# THE 3-CLOSURE OF A SOLVABLE PERMUTATION GROUP IS SOLVABLE

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Abstract. Let $m$ be a positive integer and let $\Omega$ be a finite set. The $m$-closure of $G \leq \mathrm{Sym}(\Omega)$ is the largest permutation group on $\Omega$ having the same orbits as $G$ in its induced action on the Cartesian product $\Omega^m$. The 1-closure and 2-closure of a solvable permutation group need not be solvable. We prove that the $m$-closure of a solvable permutation group is always solvable for $m \geq 3$.

*Dedicated to the memory of our friend Jan Saxl*

## 1. Introduction

Let $m$ be a positive integer and let $\Omega$ be a finite set. The *$m$-closure* $G^{(m)}$ of $G \leq \mathrm{Sym}(\Omega)$ is the largest permutation group on $\Omega$ having the same orbits as $G$ in its induced action on the Cartesian product $\Omega^m$. Wielandt [19, Theorems 5.8 and 5.12] showed that

$$G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(m)} = G^{(m+1)} = \cdots = G, \tag{1}$$

for some $m < |\Omega|$. (Since the stabilizer in $G$ of all but one point is always trivial, $G^{(n-1)} = G$ where $n = |\Omega|$; see Theorem 2.4.) In this sense, the $m$-closure can be considered as a natural approximation of $G$. Here we study the closures of solvable groups; for the nonsolvable case, see [10, 13, 20].

The 1-closure of $G$ is the direct product of symmetric groups $\mathrm{Sym}(\Delta)$, where $\Delta$ runs over the orbits of $G$. Thus the 1-closure of a solvable group is solvable if and only if each of its orbits has cardinality at most 4. The case of 2-closure is more interesting. The 2-closure of every (solvable) 2-transitive group $G \leq \mathrm{Sym}(\Omega)$ is $\mathrm{Sym}(\Omega)$; other examples of solvable $G$ and nonsolvable $G^{(2)}$ appear in [17]. But, as shown by Wielandt [19], each of the classes of finite $p$-groups and groups of odd order is closed with respect to taking the 2-closure. Currently, no characterization of solvable groups having solvable 2-closure is known.

Seress [15] observed that if $G$ is a primitive solvable group, then $G^{(5)} = G$; so the 5-closure of a primitive solvable group is solvable. Our main result is the following stronger statement.

**Theorem 1.1.** *The 3-closure of a solvable permutation group is solvable.*

Theorem 1.1 follows from Theorems 4.6, 5.2, and 6.1. The corollary below is an immediate consequence of Theorem 1.1 and the chain of inclusions (1).

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**Corollary 1.2.** *For every integer $m \geq 3$, the $m$-closure of a solvable permutation group is solvable.*

We briefly outline the structure of our proof. In Section 2 we recall the basic theory of the closure of permutation groups, as developed by Wielandt [19]. In Section 3 we deduce that the $m$-closure of the direct (respectively, imprimitive wreath) product of two permutation groups is isomorphic to a subgroup of the direct (respectively, imprimitive wreath) product of their $m$-closures, and prove (with some natural constraints) that the same holds true for the primitive wreath product. Thus, in Theorem 4.6, we reduce the proof of Theorem 1.1 from an arbitrary solvable permutation group $G$ to a linearly primitive group: a point stabilizer $G_0$ of such a group is a primitive linear group over a finite field.

A natural dichotomy arises in our treatment of linearly primitive groups. If the point stabilizer $G_0$ has a regular faithful orbit, then Wielandt's theory shows that the 3-closure of $G$ is solvable (Corollary 2.5). Otherwise, we use results from [21–23] to obtain in Theorem 5.2 an explicit list of pairs $(d, p)$ for which $G \leq \mathrm{AGL}(d, p)$.

In Section 6 we complete the proof, relying on computer calculations. Namely, following Short's approach [16], we find for each pair $(d, p)$ a set $\mathcal{H}$ of linearly primitive subgroups of $\mathrm{GL}(d, p)$ containing a conjugate of each maximal solvable primitive subgroup of $\mathrm{GL}(d, p)$; in particular, $G_0$ is a subgroup of some $H \in \mathcal{H}$. If $G$ is 2-transitive, then $G^{(3)}$ is solvable (Lemma 4.3). In the remaining cases, $G^{(3)} = G$ (and so solvable) because $H^{(2)} = H$ for every $H \in \mathcal{H}$. Verification of the latter is based on sufficient conditions given in Corollary 2.5 and Lemma 4.4.

## 2. Wielandt's theory

Let $G \leq \mathrm{Sym}(\Omega)$ and let $m$ be a positive integer. The $m$-orbits of $G$ are the orbits of componentwise action of $G$ on the Cartesian product $\Omega^m$ of $\Omega$; the set of all such orbits is denoted by $\mathrm{Orb}_m(G)$.

**Example 2.1.** *We describe the $m$-orbits of $G = \mathrm{Sym}(\Omega)$. Given an $m$-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ of $\Omega^m$, let $\pi(\alpha)$ be the partition of $I = \{1, \ldots, m\}$ such that the elements $i$ and $j$ belong to the same class of $\pi$ if and only if $\alpha_i = \alpha_j$. If $s \in \mathrm{Orb}_m(G)$, then $\pi(s) := \pi(\alpha)$ does not depend on the choice of $\alpha \in s$. If $m \leq |\Omega|$, then the mapping $s \mapsto \pi(s)$ establishes a one-to-one correspondence between the $m$-orbits of $G$ and the partitions of $I$; in particular, $|\mathrm{Orb}_m(G)| \geq m + 2$ for all $m \geq 3$.*

A permutation group $H$ on $\Omega$ is *$m$-equivalent* to $G$ if

$$\mathrm{Orb}_m(H) = \mathrm{Orb}_m(G).$$

Obviously, $m$-equivalence is an equivalence relation on the set of permutation groups on $\Omega$. If $m \geq 2$, then two $m$-equivalent groups share some properties, such as primitivity, or 2-transitivity; see [19, Theorems 4.8 and 4.10].

The following criterion for $m$-equivalence can easily be deduced from [19, Theorem 4.7].

**Lemma 2.2.** *Let $G$ and $H$ be permutation groups on $\Omega$ and assume that $G \leq H$. Then $H$ is $m$-equivalent to $G$ if and only if, for every $\alpha \in \Omega^m$ and every $h \in H$, there exists $g \in G$ such that $\alpha^h = \alpha^g$.*

Wielandt [19, Theorem 4.3 and Lemma 4.12] established the following.

**Theorem 2.3.** *Let $m \geq 2$ be an integer and let $G$ and $H$ be $m$-equivalent permutation groups on $\Omega$. The following hold:*

(i) *$G$ and $H$ are $(m-1)$-equivalent;*
(ii) *$G_\alpha$ and $H_\alpha$ are $(m-1)$-equivalent for all $\alpha \in \Omega$.*

The definition of $m$-equivalence implies that the *$m$-closure* of $G$ is the largest group in the class of $m$-equivalent groups containing $G$. In particular, $G$ and $H$ are $m$-equivalent if and only if $G^{(m)} = H^{(m)}$. If $G^{(m)} = G$ then $G$ is *$m$-closed*. Note that $m$-closure is a closure operator: namely, $G \leq G^{(m)}$, $G^{(m)} = (G^{(m)})^{(m)}$, and $G \leq H$ implies $G^{(m)} \leq H^{(m)}$.

