

CLOSURES OF PERMUTATION GROUPS WITH RESTRICTED NONABELIAN COMPOSITION FACTORS

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ABSTRACT. Given a permutation group G on a finite set Ω , let $G^{(k)}$ denote the k -closure of G , that is, the largest permutation group on Ω having the same orbits in the induced action on Ω^k as G . Recall that a group is $\text{Alt}(d)$ -free if it does not contain a section isomorphic to the alternating group of degree d . Motivated by some problems in computational group theory, we prove that the k -closure of an $\text{Alt}(d)$ -free group is again $\text{Alt}(d)$ -free for $k \geq 4$ and $d \geq 25$.

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1. INTRODUCTION

Throughout all groups and sets on which they act are assumed to be finite.

One of the motivations for the present paper is a computational problem in which one needs to find efficiently the automorphism group $\text{Aut}(\mathfrak{S})$ of a given set \mathfrak{S} of relations (generally speaking of different arities). In the case when all the relations of \mathfrak{S} are binary, this problem is equivalent to the famous Graph Isomorphism Problem and can be solved by the Babai algorithm [2] in quasipolynomial time in the size of \mathfrak{S} . It is currently unknown whether the automorphism group of a graph can be found in polynomial time. It is surprising, however, that apparently the problem does not become easier if a sufficiently large subgroup of $\text{Aut}(\mathfrak{S})$ is known in advance. In the present paper we take a step towards the latter problem by

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studying the structure of the group $\text{Aut}(\mathfrak{S})$ in the case when all the relations of \mathfrak{S} are of arity at least 4. To be more precise, we recall briefly the method of invariant relations that was introduced and developed by Wielandt in the late 1960s [32] and used in the above-mentioned Babai's paper in the framework of multidimensional coherent configurations.

The method of invariant relations was considered by Wielandt as one of the main tools for studying actions of a group on a set [33]. This method is based on the concept of k -equivalence of permutation groups, where k is a natural number. Namely, two permutation groups G and H on a set Ω are said to be k -equivalent if for every k -ary relation $s \subseteq \Omega^k$, we have $s^G = s$ if and only if $s^H = s$. In each class of k -equivalent groups there is a unique maximal (with respect to inclusion) group, which is called the k -closure of each group G belonging to this class and is denoted by $G^{(k)}$. Wielandt proved [32] (see also Proposition 2.2 below) that

$$(1.1) \quad \text{Sym}(\Omega) \geq G^{(1)} \geq G^{(2)} \geq \dots \geq G^{(k)} = G^{(k+1)} = \dots = G$$

for some k . Thus, the closures of the group G can be considered as approximations to G . On the other hand, the concept of the closure is a powerful tool for dealing with the above-mentioned computational problem, because the k -closures are exactly the groups $\text{Aut}(\mathfrak{S})$ where \mathfrak{S} is the set of relations of arity at least k left invariant by the action of G .

From the computational point of view the k -closure problem consists in finding the k -closure of a given permutation group (it is assumed that permutation groups are given by generating sets, see [27]). Note that the case $k = 1$ is trivial, because the 1-closure of any permutation group G is the direct product of symmetric groups acting on the orbits of G . In this setting, polynomial-time algorithms for computing the 2-closure were constructed for the nilpotent groups [21], the groups of odd order [7], and the supersolvable groups [22]. Very recently it was shown that the 3-closure of a solvable group can also be computed in polynomial time [23].

The above computational results are based on some "similarity" of k -equivalent groups. It is clear from (1.1) that the larger the number k is the more "similar" the k -equivalent groups are. In [32] Wielandt proved that if groups G and H are k -equivalent and $k \geq 1$ (respectively, $k \geq 2$, $k \geq m$), then G is transitive (respectively, imprimitive, m -transitive) if and only if so is H . More interestingly, however, that k -equivalent groups are "similar" not only as permutation groups, but also as abstract groups. For example, if G is an abelian group (respectively, a p -group, a group of odd order), then, for $k \geq 2$, the group H is an abelian group (respectively, a p -group, a group of odd order), see [32]. Recently it was proved [19] that a similar statement is true for solvable groups if $k \geq 3$ (the example of 2-transitive solvable groups shows that $k = 2$ cannot be taken here). Each of these statements expresses the fact that the k -closure (for the corresponding k) of any permutation group of the corresponding class also lies in it. This fact, a polynomial upper bound on the orders of solvable primitive groups [20], and the Babai–Luks algorithm [4] form a foundation of the computational results from the previous paragraph (except for the case of supersolvable groups, where the algorithm is constructed on more subtle arguments, because the 2-closure of a supersolvable group need not be even solvable).

