k = 2$ and H is 2-transitive.

Exercise 5. Prove the corollary.

Hint: Prove that $m_k \geq k + 1$ if $k \geq 3$ or $H^{(2)} \neq \operatorname{Sym}(n)$, and apply $L^{[k+1]} \leq L^{(k)}$.

Groups preserving a product decomposition

A group $G \leq \mathrm{Sym}(\Omega)$ preserves a nontrivial product decomposition if there are Δ and Σ with $|\Delta| > 1$ and $|\Sigma| > 1$ and a bijection between Ω and Δ^Σ inducing an embedding of G into a wreath product $G^\Delta \uparrow G^\Sigma$, where $G^\Delta \leq \mathrm{Sym}(\Delta)$ and $G^\Sigma \leq \mathrm{Sym}(\Sigma)$.

Note that G^Δ and G^Σ are sections of G.

A primitive group G is called nonbasic if it preserves some nontrivial product decomposition, and basic otherwise.

If $k \geq 3$ and $G \leq \mathrm{Sym}(\Omega)$ is a nonbasic group preserving a product decomposition $\Omega = \Delta^\Sigma$, then $G^{(k)} \leq (G^\Delta)^{(k)} \uparrow (G^\Sigma)^{(k)}$.

Proof. $G \leq G^\Delta \uparrow G^\Sigma \implies G^{(k)} \leq (G^\Delta \uparrow G^\Sigma)^{(k)}$. For $k \geq 3$, $(G^\Delta \uparrow G^\Sigma)^{(k)} \leq (G^\Delta)^{(k)} \uparrow (G^\Sigma)^{(k)}$.

Reduction to basic groups

A: If G is a solvable group, then $G^{(k)}$ is solvable for $k \geq 3$.

B: If G is an Alt(d)-free group with $d \geq 25$, then $G^{(k)}$ is Alt(d)-free group for $k \geq 4$.

The main achievement of the today lecture is that we are ready to prove the following

Claim. It suffices to prove (A) and (B) for basic groups.

Complete classes of groups

According to [Wielandt, 1964], a class $\mathfrak{X}$ of (abstract) groups is said to be complete if it is closed with respect to taking

- subgroups ($H \leq G$ and $G \in \mathfrak{X} \implies H \in \mathfrak{X}$),
- quotients ($N \trianglelefteq G$ and $G \in \mathfrak{X} \implies G/N \in \mathfrak{X}$),
- extensions ($N \trianglelefteq G$ and $N, G/N \in \mathfrak{X} \implies G \in \mathfrak{X}$).

Examples of complete classes:

- all (finite) groups,
- p-groups,
- solvable groups,
- Alt(d)-free groups.

Remark. The classes of abelian and nilpotent groups are not complete, because they are not closed with respect to taking extensions. But they are closed with respect to taking direct products.

Reduction for groups from complete classes

$\mathfrak{X}_n$ is the class of permutation groups from $\mathfrak{X}$ of degree at most n.

[Ponom.–V., 2024]: Let $\mathfrak{X}$ be a complete class and $k, n \in \mathbb{N}$. Then $\mathfrak{X}_n$ is closed with respect to taking k-closures if and only if one of the following hold:

(i) $\mathfrak{X}_n$ contains k-closure of every primitive group in $\mathfrak{X}_n$ for $k \geq 2$

(ii) $\mathfrak{X}_n$ contains k-closure of every basic group in $\mathfrak{X}_n$ for $k \geq 3$.

Proof. We prove only (ii) by induction on n.

Let $\star \in \{\times, \wr, \uparrow\}$, where the signs $\times, \wr, \uparrow$ denote the operations of direct and wreath product in imprimitive action and product action, respectively. Suppose that $G \in \mathfrak{X}_n$. If G is basic, then we are done. Otherwise $G \leq H \star L$ for some permutation groups of degrees strictly less than n. Since $\mathfrak{X}$ is complete, $H, L \in \mathfrak{X}$ as sections of G. By induction, $H^{(k)}, L^{(k)} \in \mathfrak{X}$. The results of previous lectures $\implies G^{(k)} \leq (H \star L)^{(k)} \leq H^{(k)} \star L^{(k)} \implies G^{(k)} \in \mathfrak{X}_n$.

Bibliography (1)

L. Babai and E. M. Luks, *Canonical labeling of graphs*, Proc. 15th ACM STOC, 1983, pp. 171–183.

L. Babai, P. J. Cameron, and P. P. Pálfy, *On the orders of primitive groups with restricted nonabelian composition factors*, J. Algebra, 1982, **79**, no. 1, 161–168.

D. V. Churikov, *Structure of k-closures of finite nilpotent permuatation groups*, Algebra Logic, 2021, **60**, no. 2, 154–159.

D. Churikov, Ch. E. Praeger, *Finite totally k-closed groups*, Tr. Inst. Mat. Mekh., 2021, **27**, no. 1, 240–245.

D. Churikov, I. Ponomarenko, *On 2-closed abelian permutation groups*, Commun. Algebra, 2022, **50**, no. 4, 1792–1801.

S. Evdokimov and I. Ponomarenko, *Two-closure of odd permutation group in polynomial time*, Discrete Math., 2001, **235**, no. 1-3, 221–232.

L. A. Kalužnin, M. H. Klin, *On some numerical invariants of permutation groups*, Latv. Mat. Ežegod., 1976, **18**, no. 1, 81–99 (in Russian).

E. A. O'Brien, I. Ponomarenko, A. V. Vasil'ev, E. Vdovin, *The 3-closure of a solvable permutation group is solvable. J. Algebra*, 2022, **607**, 618–637.

I. Ponomarenko, *Graph isomorphism problem and 2-closed permutation groups*, Appl. Algebra Engrg. Comm. Comput., 1994, **5**, 9–22.

I. Ponomarenko, S. V. Skresanov, and A. V. Vasilev, *Closures of permutation groups with restricted nonabelian composition factors*, Bulletin of Mathematical Sciences, 2025, Online Ready, DOI:10.1142/S1664360725500122

Bibliography (2)

I. Ponomarenko and A. V. Vasil'ev, *On computing the closures of solvable permutation groups*, Internat. J. Algebra Comput., 2024, **34**, no. 1, 137–145.

C. E. Praeger, J. Saxl, *Closures of finite primitive permutation groups*, Bull. London Math. Soc., 1992, **24**, no. 3, 251–258.

A. V. Vasil'ev and I. N. Ponomarenko, *The closures of wreath products in product action*, Algebra Logic, 2021, **60**, no. 3, 188–195.

H. Wielandt, *Zusammengesetzte Gruppen endlicher Ordnung*, Vorlesung an der Universität Tübingen im Wintersemester 1963/64. In: Huppert, B., Schneider, H. (eds.) Helmut Wielandt: Mathematical Works, Vol. 1: Group theory (De Gruyter, Berlin (1994) 607–655.

H. Wielandt, *Permutation groups through invariant relations and invariant functions*, The Ohio State University (1969).