**Theorem 2.4.** *Let $m \geq 2$ be an integer. If a point stabilizer of a permutation group $G$ is $(m-1)$-closed, then $G$ is $m$-closed.*

**Proof**. Let $G \leq \mathrm{Sym}(\Omega)$ and let $\alpha \in \Omega$. Since $G$ and $H := G^{(m)}$ are $m$-equivalent, $G_\alpha$ and $H_\alpha$ are $(m-1)$-equivalent (Theorem 2.3(ii)). Since $G_\alpha$ is $(m-1)$-closed,

$$H_\alpha \leq (H_\alpha)^{(m-1)} = (G_\alpha)^{(m-1)} = G_\alpha.$$

But $G_\alpha \leq H_\alpha$, so $G_\alpha = H_\alpha$. Furthermore, Theorem 2.3(i) implies that $G$ and $H$ are 1-equivalent. Therefore $\alpha^G = \alpha^H$. Hence

$$|G| = |\alpha^G| \cdot |G_\alpha| = |\alpha^H| \cdot |H_\alpha| = |H|.$$

Thus $G = G^{(m)}$ because $G \leq H$. □

A permutation group is *partly regular* if it has a faithful regular orbit. Clearly, every subgroup of a partly regular group is partly regular.

**Corollary 2.5.** *Let $G \leq \mathrm{Sym}(\Omega)$ and let $m \geq 2$ be an integer. If an $(m-1)$-point stabilizer of $G$ is partly regular, then $G$ is $(m+1)$-closed.*

**Proof**. If $G$ has a partly regular $(m-1)$-point stabilizer, then $G$ has an $m$-point stabilizer which is trivial and so $m$-closed. The claim follows from Theorem 2.4. □

## 3. Closures of permutation groups

We now study the $m$-closure operator under standard operations in permutation group theory; compare with similar results for $m = 2$ in [4].

**Theorem 3.1.** *Let $K \leq \mathrm{Sym}(\Gamma)$, let $L \leq \mathrm{Sym}(\Delta)$, and let $K \times L$ act on the disjoint union $\Gamma \cup \Delta$. For every integer $m \geq 1$,*

$$(K \times L)^{(m)} \leq K^{(m)} \times L^{(m)}.$$

**Proof**. Observe that $K \times L$ and $H := (K \times L)^{(m)}$ are 1-equivalent. Hence the sets $\Gamma$ and $\Delta$ are invariant under $H$. It follows that $K = (K \times L)^\Gamma$ is $m$-equivalent to $H^\Gamma$, the permutation group induced by the action of $H$ on $\Gamma$, and $L = (K \times L)^\Delta$ is $m$-equivalent to $H^\Delta$. In particular, $H^\Gamma \leq K^{(m)}$ and $H^\Delta \leq L^{(m)}$. Thus

$$(K \times L)^{(m)} = H \leq H^\Gamma \times H^\Delta \leq K^{(m)} \times L^{(m)},$$

as required. □

An analogue of Theorem 3.1 exists for the direct product of permutation groups acting on the Cartesian product of their underlying sets, but it is not needed here.

The following is a consequence of [8, Lemma 2.5].

**Theorem 3.2.** *Let $K \wr L$ be the imprimitive wreath product of permutation groups $K$ and $L$. For every integer $m \geq 2$,*

$$(K \wr L)^{(m)} = K^{(m)} \wr L^{(m)}.$$

The case of the wreath product in *product action* is more subtle. Recall, for example from [3, Lemma 2.7A], that $K \uparrow L$, the wreath product in product action of permutation groups $K$ and $L$, is primitive if and only if $K$ is primitive and nonregular, and $L$ is transitive and nontrivial. For the remainder of the paper, we assume that $K \uparrow L$ *is primitive* and so label this construction as the *primitive wreath product*. Even with this assumption, $(K \uparrow L)^{(m)}$ is not always a subgroup of $K^{(m)} \uparrow L^{(m)}$: consider for example $m = 2$, $K = \mathrm{Sym}(4)$, and $L = \mathrm{Alt}(3)$. But we obtain the following.

**Theorem 3.3.** *Let $K \uparrow L$ be the primitive wreath product of permutation groups $K$ and $L$ of degrees $r$ and $d$, respectively. Assume that $m \geq 3$ is an integer such that $m \leq r$, and also $m \leq d$ unless $d = 2$. Then*

$$(K \uparrow L)^{(m)} \leq K^{(m)} \uparrow L^{(m)}.$$

**Proof**. Let $K \leq \mathrm{Sym}(\Gamma)$ and $L \leq \mathrm{Sym}(\Delta)$, where $\Gamma$ and $\Delta$ are sets of cardinality $r$ and $d$, respectively. Without loss of generality, we assume that $\Delta = \{1, \ldots, d\}$. Thus $G := K \uparrow L$ acts on the Cartesian product

$$\Omega = \underbrace{\Gamma \times \cdots \times \Gamma}_{d \text{ copies}}.$$

In what follows, an $m$-tuple $x \in \Omega^m$ is considered as an $m \times d$ matrix $(x_{ij})$ with $x_{ij} \in \Gamma$; the $j$th column of $x$ is denoted by $x_{*j}$. Note that $K^d$ acts on $x$ by permuting elements inside columns, whereas $L$ permutes the columns.

As observed in [4, Proof of Proposition 3.1], $G$ is contained in the 2-closed group $\mathrm{Sym}(\Gamma) \uparrow \mathrm{Sym}(\Delta)$. It follows that $H := G^{(m)}$, being 2-equivalent to $G$, is also contained in $\mathrm{Sym}(\Gamma) \uparrow \mathrm{Sym}(\Delta)$. Therefore every permutation of $H$ can be written in the form

$$h = (h_1, \ldots, h_d; \overline{h}), \qquad h_1, \ldots, h_d \in \mathrm{Sym}(\Gamma),\ \overline{h} \in \mathrm{Sym}(\Delta). \tag{2}$$

Let $\overline{H} = \{\overline{h} \ :\ h \in H\}$ be the permutation group induced by the action of $H$ on $\Delta$. As a critical step in our proof, we establish the following.

**Claim.** $H \leq \mathrm{Sym}(\Gamma) \uparrow L^{(m)}$.

**Proof**. Without loss of generality, we may assume that $d \geq 3$. It suffices to show that $\overline{H} \leq L^{(m)}$. Equivalently (cf. Lemma 2.2), we show that, for every $\alpha \in \Delta^m$ and every $\overline{h} \in \overline{H}$, there exists $\overline{g} \in L$ such that

$$\alpha^{\overline{h}} = \alpha^{\overline{g}}. \tag{3}$$

Since by assumption $K \uparrow L$ is primitive, $L$ is transitive and a subgroup of $\overline{H}$. Consequently, $\overline{H} = L\overline{H}_d$, where $\overline{H}_d$ is the stabilizer in $\overline{H}$ of the point $d \in \Delta$. Thus we may assume that the element $\overline{h}$ in (3) belongs to $\overline{H}_d$.

Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \Delta^m$, where $\{\alpha_1, \ldots, \alpha_m\} = \{j_1, \ldots, j_t\}$ and $1 \leq j_1 < \cdots < j_t \leq m$. Let $\pi(\alpha)$ be the partition of $\{1, \ldots, m\}$ into the classes $\{\ell :\ \alpha_\ell = j_i\}$ where $i = 1, \ldots, t$ and $t = |\pi(\alpha)|$. Note that the number of $m$-orbits of $\mathrm{Sym}(\Gamma)$ is at least $m + 2$, because $|\Delta| = d \geq m \geq 3$ (see Example 2.1). Thus there are pairwise distinct $m$-orbits $s_0, s_1, \ldots, s_t$ such that $\pi(s_i) \neq \pi(\alpha)$ for all $i$. Denote by $\boldsymbol{x}(\alpha)$ the set of all tuples $x \in \Omega^m$ such that

$$\pi(x_{*j_1}) = \pi(s_1), \quad \pi(x_{*j_2}) = \pi(s_2), \quad \ldots, \quad \pi(x_{*j_t}) = \pi(s_t), \tag{4}$$

$$\pi(x_{*d}) = \pi(\alpha), \quad \pi(x_{*i}) = \pi(s_0), \quad i \notin \{j_1, \ldots, j_t, d\}. \tag{5}$$

Of course, the choice of $s_0, \ldots, s_t$ is arbitrary. However, if $\beta$ is in the $m$-orbit of $\mathrm{Sym}(\Delta)$ containing $\alpha$, then $\pi(\beta) = \pi(\alpha)$, so we can use the same $s_0, \ldots, s_t$ to define $\boldsymbol{x}(\beta)$.