In fact, the Babai–Luks algorithm [4] can be applied in a much broader situation, for the class of $\text{Alt}(d)$ -free groups. Recall that H is a *section* of a group G if H is a homomorphic image of a subgroup of G . Now a group is $\text{Alt}(d)$ -free, $d \geq 5$,

if it does not contain a section isomorphic to the alternating group of degree d . In particular, solvable groups are $\text{Alt}(5)$ -free. Furthermore, the classical result by Babai, Cameron, and Pálffy [3] states, up to the language (see, e.g., a remark before Theorem III in [26]), that the size of an $\text{Alt}(d)$ -free primitive permutation group of degree n is bounded from above by n^c , where c is a constant depending only on d . Thus, in the context of the previous paragraph, we arrive at the question when the k -closure of an $\text{Alt}(d)$ -free group is also $\text{Alt}(d)$ -free. The goal of the present paper is to answer this question even for a wider class of groups.

To proceed, we recall another concept introduced by Wielandt [31, Definition 11.3]. A class \mathfrak{X} of (abstract) groups is said to be *complete* if it is closed with respect to taking subgroups, quotients, and extensions. The examples are the class of all groups, p -groups for any fixed prime p , and solvable groups. In fact, given a fixed set S (finite or infinite) of simple groups, the class of groups whose composition factors are sections of groups from S is complete.

We are ready to state our main result.

Theorem 1.1. *Let \mathfrak{X} be a complete class including all $\text{Alt}(25)$ -free groups. Then the k -closure of every permutation group from \mathfrak{X} belongs to \mathfrak{X} for each $k \geq 4$.*

Since the class of all $\text{Alt}(d)$ -free groups is complete, the statement below is an immediate consequence of Theorem 1.1.

Corollary 1.2. *If G is an $\text{Alt}(d)$ -free group with $d \geq 25$, then $G^{(k)}$ is $\text{Alt}(d)$ -free group for $k \geq 4$.*

Note that the constant 4 in Theorem 1.1 is the best possible and, if $k = 4$, then the same holds true for the constant 25. This follows from the two examples below.

(i) The affine group $G = \text{AGL}_m(2)$ is 3-transitive in its natural action on a linear space of dimension $m \geq 2$ over the field of order 2. It follows that $G^{(3)} = \text{Sym}(2^m)$, so the theorem does not hold for $k \leq 3$.

(ii) The Mathieu group $G = M_{24}$ is $\text{Alt}(9)$ -free and acts on 24 points 5-transitively, so $G^{(4)} = \text{Sym}(24)$ is not $\text{Alt}(24)$ -free.

It should be also noted that there are a lot of complete classes of groups distinct from the class of $\text{Alt}(d)$ -free groups, where $d \geq 25$, that satisfy the hypothesis of Theorem 1.1. Indeed, consider any nonabelian simple group S containing a section isomorphic to $\text{Alt}(d)$. Then the class of all groups whose nonabelian factors are either $\text{Alt}(25)$ -free or sections of S is such an example.

Let us now briefly discuss the strategy of the proof of the main result. Suppose that \mathfrak{X} is a complete class of groups satisfying the hypothesis of Theorem 1.1 and $k \geq 4$. We are interested in the question of when $G^{(k)} \in \mathfrak{X}$ for any permutation group $G \in \mathfrak{X}$. Note that along with each pair of permutation groups, the class \mathfrak{X} contains their transitive and intransitive direct products, as well as their imprimitive and primitive wreath products. Explicit formulas for the k -closures of these products are known, see [11, 22], hence our question is reduced to the case when the group G is primitive and even basic [23]. Recall that a permutation group is *nonbasic* if it is contained in a wreath product in the product action; it is *basic* otherwise, see [5, Section 4.3].

