Closures of finite permutation groups (Lectures 5,6)

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Reduction for complete classes of groups

A class $\mathfrak{X}$ of groups is complete if it is closed with respect to taking subgroups, quotients, and extensions. Classes of the solvable and $\mathrm{Alt}(d)$-free groups are complete.

$\mathfrak{X}_n$ is the class of permutation groups from $\mathfrak{X}$ of degree at most n.

[Ponom.–V., 2024]: Let $\mathfrak{X}$ be a complete class and $k, n \in \mathbb{N}$, where $k \geq 3$. Then $\mathfrak{X}_n$ is closed with respect to taking k-closures if and only if $\mathfrak{X}_n$ contains k-closure of every basic group in $\mathfrak{X}_n$.

Permutation groups $\Longrightarrow$ Transitive groups $\Longrightarrow$
$\Longrightarrow$ Primitive groups $\Longrightarrow$ Basic groups

Preliminary lemma

Lemma. Let $G \leq \mathrm{Sym}(\Omega)$ and $\alpha \in \Omega$.

1. If $k \geq 2$ and G_α is $(k-1)$-closed, then G is k-closed.
2. If G_α has a faithful regular orbit, then G is 3-closed.

Proof. Let $H = G^{(k)}$. From Lecture 2:

For $k \geq 2$, if $G \approx_k H$ and $\alpha \in \Omega$, then $G_\alpha \approx_{k-1} H_\alpha$.

G_α is $(k-1)$-closed $\implies H_\alpha \leq H_\alpha^{(k-1)} = G_\alpha^{(k-1)} = G_\alpha$.

$H_\alpha \leq G_\alpha$ and $G \leq H \implies H_\alpha = G_\alpha$.

G and H have the same orbits $\implies \alpha^G = \alpha^H$. Hence

$|G| = |\alpha^G||G_\alpha| = |\alpha^H||H_\alpha| = |H| \implies G = H$ proving (1).

(2) follows from (1) or directly from Wielandt's criterion.

Solvable groups: outline of the proof

[O'Brien–Ponom.–V.–Vdovin, 2022]: If G is a solvable group, then $G^{(k)}$ is solvable for $k \geq 3$.

- Let G be a counterexample of the least possible degree.
- Since $G^{(k)} \leq G^{(3)}$ for $k \geq 3$, it suffices to prove that $G^{(3)}$ is solvable.
- By the above, G is basic, i. e. G is primitive and does not preserve any nontrivial product decomposition of Ω.
- Since G is primitive and solvable, G is affine.

Exercise 1. If a permutation group G is primitive and solvable, then $G = V \rtimes G_0$ is semidirect product of a regular normal elementary abelian p-subgroup V and a point stabilizer G_0.

Affine permutation groups

We say that $G = V \rtimes G_0 \leq \mathrm{AGL}_d(p) \leq \mathrm{Sym}(\Omega)$ is affine, if

- $\Omega = V \simeq (\mathbb{F}_p)^d$ is a linear space over a prime field $\mathbb{F}_p$
- $G_0 \leq \mathrm{GL}(V) = \mathrm{GL}_d(p)$
- G acts on V by maps of the form $x \mapsto ax + b$,
  where $a \in G_0 \leq \mathrm{GL}_d(p)$, $b \in V$.

By the lemma, it suffices to prove that $G_0 \leq \mathrm{GL}(V)$ is 2-closed.

Sometimes it is convenient to enlarge the field (if possible):

- $\Omega = V$ is a linear space over a field $\mathbb{F}_q$, $|V| = q^a = p^d$.
- $G = V \rtimes G_0 \leq \mathrm{A\Gamma L}(V) = \mathrm{A\Gamma L}_a(q)$,
  where $G_0 \leq \Gamma\mathrm{L}(V) = \Gamma\mathrm{L}_a(q) = \mathrm{GL}_a(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$
- G acts on V by maps of the form $x \mapsto ax^\phi + b$,
  where $a \in \mathrm{GL}_a(q)$, $\phi \in \mathrm{Aut}(\mathbb{F}_q)$, $a\phi \in G_0$, $b \in V$.

Primitive solvable linear groups

$G = V \rtimes G_0$ is affine and $G_0 \leq \mathrm{GL}(V)$.