Assume that $\boldsymbol{x}(\alpha) = \boldsymbol{x}(\beta)$ for some $\beta \in \Delta^m$. Condition (5) ensures that

$$\pi(\alpha) = \pi(\beta) \quad \text{and} \quad \{\alpha_1, \ldots, \alpha_{m-1}\} = \{\beta_1, \ldots, \beta_{m-1}\}.$$

In particular, $\alpha_i = \alpha_j$ if and only if $\beta_i = \beta_j$ for all $1 \leq i, j \leq m$. Condition (4) yields $\alpha = \beta$. Thus

$$\boldsymbol{x}(\alpha) \cap \boldsymbol{x}(\beta) \neq \varnothing \quad \text{if and only if} \quad \alpha = \beta. \tag{6}$$

Let $h \in H$ be as in (2) and let $\overline{h} \in \overline{H}_d$. In view of our definitions, given indices $i, j \in \{1, \ldots, m\}$, the column $j$ of a tuple $x \in \boldsymbol{x}(\alpha)$ belongs to $s_i$ if and only if the column $j^{\overline{h}}$ of the tuple $x^h$ belongs to $s_i$. Therefore

$$\boldsymbol{x}(\alpha)^h = \boldsymbol{x}(\alpha^{\overline{h}}). \tag{7}$$

In particular, the preimage of $\overline{H}_d$ in $H$ acts on the set $\boldsymbol{x}(\alpha)$ for each $\alpha \in \Delta^m$. Since $G \leq H$, (7) holds for every $g \in G$ with $\overline{g} \in L_d$, where $L_d$ is the stabilizer in $L$ of the point $d \in \Delta$.

Since $G$ and $H$ are $m$-equivalent and $G \leq H$, Lemma 2.2 implies that for every $x \in \boldsymbol{x}(\alpha)$ and every $h \in H$ there exists $g \in G$ such that $x^h = x^g$. But $\overline{h} \in \overline{H}_d$; so, in accord with the first equality in (5), $\overline{g} \in L_d$. In view of (7), this implies that

$$x^g = x^h \in \boldsymbol{x}(\alpha^{\overline{h}}) \cap \boldsymbol{x}(\alpha^{\overline{g}}).$$

Now (6) yields $\alpha^{\overline{h}} = \alpha^{\overline{g}}$, which proves (3). □

We return to the proof of the theorem. It remains to verify that for every $h \in H$ and every $\alpha \in \Gamma^m$,

$$\alpha^{h_j} \in \alpha^K \text{ for } j = 1, \ldots, d, \tag{8}$$

where $h_j$ is defined by (2). Without loss of generality, we prove (8) for $j = 1$ only. Take distinct $m$-orbits $s_0$ and $s_1$ of $\mathrm{Sym}(\Gamma)$ such that $\alpha_1 \in s_1$. Choose a tuple $x \in \Omega^m$ satisfying the conditions

$$x_{*1} = \alpha_1 \quad \text{and} \quad x_{*2}, \ldots, x_{*d} \in s_0. \tag{9}$$

Our claim implies that $L \leq \overline{H} \leq L^{(m)}$, so $L$ and $\overline{H}$ are $m$-equivalent and have the same orbits. Hence there exists $\overline{g} \in L$ such that $\overline{h}\overline{g} \in \mathrm{Sym}(\Delta)$ fixes $1 \in \Delta$. So

$$g = (1, \ldots, 1; \overline{g}) \in G$$

and the only column of the tuple $x^{hg}$ belonging to $s_1$ is the first one, from (9). Thus

$$(x^{hg})_{*1} = \alpha^{h_1}.$$

Since $x^H \in \mathrm{Orb}_m(H) = \mathrm{Orb}_m(G)$, there exists $f \in G$ such that $x^{hg} = x^f$; in particular, $\overline{f} \in \mathrm{Sym}(\Delta)$ fixes $1 \in \Delta$. Consequently,

$$\alpha^{h_1} = (x^{hg})_{*1} = (x^f)_{*1} = \alpha^{f_1} \in \alpha^K,$$

as required. □

If $m = 3$, then the hypothesis of Theorem 3.3 does not impose any restrictions on the degrees $r$ and $d$ of the groups $K$ and $L$ because the primitivity of $K \uparrow L$ implies that $r \geq 3$ and $d \geq 2$. Therefore the following holds.

**Corollary 3.4.** *Let $K \uparrow L$ be the primitive wreath product of permutation groups $K$ and $L$. Then*

$$(K \uparrow L)^{(3)} \leq K^{(3)} \uparrow L^{(3)}.$$

## 4. Reducing primitive solvable groups to linearly primitive groups

We summarize the well-known structure of a primitive solvable group; for a proof, see for example [18, Chap. 1, Theorem 7].

**Theorem 4.1.** *Let $G \leq \mathrm{Sym}(\Omega)$ be a primitive solvable permutation group. Now $\Omega$ has cardinality $p^d$ for a prime $p$ and integer $d \geq 1$ and can be identified with a $d$-dimensional vector space $V$ over $\mathrm{GF}(p)$. Moreover, $G \leq \mathrm{AGL}(d,p)$, and the stabilizer $H$ in $G$ of the zero vector is an irreducible subgroup of $\mathrm{GL}(d,p)$.*

We establish Theorem 1.1 for two classes of groups.

**Lemma 4.2.** *If $G \leq \mathrm{A\Gamma L}(1,p^d)$, then $G^{(3)}$ is solvable.*

**Proof**. A point stabilizer $\Gamma\mathrm{L}(1,p^d)$ of $\mathrm{A\Gamma L}(1,p^d)$ is 2-closed by [20, Proposition 3.1.1]. Theorem 2.4 implies that $\mathrm{A\Gamma L}(1,p^d)$ is 3-closed. Thus

$$G^{(3)} \leq \mathrm{A\Gamma L}(1,p^d)^{(3)} = \mathrm{A\Gamma L}(1,p^d),$$

and so $G^{(3)}$ is solvable. □

**Lemma 4.3.** *If $G$ is a 2-transitive solvable group, then $G^{(3)}$ is solvable.*

**Proof**. By a theorem of Huppert [11, Theorem 6.9], $G \leq \mathrm{A\Gamma L}(1,p^d)$, or

$$p^d \in \{3^2, 5^2, 7^2, 11^2, 23^2, 3^4\}. \tag{10}$$

The first case is settled by Lemma 4.2.

In the second case, we consider a point stabilizer $G_\alpha$ and used the `TwoClosure` command in GAP [5] and the IRREDSOL package [6] to establish that $(G_\alpha)^{(2)}$ is solvable for those $p^d$ satisfying (10). Since $(G^{(3)})_\alpha$ and $G_\alpha$ are 2-equivalent,

$$(G^{(3)})_\alpha \leq (G_\alpha)^{(2)},$$

and so $(G^{(3)})_\alpha$ is solvable. Hence $G^{(3)}$, an extension of an elementary abelian group (of order $p^d$) by $(G^{(3)})_\alpha$, is solvable. □

The following lemma gives a sufficient condition for a primitive solvable group to be 3-closed.