Apparently, the first nontrivial results on the closures of primitive groups were obtained in [15, 24], where it was proved that in most cases the socle of the group is preserved when taking the k -closure, and all the cases were described when this is

not so. These results allow us to reduce the situation to the case of the affine groups (see Section 4). When working with affine groups it is natural to use Aschbacher's classification [1], which divides them structurally into 9 classes. An important step in the study of k -closures of affine groups was made in [29, 30], where for each Aschbacher class an explicit value for the number k was found such that the k -closure of each group from this class remains in the same class. Following the Aschbacher classification, we consider the classes to which group G might belong. While excluding the Aschbacher classes related with tensor decompositions of the underlying linear space, we need some new tools to control the k -equivalence and k -closure for stabilizers of the tensor decompositions, which are the subject of Section 3. Finally (Subsections 4.1 and 4.2), we exclude the cases when the group G is of symplectic type or belongs to the last Aschbacher class (the classes \mathcal{C}_6 and \mathcal{C}_9 in the notation in [30]), thus completing the proof of the theorem.

To make the paper as self-contained as possible we cite the relevant results concerning the k -equivalence and k -closure of permutation groups in Section 2. The concluding remarks are collected at the end of the paper (Section 5).

2. PRELIMINARIES

Let G be a permutation group on Ω , and let $k\text{-rel}(G)$, $k \geq 1$, denote the set of G -invariant k -ary relations.

Lemma 2.1. [32, Theorem 4.7] *Let G and H be permutation groups on the same set Ω , and $k \geq 1$. Then $k\text{-rel}(H) \subseteq k\text{-rel}(G)$ if and only if for any $x_1, \dots, x_k \in \Omega$ and any $g \in G$ there exists some $h \in H$ such that $x_i^g = x_i^h$ for all $i = 1, \dots, k$.*

We say that G is k -equivalent to H if $k\text{-rel}(G) = k\text{-rel}(H)$. Observe that the k -closure of G is the largest permutation group on Ω which is k -equivalent to G . Recall that a *base* of a permutation group is a subset whose pointwise stabilizer is trivial. We collect some basic properties of k -closures and k -equivalence which will be used all throughout the text.

Proposition 2.2. *Let G and H be permutation groups on Ω .*

- (i) *For $k \geq 1$, if G and H are $(k+1)$ -equivalent, then they are k -equivalent. In particular, $G^{(k+1)} \leq G^{(k)}$.*
- (ii) *For $k \geq 2$, if G and H are k -equivalent, then their orbits and systems of imprimitivity are the same.*
- (iii) *For $k \geq 2$ and $\alpha \in \Omega$, if G and H are k -equivalent, then G_α is $(k-1)$ -equivalent to H_α .*
- (iv) *For $k \geq 1$, if $G \leq H$, then $G^{(k)} \leq H^{(k)}$.*
- (v) *For $k \geq 1$, we have $(G^{(k)})^{(k)} = G^{(k)}$.*
- (vi) *Suppose G has a base of size $k \geq 1$, then $G^{(k+1)} = G$. In particular, if G has a faithful regular orbit, then $G^{(2)} = G$.*

Proof. Parts (i)-(vi) follow from Theorems 4.3, 4.11, 4.12, 5.7, 5.9, and 5.12 of [32], respectively. \square

For $\Delta \subseteq \Omega$ let G^Δ denote the group induced on Δ by the setwise stabilizer of Δ in G . We may view G^Δ as a subgroup of $\text{Sym}(\Delta)$.

Let Δ be some block of imprimitivity for a transitive permutation group G . Translates of Δ give a system of imprimitivity of G which we will denote by Ω/Δ . Let $G^{\Omega/\Delta}$ denote the permutation group induced by G on Ω/Δ by permuting the

blocks. Clearly, G^Δ and $G^{\Omega/\Delta}$ are sections of G , so if G lies in some complete class \mathfrak{X} then $G^\Delta \in \mathfrak{X}$ and $G^{\Omega/\Delta} \in \mathfrak{X}$. Moreover, G can be embedded into the wreath product in imprimitive action $G^\Delta \wr G^{\Omega/\Delta}$.

Given two permutation groups $L \leq \text{Sym}(\Delta)$ and $K \leq \text{Sym}(\Gamma)$, let $L \uparrow K \leq \text{Sym}(\Delta^\Gamma)$ denote the wreath product in product action. As an abstract group it is isomorphic to $L^{|\Gamma|} \rtimes K$, where $L^{|\Gamma|}$ acts on Δ^Γ coordinatewise, while K permutes the coordinates. A primitive group G is called *basic* if it cannot be embedded into the wreath product in product action for $|\Delta| > 1$, $|\Gamma| > 1$.