- Since G is primitive, G_0 is irreducible,
- Since G is basic, G_0 is primitive.

An irreducible group $H \leq \mathrm{GL}(V)$ is imprimitive (as a linear group), if there is a subspace $U \subset V$ such that V is a direct sum of U^h, $h \in H$, and primitive otherwise.

[Yang–Vasil'ev A.S.–Vdovin, 2020]: If $H \leq \mathrm{GL}(V) = \mathrm{GL}_d(p)$ is a primitive solvable linear group, then one of the following hold:

1. H has a faithful regular orbit;
2. $H \leq \Gamma \mathrm{L}_1(p^d)$;
3. H is transitive on $V \setminus \{0\}$;
4. H lies in one of 102 known 'small' groups.

[Suprunenko, 1976]: description of max prim solv subgr in $\mathrm{GL}(V)$.

End of the proof

$G = V \rtimes G_0$ is affine and $G_0 \leq \mathrm{GL}(V) = \mathrm{GL}_d(p)$ primitive solvable.

1. G_0 has a faithful regular orbit $\implies$ G_0 is 2-closed.
2. $G_0 \leq \Gamma\mathrm{L}_1(p^d)$ $\implies$ G_0 is 2-closed [Xu–Giudichi–Li–Praeger, 2011].
3. G_0 is transitive on $V \setminus \{0\}$ and $G_0 \not\leq \Gamma\mathrm{L}_1(p^d)$ $\implies$ $p^d \in \{3^2, 5^2, 7^2, 11^2, 23^2, 3^4\}$ [Huppert, 1957].

If G is a sufficiently large basic solvable group, then G is 3-closed.

In order to complete the proof of the main theorem, we check with the help of computer computations that $G^{(3)}$ is solvable for all 'small' exceptions ($|V| = 5^{18}$ for the largest of them).

Tools: GAP packages IRREDSOL and COCO2, and for some 'large small' groups additional computations in MAGMA.

Alt(d)-free groups

(Abstract) group is called Alt(d)-free, $d \geq 5$, if it does not contain section isomorphic to the alternating group of degree d.

If $d \geq 25$, then list of simple Alt(d)-free groups includes

- the groups of order p for all primes p;
- all sporadic groups;
- all exceptional groups of Lie type;
- all classical groups of dimension less than $d-2$;
- all alternating groups of degree less than d.

If the composition width cw(G) of G is less than d, then G is Alt(d)-free, but not vice versa. Nevertheless, we have

[Babai–Cameron–Palfy, 1982]: The order of a primitive Alt(d)-free group of degree n is at most $n^{f(d)}$.

In this form the statement is taken from [Pyber–Shalev, 1997].

Closures of Alt(d)-free groups

[Ponom.–Skresanov–V., 2025]: Let $\mathfrak{X}$ be a complete class including all Alt(25)-free groups. Then the k-closure of every permutation group from $\mathfrak{X}$ belongs to $\mathfrak{X}$ for each $k \geq 4$.

Corollary. If G is an Alt(d)-free group with $d \geq 25$, then $G^{(k)}$ is Alt(d)-free group for $k \geq 4$.

$k = 4$ is the best possible, because $\mathrm{AGL}_m(2)^{(3)} = \mathrm{Sym}(2^m)$.

$d = 25$ is the best possible (for $k = 4$), because 4-closure of the Alt(9)-free Mathieu group M_{24} is Sym(24).

Corollary. If $\mathrm{cw}(G) < d$, then $\mathrm{cw}(G^{(k)}) < d$ for $k \geq 15, d \geq 25$.

Outline of the proof

- Let $G \in \mathfrak{X}$ be a counterexample of the least possible degree n.
- Since $\mathfrak{X}$ includes all Alt(25)-free groups, $n > 24$.
- Since $G^{(k)} \leq G^{(4)}$ for $k \geq 4$, it suffices to prove that $H := G^{(4)} \in \mathfrak{X}$.
- Since $k \geq 3$ and $\mathfrak{X}$ is complete, G is basic.
- G is nonsolvable, otherwise H is solvable and $H \in \mathfrak{X}$.