**Lemma 4.4.** *In the notation of Theorem 4.1, if there exists a nonzero $\alpha \in V$ such that the restriction of $H$ to $\alpha^H$ is 2-closed, then $G$ is 3-closed.*

**Proof**. Since $H$ is an irreducible linear group, the orbit $\Delta = \alpha^H$ contains a basis of $V$. It follows that $H \cong H^\Delta$, the restriction of $H$ to $\Delta$. On the other hand, $G^{(3)}$ and $G$ are 2-equivalent by Theorem 2.3(i). Consequently, $G^{(3)}$ is also a primitive subgroup of $\mathrm{AGL}(d,p)$ [20, Theorem 1.4]. Hence the stabilizer $L$ in $G^{(3)}$ of the zero vector acts faithfully on $\Delta$. Thus

$$H \cong H^\Delta \quad \text{and} \quad L \cong L^\Delta. \tag{11}$$

Since $H$ and $L$ are point stabilizers of 3-equivalent groups, they are 2-equivalent by Theorem 2.3(ii). Therefore

$$L \le H^{(2)}. \tag{12}$$

Following [12, Lemma 2.1(iii)], we verify that

$$(H^{(2)})^{\Delta} \le (H^{\Delta})^{(2)}. \tag{13}$$

By hypothesis $(H^{\Delta})^{(2)} = H^{\Delta}$. This, together with (11), (12), and (13), implies that

$$L \cong L^{\Delta} \le (H^{(2)})^{\Delta} \le (H^{\Delta})^{(2)} = H^{\Delta} \cong H.$$

Since $H \le L$, this yields $H = L$. Thus $G = G^{(3)}$. □

If the point stabilizer $H$ in Theorem 4.1 is primitive as a linear group, then $G$ is *linearly primitive*, otherwise it is *linearly imprimitive*. As is well-known, the latter case reduces to the primitive wreath product; see for example [4, Proposition 4.1].

**Lemma 4.5.** *Every linearly imprimitive solvable permutation group $G$ is isomorphic to a subgroup of a primitive wreath product of two solvable permutation groups of degrees smaller than the degree of $G$.*

We now reduce the proof of Theorem 1.1 to linearly primitive groups.

**Theorem 4.6.** *A counterexample of minimal degree to the statement of Theorem* 1.1 *is linearly primitive.*

**Proof**. We consider separately the cases where the permutation group $G$ is intransitive, imprimitive, or linearly imprimitive.

**Case 1:** $G$ is intransitive. Now $G$ is a subdirect product of (solvable) constituents, say $K$ and $L$, and their degrees are less than that of $G$. Therefore their 3-closures are solvable. By Theorem 3.1 so is $(K \times L)^{(3)}$. Thus

$$G^{(3)} \le (K \times L)^{(3)}$$

is also solvable.

**Case 2:** $G$ is imprimitive. Now $G$ can be identified with a subgroup of the imprimitive wreath product $K \wr L$, where $K$ and $L$ are solvable permutation groups, and their degrees are less than that of $G$. Therefore their 3-closures are solvable. By Theorem 3.2 so is $(K \wr L)^{(3)}$. Thus

$$G^{(3)} \le (K \wr L)^{(3)}$$

is also solvable.

**Case 3:** $G$ is primitive, but linearly imprimitive. Now, by Lemma 4.5, it can be identified with a subgroup of the primitive wreath product $K \uparrow L$, where $K$ and $L$ are solvable permutation groups, and their degrees are less than that of $G$. Therefore their 3-closures are solvable. By Corollary 3.4 so is $(K \uparrow L)^{(3)}$. Thus

$$G^{(3)} \le (K \uparrow L)^{(3)}$$

is also solvable. □

## 5. Linearly primitive solvable groups: background theory

Let $G$ be a linearly primitive solvable permutation group with point stabilizer $G_0$. Often essential information about $G_0$ can be obtained from a maximal solvable primitive linear group $H$ containing $G_0$. Basic information on the structure of $H$ is collected in the following theorem; for its proof, see for example [15, Lemma 2.2].

**Theorem 5.1.** *Let $H \leq \mathrm{GL}(d,p)$ be a maximal solvable primitive group. It has a series $1 < U \leq F \leq A \leq H$ satisfying the following:*

(i) *$U$ is the unique maximal abelian normal subgroup of $H$, the linear span of $U$ in $\mathrm{Mat}(d,p)$ is $\mathrm{GF}(p^a)$ where $a$ divides $d$, and $U$ is cyclic of order $p^a - 1$;*
(ii) *$F = \mathrm{Fit}(C_H(U))$ is the Fitting subgroup of the centralizer $C_H(U)$, and $|F\,/\,U| = e^2$ where $d = ae$ and each prime divisor of $e$ divides $p^a - 1$;*
(iii) *$A = C_H(U)$ and $A/F$ is isomorphic to a completely reducible subgroup of the direct product $\prod_{i=1}^m \mathrm{Sp}(2n_i, p_i)$ where the $p_i$ and $n_i$ are defined by the prime power decomposition $e = \prod_{i=1}^m p_i^{n_i}$;*
(iv) *$H/A$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{GF}(p^a))$ and so $|H/A|$ divides $a$.*

Let $G$ be a linearly primitive solvable permutation group. We fix an embedding of a point stabilizer $G_0$ of $G$ into a specific maximal solvable primitive linear group $H$. We call the integers $p$, $d$, $a$, $e$ defined in Theorem 5.1 for $H$ the *parameters* of $G$. Although the parameters of $G$ depend on the choice of $H$, the monotonicity of the $m$-closure operator guarantees that our results are independent of it.

**Theorem 5.2.** *The 3-closure of a linearly primitive solvable permutation group is solvable, except possibly for those groups whose parameters are listed in columns 2–5 of Table 2.*

**Proof**. Let $G$ be a linearly primitive solvable permutation group with parameters $p$, $d$, $a$, $e$, and let $H \leq \mathrm{GL}(d,p)$ be a maximal solvable primitive group containing a point stabilizer $G_0$ of $G$.

If $e = 1$, then $G \leq \mathrm{AGL}(1, p^d)$, and the result follows by Lemma 4.2. So we may assume that $e > 1$. If $H$ is partly regular, then so is $G_0 \leq H$; thus $G$ is 3-closed by Corollary 2.5 (take $m = 2$). If $H$ is not partly regular, then its parameters are listed in [23, Corollary 3.2]; columns 2–5 of Table 2 are taken from [23, Table 2]. □

The data in [23, Table 2] was obtained using [22, Theorem 4.1] which states the following: if $H$ is not partly regular, then

$$e = 2,\ 3,\ 4,\ 8,\ 9,\ 16. \tag{14}$$

Hence $e = r^k$ where $2 \leq r \leq 3$ and $1 \leq k \leq 4$. Let $b$ be the least positive integer with $p^b \equiv 1 \pmod{r^c}$ where $c = 2$ for $r = 2$ and $c = 1$ otherwise. We denote the general orthogonal group of degree $2k$ over the field of order $q$ by $O^\varepsilon(2k, q)$ where $\varepsilon \in \{+,-\}$ depends on the Witt index of the corresponding quadratic form. The *$r$-radical* of a group is its largest normal $r$-subgroup.