Let $G \leq \text{Sym}(\Omega)$ be a primitive permutation group preserving a nontrivial product decomposition $\Omega = \Delta^\Gamma$. Without loss of generality we may assume that $\Gamma = \{1, \dots, m\}$ for some $m > 1$. For $\gamma \in \Gamma$ define a partition of Ω into $|\Delta|$ parts

$$\Delta_\gamma = \{\Delta \times \dots \times \Delta \times \{x\} \times \Delta \times \dots \times \Delta \mid x \in \Delta\},$$

i.e. each part of Δ_γ consists of those tuples from Δ^Γ which have its γ -th coordinate constant. There is a natural bijective correspondence between the partitions Δ_γ , $\gamma \in \Gamma$, and the points of Γ . The group G permutes the partitions Δ_γ , $\gamma \in \Gamma$, so let G^Γ denote the permutation group on Γ obtained through this correspondence. With some abuse of notation, let G^{Δ_γ} denote the permutation group on Δ induced by the stabilizer of Δ_γ in G . In the terminology of [25, Section 5.3], the group G^{Δ_γ} is the γ -component of G , and there is an explicit formula for it [25, (5.11)].

Clearly, G^Γ and G^{Δ_γ} are sections of G , and [25, Theorem 5.14 (ii)] implies that G can be embedded into the wreath product $G^{\Delta_\gamma} \uparrow G^\Gamma$ in product action.

To compute the k -closure of a group in product action we need another closure operator. Observe that a permutation group $K \leq \text{Sym}(\Gamma)$ acts on the set of ordered partitions of Γ in at most $r \geq 1$ classes. Let $K^{[r]}$ denote the largest permutation group on Γ having the same orbits on the set of ordered partitions into at most r parts as K . This new operator, which we call the r -closure with respect to partitions, is indeed a closure operator and enjoys some properties similar to the k -closure, see [28, Section 3]. In particular, $K^{[r+1]} \geq K^{[r]}$ and $K^{[m]} = K$ for $m = |\Gamma|$.

There is a direct connection between these two types of closures.

Proposition 2.3 ([28, Lemma 3.2]). *Let $K \leq \text{Sym}(\Gamma)$ be a permutation group. For $k \geq 1$ we have $K^{[k+1]} \leq K^{(k)}$.*

Let $\text{Orb}_k(K)$ denote the set of orbits of $K \leq \text{Sym}(\Gamma)$ on Γ^k .

Proposition 2.4. *Let G and H be permutation groups on Ω .*

- (i) *Let $\Delta \subset \Omega$ be an orbit of G . For any $k \geq 1$ we have $(G^{(k)})^\Delta \leq (G^\Delta)^{(k)}$. Moreover, if H is k -equivalent to G then G^Δ is k -equivalent to H^Δ .*
- (ii) *Let G be a transitive imprimitive group with a nontrivial block $\Delta \subset \Omega$. For any $k \geq 2$,*

$$(G^\Delta \wr G^{\Omega/\Delta})^{(k)} = (G^\Delta)^{(k)} \wr (G^{\Omega/\Delta})^{(k)}.$$

Moreover, if H is k -equivalent to G , then H is also imprimitive, and G^Δ is k -equivalent to H^Δ and $G^{\Omega/\Delta}$ is k -equivalent to $H^{\Omega/\Delta}$.

- (iii) *Let G be a primitive nonbasic group preserving a nontrivial product decomposition $\Omega = \Delta^\Gamma$, and let $\gamma \in \Gamma$. For any $k \geq 2$ we have*

$$(G^{\Delta_\gamma} \uparrow G^\Gamma)^{(k)} = (G^{\Delta_\gamma})^{(k)} \uparrow (G^\Gamma)^{[r]},$$

where $r = \min\{|\text{Orb}_k(G^{\Delta_\gamma})|, |\Gamma|\}$. Moreover, if H is k -equivalent to G , then H is also nonbasic, and G^{Δ_γ} is k -equivalent to H^{Δ_γ} and $(G^\Gamma)^{[r]} = (H^\Gamma)^{[r]}$.

Proof. (i) Since $k\text{-rel}(G^\Delta) = k\text{-rel}(G) \cap \Delta^k$, groups G^Δ and H^Δ are k -equivalent and the second part of the claim follows. Now take $H = G^{(k)}$ and observe that $(G^{(k)})^\Delta$ is k -equivalent to G^Δ , hence $(G^{(k)})^\Delta \leq (G^\Delta)^{(k)}$.