$\mathrm{Soc}(G) =$ the product of all minimal normal subgroups of G.

$$\mathrm{Soc}(G) = \mathrm{Soc}(H) \tag{1}$$

Proof. By [Praeger–Saxl, 1992], $\mathrm{Soc}(G) = \mathrm{Soc}(H)$ for $k \geq 4$ unless G is 4-transitive. If G is 4-transitive and $n > 24$, then $G \geq \mathrm{Alt}(n)$, so $\mathrm{Alt}(n) \in \mathfrak{X}$ and hence $H = \mathrm{Sym}(n) \in \mathfrak{X}$, a contradiction.

O'Nan–Scott theorem for basic groups

Recall that a group G is basic, if G is primitive and does not preserve any nontrivial product decomposition.

[Liebeck–Praeger–Saxl, 1988]: If G is a basic permutation group, then one of the following hold:

(i) G is almost simple,

(ii) G is in a diagonal action,

(iii) G is an affine group.

G is almost simple if $S = \mathrm{Soc}(G)$ is a nonabelian simple group, in this case $S \leq G \leq \mathrm{Aut}(S)$, so $G/S \leq \mathrm{Out}(S)$.

G is in a diagonal action, if $S = \mathrm{Soc}(G) = T^m$, where T is a nonabelian simple group, and G/S is a subgroup of $\mathrm{Out}(T) \times L$, where $L = \mathrm{Sym}(m)$ is the symmetric group acting faithfully on the set of simple factors of the socle by conjugation.

Reduction to affine groups

G is not almost simple.

By (1), $S = \mathrm{Soc}(H) = \mathrm{Soc}(G)$. In view of the validity of the Schreier conjecture, $H/S \leq \mathrm{Out}(S)$ is solvable $\implies H \in \mathfrak{X}$.

G is not in a diagonal action.

Otherwise, $S = \mathrm{Soc}(G) = T^m$, T is a nonabelian simple group, and $G/S \leq \mathrm{Out}(T) \times L$, and $L = \mathrm{Sym}(m)$ acts faithfully on the set of simple factors of S by conjugation.

By (1), $\mathrm{Soc}(H) = S$, so $H/S \leq \mathrm{Out}(T) \times L$. If $m \leq 4$ or G/S includes the alternating subgroup of L, then the nonabelian composition factors of G and H are the same, and we are done.

Otherwise, G has a base of size 2 in view of [Fawcett, 2013], and $G^{(3)} = G$ by Wielandt's criterion. Hence $H = G \in \mathfrak{X}$.

Affinity of the closure

By above, $G = V \rtimes G_0$ is affine, and $G_0 \leq \mathrm{GL}(V) = \mathrm{GL}_p(d)$.

For $H = G^{(4)}$, we have $H = V \rtimes H_0$ and $H_0 = G_0^{(3)} \leq \mathrm{GL}(V)$.

Since G is primitive, V is the unique minimal normal subgroup of G. Hence $V = \mathrm{Soc}(G)$. Then $\mathrm{Soc}(H) = \mathrm{Soc}(G) = V$ is the regular normal subgroup of H. Therefore, for a point stabilizer H_0 in H, we have $H_0 \cap V = 1$ and $n = |V| = |H : H_0|$. Thus, $H = V \rtimes H_0$, and we may identify H_0 with a subgroup of $\mathrm{GL}(V)$.

The equality $H_0 = G_0^{(3)}$ can be verified by applying the closure argument.

Corollary. It suffice to prove that $H_0 = G_0^{(3)} \in \mathfrak{X}$.

We reduce our problem to the matrix groups.