**Lemma 5.3.** *We retain the notation of Theorem 5.1 where $e = r^k$.*

(i) *$F$ is the central product of $U$ and an extraspecial group $E$ of order $r^{2k+1}$; if $b \mid a$, then all such subgroups $F$ are conjugate in $\mathrm{GL}(d,p)$, else $r = b = 2$, $a$ is odd, and there are two conjugacy classes of such subgroups;*
(ii) *$A/F$ is a maximal solvable subgroup of $N/F$, where $N = N_L(F)$ and $L = C_{\mathrm{GL}(d,p)}(U) \cong \mathrm{GL}(e, p^a)$; if $b \mid a$, then $N/F \cong \mathrm{Sp}(2k, r)$, else $N/F$ is isomorphic to one of $O^\varepsilon(2k, 2)$ depending on the conjugacy class of $F$;*

(iii) *the $r$-radical of $A/F$ has order at most 2 and is trivial if $b \mid a$.*

**Proof**. Item (i) is well known; see for example [16, Theorem 2.4.7] and subsequent remarks. By [16, Theorems 2.5.31, 2.5.34 and 2.4.12], $A/F$ is isomorphic to a maximal solvable subgroup $M$ of $S = N/F$, where $S = \mathrm{Sp}(2k, r)$ if $b \mid a$ and $S$ is one of $O^{\varepsilon}(2k, 2)$ otherwise; this proves (ii). Moreover, $M$ fixes no nonzero isotropic subspace of the natural $\mathrm{GF}(r)$-module of $S$. Therefore $S$ is not contained in any parabolic subgroup of $S$, so the $r$-radical of $M$ is either trivial or has order 2, and the latter is possible only if $M$ is an orthogonal group. □

The following lemma can be established using MAGMA [1]. Note that $O^+(2, 2)$, $\mathrm{Sp}(2, 2) \cong O^-(2, 2)$, $\mathrm{Sp}(2, 3)$, and $O^+(4, 3)$ are solvable.

**Lemma 5.4.** *Let $M$ be a maximal solvable subgroup of $S \leq \mathrm{GL}(d, r)$ and let the $r$-radical of $M$ satisfy the order conditions of Lemma 5.3(iii).*

(i) *If $S = \mathrm{Sp}(2, 2)$, $\mathrm{Sp}(2, 3)$, $O^{\varepsilon}(2, 2)$, $O^+(4, 3)$, then $M = S$.*

(ii) *If $S = O^-(4, 2)$, then $M$ is conjugate to a subgroup isomorphic to either $5 : 4$ of order 20, or $S_3 \times S_2$ of order 12.*

(iii) *If $S = \mathrm{Sp}(4, 2)$, then $M$ is conjugate to a subgroup isomorphic to either $O^+(4, 2) \cong S_3 \wr S_2$ of order 72, or the normalizer of a Sylow 5-subgroup of $\mathrm{Sp}(4, 2)$ which is isomorphic to $5 : 4$ and has order 20.*

(iv) *If $S = \mathrm{Sp}(4, 3)$, then $M$ is conjugate to one of the following:*
- *the normalizer of a Sylow 5-subgroup of $\mathrm{Sp}(4, 3)$ which is isomorphic to $D_{20}.2$ and has order 40;*
- *a subgroup isomorphic to $2^{1+4} : S_3$ of order 192;*
- *a subgroup isomorphic to $2^{1+4} : D_{10}$ of order 320;*
- *a subgroup isomorphic to $\mathrm{Sp}(2, 3) \wr S_2$ of order 1152.*

(v) *If $S = O^+(6, 2)$, then $M$ is conjugate to one of the following:*
- *the normalizer of a Sylow 3-subgroup of $O^+(6, 2)$ which is isomorphic to $O^+(4, 2) \times O^+(2, 2) \cong (S_3 \wr S_2) \times S_2$ and has order 144;*
- *the normalizer of a Sylow 5-subgroup of $O^+(6, 2)$ which is isomorphic to $(5 : 4) \times S_3$ and has order 120;*
- *the normalizer of a Sylow 7-subgroup of $O^+(6, 2)$ which is isomorphic to $7 : 6$ and has order 42.*

(vi) *If $S = O^-(6, 2)$, then $M$ is conjugate to one of the following:*
- *a subgroup isomorphic to $3^{1+2} : (2.S_4)$ of order 1296;*
- *a subgroup isomorphic to $3^3 : (S_4 \times S_2)$ of order 1296;*
- *the normalizer of a Sylow 5-subgroup of $O^-(6, 2)$ which is isomorphic to $(5 : 4) \times S_2$ and has order 40.*

(vii) *If $S = \mathrm{Sp}(6, 2)$, then $M$ is conjugate to one of the following:*
- *a subgroup isomorphic to $3^{1+2} : (2.S_4)$ of order 1296;*
- *a subgroup isomorphic to $3^3 : (S_4 \times S_2)$ of order 1296;*
- *the normalizer of a Sylow 5-subgroup of $\mathrm{Sp}(6, 2)$ which is isomorphic to $(5 : 4) \times S_3$ and has order 120;*
- *the normalizer of a Sylow 7-subgroup of $\mathrm{Sp}(6, 2)$ which is isomorphic to $7 : 6$ and has order 42.*

In Table 1, we summarize the orders of the maximal solvable subgroups listed in Lemma 5.4.

TABLE 1. The orders of certain maximal solvable subgroups

| $e$ | $S$ | $\lvert M \rvert$ |
|---|---|---|
| 9 | $\mathrm{Sp}(4,3)$ | $40, 192, 320, 1152$ |
| 8 | $\mathrm{Sp}(6,2)$ | $42, 120, 1296$ |
| | $O^+(6,2)$ | $42, 120, 144$ |
| | $O^-(6,2)$ | $40, 1296$ |
| 4 | $\mathrm{Sp}(4,2)$ | $20, 72$ |
| | $O^+(4,2)$ | $72$ |
| | $O^-(4,2)$ | $12, 20$ |
| 3 | $\mathrm{Sp}(2,3)$ | $24$ |
| 2 | $\mathrm{Sp}(2,2)$ | $6$ |
| | $O^+(2,2)$ | $2$ |
| | $O^-(2,2)$ | $6$ |

To state the next lemma we introduce some additional notation. Let $I_m$ be the identity $m \times m$ matrix, and let $\otimes$ denote the Kronecker product of matrices. If $y$ and $z$ are $k \times k$ and $m \times m$ matrices respectively, then $y \otimes I_m$ commutes with $I_k \otimes z$. For $G \le \mathrm{GL}(k,p)$ let $G \otimes I_m$ be the subgroup $\{g \otimes I_m \mid g \in G\}$ of $\mathrm{GL}(k \cdot m, p)$. Clearly $G \cong G \otimes I_m$. For positive integers $k, m$ there exists a natural embedding

$$\mathrm{GL}(k, p^m) : \langle \varphi \rangle \le \mathrm{GL}(k \cdot m, p), \tag{15}$$

where $\varphi$ is a field automorphism of $\mathrm{GL}(k, p^m)$. The uniqueness of a field of order $p^m$ implies that all such embeddings are conjugate in $\mathrm{GL}(k \cdot m, p)$. Below we assume that we fix such an embedding and so realise $\mathrm{GL}(k, p^m) : \langle \varphi \rangle$ as a subgroup of $\mathrm{GL}(k \cdot m, p)$.