(ii) Follows from [11, Theorem 2.4 and Lemma 2.5].

(iii) The closure formula is the main result of [28]. It follows from the closure formula that $H \leq (G^{\Delta_\gamma})^{(k)} \uparrow (G^\Gamma)^{[r]}$ so H is nonbasic. It also follows that $H^{\Delta_\gamma} \leq (G^{\Delta_\gamma})^{(k)}$ and $H^\Gamma \leq (G^\Gamma)^{[r]}$. Therefore $(H^{\Delta_\gamma})^{(k)} \leq (G^{\Delta_\gamma})^{(k)}$ and $(H^\Gamma)^{[r]} \leq (G^\Gamma)^{[r]}$. By exchanging the roles of H and G we derive $(G^{\Delta_\gamma})^{(k)} \leq (H^{\Delta_\gamma})^{(k)}$ and $(G^\Gamma)^{[r]} \leq (H^\Gamma)^{[r]}$. Hence $(G^{\Delta_\gamma})^{(k)} = (H^{\Delta_\gamma})^{(k)}$ which implies that G^{Δ_γ} is k -equivalent to H^{Δ_γ} , and $(G^\Gamma)^{[r]} = (H^\Gamma)^{[r]}$ as claimed. \square

Note that in Proposition 2.4 (iii) when $k \geq 3$ we have $|\text{Orb}_k(G^{\Delta_\gamma})| \geq k+1$ since $\Delta \geq 2$, so Proposition 2.3 implies that $(G^{\Delta_\gamma} \uparrow G^\Gamma)^{(k)} \leq (G^{\Delta_\gamma})^{(k)} \uparrow (G^\Gamma)^{(k)}$, cf. [28, Theorem 1.2].

Recall that a permutation group G on Ω is called *affine* if it contains a normal elementary abelian subgroup V acting regularly on Ω . In that case Ω can be identified with V and G decomposes into a semidirect product $G = V \rtimes G_0$, where G_0 is the zero stabilizer. The group G_0 acting on V can be viewed as a subgroup of $\text{GL}(V)$, and G is primitive if and only if G_0 acts irreducibly on V . In this case, $V = \text{Soc}(G)$.

Lemma 2.5. *Let V be a vector space over a prime field, and let $G = V \rtimes G_0$, $G_0 \leq \text{GL}(V)$, be an affine group with socle V . Suppose that $G^{(k)}$, $k \geq 2$, is also an affine group with socle V , i.e. $G^{(k)} = V \rtimes H_0$ for some $H_0 \leq \text{GL}(V)$. Then $H_0 = G_0^{(k-1)} \cap \text{GL}(V)$.*

Proof. By Proposition 2.2 (iii), G_0 is $(k-1)$ -equivalent to H_0 , hence $H_0 \leq G_0^{(k-1)} \cap \text{GL}(V)$. To prove the converse inclusion, let $h \in G_0^{(k-1)} \cap \text{GL}(V)$ be arbitrary. Let $v_1, \dots, v_k \in V$ be arbitrary vectors. Since $h \in G_0^{(k-1)}$, by Lemma 2.1 there exists $g \in G_0$ such that $(v_i - v_1)^g = (v_i - v_1)^h$ for $i = 2, \dots, k$. As elements h and g lie in $\text{GL}(V)$, we have $v_i^h = v_i^g + v_1^h - v_1^g$ for all $i = 2, \dots, k$. There exists some element $t \in V \leq G$ acting on V by translation by the vector $v_1^h - v_1^g$, that is, $v^t = v + v_1^h - v_1^g$ for all $v \in V$. Therefore $v_i^h = v_i^{gt}$ for all $i = 2, \dots, k$. Notice that this equality holds for $i = 1$ as well, indeed, $v_1^{gt} = v_1^g + v_1^h - v_1^g = v_1^h$. Finally, gt lies in G , so h lies in $G^{(k)}$ by Lemma 2.1, as the choice of $v_1, \dots, v_k \in V$ was arbitrary. Hence $G_0^{(k-1)} \cap \text{GL}(V) \leq G^{(k)}$ and since $G_0^{(k-1)} \cap \text{GL}(V)$ stabilizes the zero vector, we derive $G_0^{(k-1)} \cap \text{GL}(V) \leq H_0$. \square

The k -closure of a primitive affine group is also an affine group with the same socle for $k \geq 4$ according to the following result.