Aschbacher's classification

[Aschbacher, 1984]: If $\mathrm{SL}(V) \nleq G_0 \leq \mathrm{GL}(V)$, where V is vector space over a finite field, then G_0 belongs to one of the classes $\mathcal{C}_i$:
$\mathcal{C}_1$: Groups preserving a nontrivial proper subspace.
$\mathcal{C}_2$: Groups acting imprimitively on the vector space.
$\mathcal{C}_3$: Groups preserving the structure of an extension field.
$\mathcal{C}_4$: Groups preserving a nontrivial decomposition of the vector space into the tensor product of two spaces of unequal dimensions.
$\mathcal{C}_5$: Groups preserving the structure of a proper subfield.
$\mathcal{C}_6$: Groups normalizing a subgroup of symplectic type.
$\mathcal{C}_7$: Groups preserving a nontrivial decomposition of the vector space into the tensor product of several spaces of equal dimensions.
$\mathcal{C}_8$: Groups preserving a nondegenerate symplectic, unitary or quadratic form. We note the subclasses of $\mathcal{C}_8$ as $\mathcal{C}_{\mathrm{Sp}}$, $\mathcal{C}_{\mathrm{U}}$ and $\mathcal{C}_{\mathrm{O}}$, respectively, depending on the form preserved.
$\mathcal{C}_9$: Groups which are not contained in any of $\mathcal{C}_1, \ldots, \mathcal{C}_8$. These groups are almost simple modulo center.

Xu's theorem and further reduction

[Xu–Giudichi–Li–Praeger, 2011]: If $G_0 \in \mathcal{C}_i$, then $H_0 \in \mathcal{C}_i$.

Let $|V| = p^s$, where p is a prime. Choose the minimal $a \geq 1$ dividing s such that $G_0 \leq \Gamma\mathrm{L}_a(q)$, where $q = p^{s/a}$. Note that $a \geq 2$ since G is not solvable. Applying Xu's theorem, we conclude that $H_0 \leq \Gamma\mathrm{L}_a(q)$ (class $\mathcal{C}_3$ in $\mathrm{GL}_d(p)$).

We also assume that $\mathbb{F}_q^\times \leq G_0$, since $\mathbb{F}_q^\times \cdot G_0$ still lies in the class $\mathfrak{X}$ and the group $V \rtimes (\mathbb{F}_q^\times \cdot G_0)$ is a counterexample of degree n.

If $\mathrm{SL}_a(q) \leq G_0$, then $\mathrm{SL}_a(q) \leq H_0 \leq \Gamma\mathrm{L}_a(q)$, so G_0 and H_0 have the same nonabelian composition factors, and $H_0 \in \mathfrak{X}$, a contradiction. No we apply Aschbacher's classification and Xu's theorem in $\Gamma\mathrm{L}_a(q)$.

Since G is basic, G_0 is primitive, so $G_0, H_0 \notin \mathcal{C}_i, i = 1, 2$.

The minimality of a yields $G_0, H_0 \notin \mathcal{C}_3$.

If $G_0, H_0 \in \mathcal{C}_i, i = 4, 5, 7$, then we are done by induction, applying results on the closure of a tensor decomposition of V (see further).

Groups preserving a tensor decomposition

Lemma. Let X and Y be vector spaces over a finite field $\mathbb{F}$, $V = X \otimes Y$, and $G, H \leq (\mathrm{GL}(X) \circ \mathrm{GL}(Y)) \rtimes \mathrm{Aut}(\mathbb{F}) \leq \Gamma\mathrm{L}(V)$. Assume that $\mathbb{F}^\times \leq G$ and $\mathbb{F}^\times \leq H$. If $G \approx_k H$ for some $k \geq 1$. Then $G_X \approx_k H_X$, and $G_Y \approx_k H_Y$.

Theorem. Let V be a vector space over a finite field $\mathbb{F}$, G a primitive affine group with socle V. Set $G = V \rtimes G_0$, where G_0 is the zero stabilizer, and assume that $\mathbb{F}^\times \leq G_0$. Suppose that G_0 preserves a nontrivial tensor decomposition $V = X \otimes Y$ over $\mathbb{F}$, where $\dim X \neq \dim Y$. If $k \geq 4$ and the k-closures of $X \rtimes (G_0)_X$ and $Y \rtimes (G_0)_Y$ lie in the class $\mathfrak{X}$, then $G^{(k)} \in \mathfrak{X}$.

Exercise 2. Derive the theorem from the lemma.

Groups normalizing a subgroup of symplectic type

$G_0 \in \mathcal{C}_6$, so there is a prime r and an r-subgroup R such that $R/Z(R)$ is elementary abelian of order r^{2m} and $G_0 \leq N_{\Gamma\mathrm{L}_a(q)}(R)$.