**Lemma 5.5.** *We retain the notation of Theorem* 5.1 *where* $e = r^k$.

(i) *If $b$ divides $a$, then there exists a maximal solvable primitive subgroup of $\mathrm{GL}(e, p^b)$ with generators $x_1, \dots, x_l$, and matrices $t \in \mathrm{GL}(a/b, p^b)$ of order $p^a - 1$ and $s \in \mathrm{GL}(a/b, p^b)$ of order $a/b$ satisfying $t^s = t^{p^b}$, such that the subgroup*

$$\langle t \otimes I_e, s \otimes I_e, I_{a/b} \otimes x_1, \dots, I_{a/b} \otimes x_l \rangle \le \mathrm{GL}(d/b, p^b) : \langle s \otimes I_e \rangle \le \mathrm{GL}(d,p)$$

*is conjugate to a normal subgroup of $H$ containing $A$ and of index dividing $b$.*

(ii) *If $b$ does not divide $a$, then there exists a maximal solvable primitive subgroup of $\mathrm{GL}(e,p)$ with generators $x_1, \dots, x_l$, and matrices $t \in \mathrm{GL}(a,p)$ of order $p^a - 1$ and $s \in \mathrm{GL}(a,p)$ of order $a$ satisfying $t^s = t^p$, such that $H$ is conjugate to the subgroup*

$$\langle t \otimes I_e, s \otimes I_e, I_a \otimes x_1, \dots, I_a \otimes x_l \rangle \le \mathrm{GL}(d,p).$$

**Proof**. (i) By Lemma 5.3(i), $F = U \circ E$, where $E$ is an extraspecial group of order $r^{2k+1}$; moreover, $F$ is unique up to conjugation in $\mathrm{GL}(d,p)$. By [16, 2.5.14], $U = \langle z \otimes I_e \rangle$ where $z$ is a Singer cycle of $\mathrm{GL}(a,p)$. By [16, Theorem 2.5.15], $L = C_{\mathrm{GL}(d,p)}(U) \cong \mathrm{GL}(e, p^a)$, and $N_{\mathrm{GL}(d,p)}(U) = \mathrm{GL}(e,p^a) : \langle \psi \rangle$, where $\psi$ is a field automorphism of order $a$. Identifying $L$ with $\mathrm{GL}(e,p^a)$, we have the series of subgroups

$$1 < U < F < A \le N_L(F) \le L = \mathrm{GL}(e, p^a).$$

Furthermore, $N_L(F) \cong \mathrm{Sp}(2k, r)$ (cf. Lemma 5.3(ii)).

Since $b$ divides $a$, there is an embedding $\mathrm{GL}(a/b, p^b) \le \mathrm{GL}(a, p)$ and we may choose it so that $z$ lies in its image. Let $t$ be the preimage of $z$ under this embedding. By [14, Lemma 2.7], there exists $s \in \mathrm{GL}(a/b, p^b)$ of order $a/b$ such that $t^s = t^{p^b}$.

The embeddings $\mathrm{GF}(p) \le \mathrm{GF}(p^b) \le \mathrm{GF}(p^a)$ yield embeddings

$$\mathrm{GL}(e, p^a) \le \mathrm{GL}(e \cdot (a/b), p^b) \le \mathrm{GL}(d, p).$$

In particular, $A = C_H(U)$ is a subgroup of $\mathrm{GL}(e \cdot (a/b), p^b)$.

By Lemma 5.3(i), $L_1 = \mathrm{GL}(e, p^b)$ contains a subgroup $F_1 = U_1 \circ E$, where $U_1 = Z(L_1)$ and $E$ is extraspecial of order $r^{2k+1}$. Hence we have the series of subgroups

$$1 < U_1 < F_1 < A_1 \le N_{L_1}(F_1) \le L_1 = \mathrm{GL}(e, p^b).$$

It follows from Lemma 5.3(ii) that $N_{L_1}(F_1) \cong \mathrm{Sp}(2k, r) \cong N_L(F)$. Since $U = \langle t \otimes I_e\rangle$ commutes with all matrices from $I_{a/b} \otimes N_{L_1}(F_1)$,

$$N_{\mathrm{GL}(e\cdot(a/b), p^b)}(F) = \langle U, I_{a/b} \otimes N_{L_1}(F_1)\rangle.$$

Therefore $A \le \langle U, I_{a/b} \otimes N_{L_1}(F_1)\rangle$ and there exists $A_1 = \langle x_1, \dots, x_l\rangle \le L_1$ such that $A = U \circ (I_{a/b} \otimes A_1)$.

Since $H/A$ is cyclic of order dividing $a$ and $s \otimes I_e$ commutes with all matrices from $I_{a/b} \otimes A_1$, it follows that $\langle A, s \otimes I_e\rangle$ is conjugate to a normal subgroup of $H$ of index dividing $b$, as required.

It remains to show that $A_1$ is primitive in $\mathrm{GL}(e, p^b)$. By way of contradiction, assume that there is a proper $\mathrm{GF}(p^b)$-subspace $\overline{W}$ of the natural $\mathrm{GF}(p^b)A_1$-module $\overline{V}$ with $\overline{V} = \bigoplus_{\overline{g} \in A_1} \overline{W}^g$. Consider an embedding of $\mathrm{GL}(e, p^b)$ into $\mathrm{GL}(e, p^a)$ such that $A = A_1 \circ U$. Therefore

$$\widetilde{V} = \mathrm{GF}(p^a) \otimes_{\mathrm{GF}(p^b)} \overline{V} = \bigoplus_{g \in A} \widetilde{W}^g,$$

where $\widetilde{V}$ is the natural $\mathrm{GF}(p^a)A$-module and $\widetilde{W} = \mathrm{GF}(p^a) \otimes_{\mathrm{GF}(p^b)} \overline{W}$. Hence $A$ is imprimitive in $\mathrm{GL}(e, p^a)$.

Since $U = \langle z \otimes I_e\rangle$ where $z$ is a Singer cycle of $\mathrm{GL}(a, p)$, the $\mathrm{GF}(p^a)$-subspaces $\widetilde{W}^g$ are $U$-invariant for all $g \in A$. Therefore

$$V = \mathrm{GF}(p^a) \otimes_{\mathrm{GF}(p)} \widetilde{V} = \bigoplus_{g \in A} W^g,$$

where $V$ is the natural $\mathrm{GF}(p)A$-module and $W = \mathrm{GF}(p^a) \otimes_{\mathrm{GF}(p)} \widetilde{W}$.

Now $H = \langle A, x\rangle$, where $x$ induces a field automorphism $\psi$ of $U$ (cf. Theorem 5.1(iv)). Since $W^g$ is $\psi$-invariant for each $g \in A$,

$$V = \bigoplus_{h \in H} W^h,$$

which contradicts the primitivity of $H$ in $\mathrm{GL}(d, p)$.

(ii) We put $b = 1$ and choose an extraspecial subgroup $F_1 \le L_1 = \mathrm{GL}(e, p)$ such that $N_{L_1}(F_1) \cong N_L(F)$ (cf. Lemma 5.3); the choice depends on which of two conjugacy classes contains $F$. The rest of the proof is similar to that of (i). $\square$

## 6. Primitive solvable linear groups: computations

To complete the proof of Theorem 1.1, it suffices to establish the following.

**Theorem 6.1.** *Let $G$ be a linearly primitive solvable permutation group with exceptional parameters listed in columns 2–5 of Table 2. Then $G^{(3)}$ is solvable.*

**Proof**. Let $G_0$ be a point stabilizer of $G$ with underlying vector space $V$. By the monotonicity of the 3-closure operator, we may assume that $H = G_0$ is a maximal solvable primitive linear group.

Our proof is computational; more details are given in Section 6.1. For each choice of the parameters $p$, $d$, $a$, $e$, we compute a list $\mathcal{H} = \mathcal{H}(p, d, a, e)$ of solvable primitive subgroups $H$ of $\mathrm{GL}(d, p)$, ensuring that $\mathcal{H}$ includes representatives of all conjugacy classes of such *maximal* solvable subgroups for specified $a$ and $e$.

For every $H \in \mathcal{H}$, we search for nonzero $\alpha \in V$ such that one of the following conditions is satisfied:

(A) $\alpha^H$ is a regular orbit of $H$, so Corollary 2.5 can be applied;
(B) the restriction of $H$ to $\alpha^H$ is 2-closed, so Lemma 4.4 can be applied.

Of course, (A) implies (B). If we find such $\alpha$, then $G$ is 3-closed by Corollary 2.5 and Lemma 4.4, respectively; so $G^{(3)} = G$ is solvable. For those groups $H$ where no such $\alpha$ is found, we verify that $H$ acts transitively on the nonzero vectors of $V$. In this case, $G^{(3)}$ is solvable by Lemma 4.3.