Lemma 2.6. [24, Lemma 4.1] *Let G be a primitive affine permutation group with socle V , and assume $k \geq 4$. Then $G^{(k)}$ is also an affine group with socle V .*

The following proposition implies that the groups of Lie type of bounded rank are $\text{Alt}(d)$ -free for d bounded in terms of the rank. A less precise inequality but with a more elementary proof can also be found in [6, Theorem 5.7A].

Proposition 2.7. *Suppose that $\text{Alt}(d)$ is a section of $\text{GL}_a(q)$ for some $a \geq 1$, $q \geq 2$. Then $a \geq d - 2$ for $d \geq 9$.*

Proof. Assume that a is the minimal integer such that $\mathrm{GL}_a(\mathbb{F})$ contains a section isomorphic to $\mathrm{Alt}(d)$, $d \geq 9$, for some field \mathbb{F} of positive characteristic. By [6, Lemma 5.7D], for some algebraically closed field \mathbb{K} of characteristic $p > 0$ there exists a subgroup G of $\mathrm{GL}_a(\mathbb{K})$ such that $G/Z(G) \simeq \mathrm{Alt}(d)$, and $Z(G)$ is a group of scalars. In particular, $\mathrm{Alt}(d)$ has a faithful projective p -modular representation of degree a . By [12, Proposition 5.3.7 (i)] we have $a \geq d - 2$. \square

When dealing with k -closures of affine permutation groups, we will use Aschbacher's classification of subgroups of linear groups [1]. Xu, Giudici, Li and Praeger [30] proved that Aschbacher's classes in $\Gamma\mathrm{L}_a(q)$ are preserved by k -closures for suitably chosen k , depending on the class; here we view a subgroup of $\Gamma\mathrm{L}_a(q)$ as a permutation group on \mathbb{F}_q^a .

We will follow [30] in our definitions and notations of Aschbacher classes. Recall that a subgroup of $\Gamma\mathrm{L}_a(q)$ not containing $\mathrm{SL}_a(q)$ lies in one of the nine classes $\mathcal{C}_1, \dots, \mathcal{C}_9$, defined mostly in terms of some geometric structure preserved by the groups lying in the class. We briefly outline the classes and relevant structures:

- \mathcal{C}_1 : Groups preserving a nontrivial proper subspace.
 - \mathcal{C}_2 : Groups acting imprimitively on the vector space.
 - \mathcal{C}_3 : Groups preserving the structure of an extension field.
 - \mathcal{C}_4 : Groups preserving a nontrivial decomposition of the vector space into the tensor product of two spaces of unequal dimensions.
 - \mathcal{C}_5 : Groups preserving the structure of a proper subfield.
 - \mathcal{C}_6 : Groups normalizing a subgroup of symplectic type.
 - \mathcal{C}_7 : Groups preserving a nontrivial decomposition of the vector space into the tensor product of several spaces of equal dimensions.
 - \mathcal{C}_8 : Groups preserving a nondegenerate alternating, hermitian or quadratic form.
- We will use notation $\mathcal{C}_{\mathrm{Sp}}$, \mathcal{C}_{U} and \mathcal{C}_{O} , respectively, depending on the form preserved.
- \mathcal{C}_9 : Groups which are not contained in any of $\mathcal{C}_1, \dots, \mathcal{C}_8$. These groups are almost simple modulo center.

Strictly speaking, each class \mathcal{C}_i , $i = 1, \dots, 8$, is defined by its maximal members. For example, $G \leq \Gamma\mathrm{L}_a(q)$ lies in \mathcal{C}_4 if $V = \mathbb{F}_q^a$ can be decomposed as $V = U \otimes W$, $\dim U > \dim W > 1$, and G can be conjugated inside the central product $(\mathrm{GL}(U) \otimes \mathrm{GL}(W)) \rtimes \mathrm{Aut}(\mathbb{F}_q)$ acting naturally on V . By [30, Proposition 4.4.1], this is equivalent to saying that G preserves the set of simple tensors, i.e. a unary relation of the form $\{u \otimes w \mid u \in U, w \in W\}$. This implies that $G^{(1)} \cap \Gamma\mathrm{L}_a(q)$ also lies in the class \mathcal{C}_4 . Essentially [30, Theorem 1.1] says that one can find similar relations for all of Aschbacher classes. Here we give a summary of the main results of [30].