$H_0 \leq N := N_{\Gamma\mathrm{L}_a(q)}(R)$, because $|R|$ depends on a and $H_0 \in \mathcal{C}_6$.

$\mathbb{F}_q^\times \leq G_0$, so G_0 includes the Fitting subgroup $F = \mathbb{F}_q^\times \cdot R$ of N. Note that $N_0/F \leq \mathrm{GL}_{2m}(r)$, where $N_0 = N \cap \mathrm{GL}_a(q)$.

Lemma. If $\mathrm{Alt}(d)$ is a section of $\mathrm{GL}_a(q)$, then $a \geq d-2$ for $d \geq 9$.

If $\mathrm{Alt}(25)$ is a section of H_0, then it is a section of $\mathrm{GL}_{2m}(r)$, so $2m \geq 23$. Thus, if $m < 12$, then N and so H_0 are $\mathrm{Alt}(25)$-free.

If $m \geq 12$, we show that there is $v \in V$ such that the N_v has a faithful regular orbit on V, so $b(G_0) = 2$ and $G_0 = H_0$.

In both cases, $H \in \mathfrak{X}$, a contradiction.

Groups preserving a form

Let $G_0 \in \mathcal{C}_8$ and, for definiteness, G preserves the nondegenerate symplectic form, that is $G_0 \in \mathcal{C}_{\mathrm{Sp}}$. By Xu's theorem, $H_0 \in \mathcal{C}_{\mathrm{Sp}}$ too.

If $\mathrm{Sp}_a(q) \leq G_0$, then $\mathrm{Sp}_a(q) \leq H_0 \leq \Gamma\mathrm{L}_a(q)$, so G_0 and H_0 have the same nonabelian composition factors, and $H_0 \in \mathfrak{X}$.

Otherwise, we apply Aschbacher classification inside the corresponding symplectic group. However, if $G_0 \in \mathcal{C}_i, i = 1, \ldots, 7$, then we have classified that group as belonging to the corresponding class C_i on one of the previous steps.

Thus, either G_0 preserves another form as well, or $G_0 \in \mathcal{C}_9$. Repeating, if necessary, the arguments for another form, we eventually comes to $G \in \mathcal{C}_9$.

End of the proof

If $G_0 \in \mathcal{C}_9$, then $H_0 \in \mathcal{C}_9$ by Xu's theorem. By Aschbacher's classification, $G_0/Z(G_0)$ and $H_0/Z(H_0)$ are almost simple groups. Therefore, the following statement completes the proof.

Lemma. Let $P, Q \leq \mathrm{Sym}(\Omega)$ and $P \approx_3 Q$. If $P/Z(P)$ and $Q/Z(Q)$ are almost simple, then either

- $\mathrm{Soc}(P/Z(P)) = \mathrm{Soc}(Q/Z(Q))$, or
- $\mathrm{Soc}(P/Z(P))$ and $\mathrm{Soc}(Q/Z(Q))$ are subgroups of $\mathrm{Alt}(24)$.

There is $\Delta \in \mathrm{Orb}(P) = \mathrm{Orb}(Q)$ such that P^Δ is nonsolvable and so $P^\Delta/Z(P^\Delta)$ is almost simple. Then the same holds for Q^Δ. So we can replace P and Q on P^Δ and Q^Δ. Similarly, we reduce to the case, when P and Q are primitive.

Since P is primitive, P is almost simple. If $|\Omega| > 24$, then [Liebeck–Praeger–Saxl, 1988b] $\implies$ $\mathrm{Soc}(P) = \mathrm{Soc}(P^{(3)})$.

Today's achievments

[O'Brien–Ponom.–V.–Vdovin, 2022]: If G is a solvable group, then $G^{(k)}$ is solvable for $k \geq 3$.

[Ponom.–Skresanov–V., 2025]: Let $\mathfrak{X}$ be a complete class including all Alt(25)-free groups. Then the k-closure of every permutation group from $\mathfrak{X}$ belongs to $\mathfrak{X}$ for each $k \geq 4$.

Corollary. If G is an Alt(d)-free group with $d \geq 25$, then $G^{(k)}$ is Alt(d)-free group for $k \geq 4$.

Next lecture: Algorithms for these classes...

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