Our results are summarized in Table 2. The 7th column lists the numbers of groups $H \in \mathcal{H}$ for which conditions (A) or (B) are satisfied; we indicate (A) or (B) by writing "partly regular" or "2-closed constituent", respectively; we write "transitive" if $H \in \mathcal{H}$ acts transitively on the nonzero vectors of $V$. Detailed results, including the GAP procedures used, generators of $H$, vectors $\alpha$, and certificates for (A) and (B), are available at [24]. □

6.1. **Constructing $\mathcal{H}$.** The construction of the list $\mathcal{H}$ for given parameters $p$, $d$, $a$, $e$ naturally divides into two cases I and II; the 6th column of Table 2 indicates which of them is applied to the stated parameters $p$, $d$, $a$, $e$.

**Case I.** Where possible, using the GAP package IRREDSOL, we constructed the list $\mathcal{L}$ of all solvable primitive subgroups of $\mathrm{GL}(d, p)$. By Theorem 5.1, every group $H \in \mathcal{H}$ has order $(p^a - 1)\, e^2\, s(H)\, a'$, where $s(H) = |M|$ is the order of a maximal solvable subgroup $M \cong A/F$ of the corresponding linear group $S$ from Lemma 5.4, and $a'$ is a divisor of $a$. Table 1 lists the possible values of $s(H)$ for all relevant values of $e$. By filtering $\mathcal{L}$ with respect to the possible orders, we obtain $\mathcal{H}$.

**Case II.** We use auxiliary results from Section 5 and [16, Section 2.5] together with computations in Magma to construct $\mathcal{H}$. Recall that $e = r^k$ where $p \neq r \in \{2, 3\}$, $d = ae$, and $b$ is the least positive integer with $p^b \equiv 1 \pmod{r^c}$ where $c = 2$ for $r = 2$ and $c = 1$ otherwise. Since $r \in \{2, 3\}$, we deduce that $b \leq 2$. For each remaining set of parameters, we proceed as follows.

1. If $b$ divides $a$ and IRREDSOL contains the solvable primitive subgroups of $\mathrm{GL}(e, p^b)$, or $b$ does not divide $a$ and IRREDSOL contains the solvable primitive subgroups of $\mathrm{GL}(e, p)$, then we construct a list $\mathcal{H}_0$ by extracting from the relevant output those groups containing an extraspecial subgroup of order $r^{2k+1}$ and proceed to Step 5.

2. Otherwise, using Holt's implementation in MAGMA of the algorithm of [7], we construct in $\mathrm{GL}(e,p^b)$ extraspecial subgroups $E$ of order $r^{2k+1}$ (one if $b$ divides $a$, and two if not) and the corresponding subgroups $F = E \circ U$, where $U = Z(\mathrm{GL}(e,p^b))$, and normalizers $N = N_{\mathrm{GL}(e,p^b)}(F)$, where $N/F \cong \mathrm{Sp}(2k,r)$ or $O^\varepsilon(2k,2)$ (cf. items (i) and (ii) of Lemma 5.3).
3. Lemma 5.3(iii) implies that $A/F$ is a maximal solvable subgroup of $N/F$ such that the $r$-radical of $A/F$ is trivial if $N/F \cong \mathrm{Sp}(2k,r)$ and otherwise has order at most 2. Using standard tools in MAGMA, we construct the list $\mathcal{L}$ consisting of all maximal solvable subgroups of $\mathrm{Sp}(2k,r)$ with trivial $r$-radical (if $b$ divides $a$), or of all maximal solvable subgroups of $O^+(2k,2)$ and $O^-(2k,2)$ with $r$-radical of order at most 2 (if $b$ does not divide $a$).
4. Following [16, Theorems 2.5.35 and 2.5.37], for each subgroup in $\mathcal{L}$, we produce generators of its complete preimage in $N$. Thus we obtain the list $\mathcal{H}_0$ containing up to conjugation all maximal solvable primitive subgroups of $\mathrm{GL}(e,p^b)$ if $b$ divides $a$, and an equivalent list in $\mathrm{GL}(e,p)$ if not.
5. Suppose $b$ divides $a$. By Lemma 5.5(i), every maximal solvable primitive subgroup $H$ of $\mathrm{GL}(d,p)$ contains up to conjugation an appropriate normal subgroup $H_1$ of index $b \le 2$ in $H$. We construct $t, s$ as in Lemma 5.5(i). For $H_0 \le \mathrm{GL}(e,p^b)$ in $\mathcal{H}_0$, define
$$H_1 = \langle t \otimes I_e, s \otimes I_e, I_{a/b} \otimes H_0 \rangle \le \mathrm{GL}(d,p).$$
If $b = 1$, then we take as $\mathcal{H}$ the set consisting of $H := H_1$ for every $H_0 \in \mathcal{H}_0$. If $b = 2$, then let $F = \mathrm{Fit}(H_1)$ and define
$$L = N_{\mathrm{GL}(d,p)}(F) \cong F : (\mathrm{Sp}(2k,r) : \mathbb{Z}_a).$$
Now we take as $\mathcal{H}$ the set consisting of $H := N_L(H_1)$ for every $H_0 \in \mathcal{H}_0$.
   Suppose $b$ does not divide $a$. We construct $t, s$ as in Lemma 5.5(ii), and take as $\mathcal{H}$ the set
$$\{\langle t \otimes I_e, s \otimes I_e, I_a \otimes H_0 \rangle \mid H_0 \in \mathcal{H}_0\}.$$
The same lemma guarantees that up to conjugation all maximal solvable primitive subgroups of $\mathrm{GL}(d,p)$ are in $\mathcal{H}$.

6.2. **Processing $\mathcal{H}$.** We discuss briefly how we process each $H \in \mathcal{H}$. For given $\alpha \in V$, condition (A) is readily checked. Since the `TwoClosure` command in GAP is time consuming, we use the GAP package COCO2P [9] to verify (B). Namely, we compute the automorphism group $\widetilde{H}$ of the coherent configuration associated with the restriction of $H$ to $\alpha^H$ using the COCO2P commands `ColorGraph` and `AutomorphismGroup`. Now $H$ is 2-closed if and only if $H = \widetilde{H}$ [2, Corollary 2.2.18].

It was sometimes infeasible to compute all $H$-orbits on $V$, so we randomly selected vectors $\alpha$ until we found one which satisfies either (A) or (B). Hence, in principle, some cases resolved by (B) could also be resolved by (A).

Table 2: Results for groups with exceptional parameters

| No. | $e$ | $p$ | $d$ | $a$ | case | results |
|---|---|---|---|---|---|---|
| 1 | 16 | 3 | 16 | 1 | II | 780, partly regular<br>21, 2-closed constituent |
| 2 | 16 | 5 | 16 | 1 | II | 1085, partly regular |
| 3 | 9 | 2 | 18 | 2 | I | 31, 2-closed constituent |

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | 9 | 7 | 9 | 1 | II | 44, partly regular |
| 5 | 9 | 13 | 9 | 1 | II | 44, partly regular |
| 6 | 9 | 2 | 36 | 4 | II | 7, partly regular |
| 7 | 9 | 19 | 9 | 1 | II | 44, partly regular |
| 8 | 9 | 5 | 18 | 2 | II | 44, partly regular |
| 9 | 8 | 3 | 8 | 1 | I | 6, 2-closed constituent |
| 10 | 8 | 5 | 8 | 1 | I | 4, 2-closed constituent |
| 11 | 8 | 7 | 8 | 1 | II | 30, partly regular<br>1, 2-closed constituent |
| 12 | 8 | 3 | 16 | 2 | II | 63, partly regular |
| 13 | 8 | 11 | 8 | 1 | II | 122, partly regular |
| 14 | 8 | 13 | 8 | 1 | II | 63, partly regular |
| 15 | 8 | 17 | 8 | 1 | II | 63, partly regular |
| 16 | 8 | 19 | 8 | 1 | II | 123, partly regular |
| 17 | 8 | 5 | 16 | 2 | II | 4, partly regular |
| 18 | 8 | 3 | 24 | 3 | II | 6, partly regular |
| 19 | 4 | 3 | 4 | 1 | I | 3, 2-closed constituent |
| 20 | 4 | 5 | 4 | 1 | I | 2, 2-closed constituent |
| 21 | 4 | 7 | 4 | 1 | I | 13, 2-closed constituent |
| 22 | 4 | 3 | 8 | 2 | I | 22, 2-closed constituent |
| 23 | 4 | 11 | 4 | 1 | I | 3, 2-closed constituent |
| 24 | 4 | 13 | 4 | 1 | I | 2, 2-closed constituent |
| 25 | 4 | 17 | 4 | 1 | I | 2, 2-closed constituent |
| 26 | 4 | 19 | 4 | 1 | I | 7, partly regular |
| 27 | 4 | 23 | 4 | 1 | I | 11, partly regular<br>1, 2-closed constituent |
| 28 | 4 | 5 | 8 | 2 | I | 41, partly regular<br>21, 2-closed constituent |
| 29 | 4 | 3 | 12 | 3 | I | 9, partly regular |
| 30 | 4 | 29 | 4 | 1 | I | 2, 2-closed constituent |
| 31 | 4 | 31 | 4 | 1 | I | 16, partly regular |
| 32 | 4 | 37 | 4 | 1 | I | 2, 2-closed constituent |
| 33 | 4 | 41 | 4 | 1 | I | 2, 2-closed constituent |
| 34 | 4 | 43 | 4 | 1 | I | 6, 2-closed constituent |
| 35 | 4 | 47 | 4 | 1 | I | 24, partly regular<br>2, 2-closed constituent |
| 36 | 4 | 7 | 8 | 2 | I | 11, partly regular<br>17, 2-closed constituent |
| 37 | 4 | 53 | 4 | 1 | I | 2, 2-closed constituent |
| 38 | 4 | 59 | 4 | 1 | I | 2, 2-closed constituent<br>1, partly regular |
| 39 | 4 | 61 | 4 | 1 | I | 2, 2-closed constituent |
| 40 | 4 | 67 | 4 | 1 | II | 2, partly regular |
| 41 | 4 | 71 | 4 | 1 | II | 2, partly regular |
| 42 | 4 | 3 | 16 | 4 | II | 2, partly regular |
| 43 | 4 | 11 | 8 | 2 | II | 2, partly regular |
| 44 | 4 | 5 | 12 | 3 | II | 2, partly regular |
| 45 | 4 | 13 | 8 | 2 | II | 2, partly regular |
| 46 | 4 | 3 | 20 | 5 | II | 3, partly regular |
| 47 | 3 | 2 | 6 | 2 | I | 2, 2-closed constituent |

| | | | | | | |
|---|---|---|---|---|---|---|
| 48 | 3 | 7 | 3 | 1 | I | 1, 2-closed constituent |
| 49 | 3 | 13 | 3 | 1 | I | 1, 2-closed constituent |
| 50 | 3 | 2 | 12 | 4 | I | 3, 2-closed constituent |
| 51 | 3 | 19 | 3 | 1 | I | 1, 2-closed constituent |
| 52 | 3 | 5 | 6 | 2 | I | 4, 2-closed constituent |
| 53 | 3 | 7 | 6 | 2 | I | 5, partly regular |
| 54 | 3 | 2 | 18 | 6 | I | 7, partly regular |
| | | | | | I | 11, 2-closed constituent |
| 55 | 3 | 11 | 6 | 2 | I | 4, partly regular |
| 56 | 3 | 13 | 6 | 2 | I | 4, partly regular |
| 57 | 3 | 2 | 24 | 8 | II | 1, partly regular |
| 58 | 3 | 17 | 6 | 2 | II | 1, partly regular |
| 59 | 3 | 7 | 9 | 3 | II | 1, partly regular |
| 60 | 3 | 19 | 6 | 2 | II | 1, partly regular |
| 61 | 2 | 3 | 2 | 1 | I | 2, transitive |
| 62 | 2 | 5 | 2 | 1 | I | 1, transitive |
| 63 | 2 | 7 | 2 | 1 | I | 5, 2-closed constituent |
| 64 | 2 | 3 | 4 | 2 | I | 6, 2-closed constituent |
| 65 | 2 | 11 | 2 | 1 | I | 3, 2-closed constituent |
| 66 | 2 | 13 | 2 | 1 | I | 2, 2-closed constituent |
| 67 | 2 | 17 | 2 | 1 | I | 1, 2-closed constituent |
| 68 | 2 | 19 | 2 | 1 | I | 3, 2-closed constituent |
| 69 | 2 | 23 | 2 | 1 | I | 7, 2-closed constituent |
| 70 | 2 | 5 | 4 | 2 | I | 20, 2-closed constituent |
| 71 | 2 | 3 | 6 | 3 | I | 3, partly regular |
| | | | | | | 1, 2-closed constituent |
| 72 | 2 | 29 | 2 | 1 | I | 1, 2-closed constituent |
| 73 | 2 | 7 | 4 | 2 | I | 12, 2-closed constituent |
| 74 | 2 | 3 | 8 | 4 | I | 30, 2-closed constituent |
| 75 | 2 | 11 | 4 | 2 | I | 14, 2-closed constituent |
| 76 | 2 | 5 | 6 | 3 | I | 2, 2-closed constituent |
| 77 | 2 | 13 | 4 | 2 | I | 8, partly regular |
| 78 | 2 | 3 | 10 | 5 | I | 4, partly regular |
| 79 | 2 | 17 | 4 | 2 | I | 8, 2-closed constituent |
| 80 | 2 | 7 | 6 | 3 | I | 22, partly regular |
| 81 | 2 | 19 | 4 | 2 | I | 19, 2-closed constituent |
| 82 | 2 | 23 | 4 | 2 | I | 12, partly regular |
| 83 | 2 | 5 | 8 | 4 | I | 27, partly regular |
| 84 | 2 | 3 | 12 | 6 | I | 17, partly regular |
| 85 | 2 | 29 | 4 | 2 | I | 8, partly regular |
| 86 | 2 | 31 | 4 | 2 | I | 8, partly regular |
| 87 | 2 | 11 | 6 | 3 | I | 7, partly regular |
| 88 | 2 | 37 | 4 | 2 | I | 8, partly regular |
| 89 | 2 | 41 | 4 | 2 | I | 8, partly regular |
| 90 | 2 | 43 | 4 | 2 | I | 8, partly regular |
| 91 | 2 | 3 | 14 | 7 | I | 4, partly regular |
| 92 | 2 | 13 | 6 | 3 | I | 5, partly regular |
| 93 | 2 | 47 | 4 | 2 | I | 8, partly regular |
| | | | | | I | 1, 2-closed constituent |
| 94 | 2 | 7 | 8 | 4 | I | 23, partly regular |

| | | | | | | |
|---|---|---|---|---|---|---|
| 95 | 2 | 53 | 4 | 2 | I | 8, partly regular |
| 96 | 2 | 5 | 10 | 5 | I | 2, partly regular |
| 97 | 2 | 59 | 4 | 2 | I | 8, partly regular |
| 98 | 2 | 61 | 4 | 2 | I | 8, partly regular |
| 99 | 2 | 67 | 4 | 2 | II | 1, partly regular |
| 100 | 2 | 17 | 6 | 3 | II | 1, partly regular |
| 101 | 2 | 71 | 4 | 2 | II | 4, partly regular |
| 102 | 2 | 73 | 4 | 2 | II | 1, partly regular |

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