Proposition 2.8. *Let $G \leq \Gamma\mathrm{L}_a(q)$, $a \geq 2$, act on \mathbb{F}_q^a naturally.*

- (i) *If $G \in \mathcal{C}_i$ for $i \in \{1, \dots, 7, \mathrm{Sp}, \mathrm{U}, \mathrm{O}\}$, then $G^{(k)} \cap \Gamma\mathrm{L}_a(q) \in \mathcal{C}_i$ for $k = 2$ if $i \in \{3, 6, \mathrm{Sp}\}$ and $k = 1$ otherwise.*
- (ii) *If $G \in \mathcal{C}_9$, then $G^{(2)} \cap \Gamma\mathrm{L}_a(q) \in \mathcal{C}_9$, unless $a = 4$, $q = 2$ and $G = \mathrm{Alt}(7)$.*

Proof. Part (i) follows from [30, Theorem 1.1], and part (ii) from [30, Proposition 3.3.1]. \square

Note that Aschbacher's theorem [1] applies not only to $\Gamma\mathrm{L}_a(q)$ but to the automorphism groups of the classical groups in general. Classes $\mathcal{C}_1, \dots, \mathcal{C}_9$ are defined similarly for the classical groups, and if a group G belongs to some \mathcal{C}_i , $i = 1, \dots, 8$,

defined for a classical group, then G lies in the same class but defined for $\Gamma L_a(q)$, if one forgets about the classical geometry preserved by G .

3. CLOSURES OF TENSOR PRODUCTS

In this section \mathfrak{X} is an arbitrary complete class which contains all solvable groups. In our descriptions of subgroups stabilizing tensor decompositions we follow [30, Sections 4.4 and 4.5].

3.1. Stabilizers of $X \otimes Y$. Let $V = X \otimes Y$ be a tensor product of vector spaces over a finite field \mathbb{F} . We will also use the symbol \otimes to denote the central product of two linear groups acting naturally on the corresponding tensor product of vector spaces. Recall that the central product $\text{GL}(X) \otimes \text{GL}(Y)$ acts on V by the following rule, if $g_X \in \text{GL}(X)$, $g_Y \in \text{GL}(Y)$ and $x \in X$, $y \in Y$, then $(x \otimes y)^{g_X \otimes g_Y} = x^{g_X} \otimes y^{g_Y}$.

The group of field automorphisms $\text{Aut}(\mathbb{F})$ acts on V as follows. First, choose an \mathbb{F} -basis x_1, \dots, x_r of X . Every $x \in X$ can be decomposed uniquely into a sum $x = \alpha_1 x_1 + \dots + \alpha_r x_r$, and one can define the action of $\sigma \in \text{Aut}(\mathbb{F})$ on X by $x^\sigma = \alpha_1^\sigma x_1 + \dots + \alpha_r^\sigma x_r$. By choosing some basis of Y one defines the action of $\text{Aut}(\mathbb{F})$ in the similar way. Finally, the action of $\sigma \in \text{Aut}(\mathbb{F})$ on $V = X \otimes Y$ is defined by $(x \otimes y)^\sigma = x^\sigma \otimes y^\sigma$, where $x \in X$, $y \in Y$.

It follows that the group $(\text{GL}(X) \otimes \text{GL}(Y)) \rtimes \text{Aut}(\mathbb{F})$ preserves the tensor decomposition $V = X \otimes Y$, and when $\dim X \neq \dim Y$ it is the largest group preserving that decomposition, see [30, Formula (4.4.1)]. Every element g of that group can be written in the form $g = (g_X \otimes g_Y)\sigma$, where $g_X \in \text{GL}(X)$, $g_Y \in \text{GL}(Y)$ and $\sigma \in \text{Aut}(\mathbb{F})$. If g admits a second decomposition $g = (g'_X \otimes g'_Y)\sigma'$, then $\sigma' = \sigma$ and there exists some nonzero $\alpha \in \mathbb{F}$ such that $g'_X = \alpha \cdot g_X$ and $g'_Y = \frac{1}{\alpha} \cdot g_Y$.

Take $G \leq (\text{GL}(X) \otimes \text{GL}(Y)) \rtimes \text{Aut}(\mathbb{F})$ where X and Y are vector spaces over \mathbb{F} . Recall that we can identify \mathbb{F}^\times with the center of $\text{GL}(X)$ and $\text{GL}(Y)$. For $g = (g_X \otimes g_Y)\sigma \in G$ define the cosets $\pi_X(g) = \mathbb{F}^\times \cdot g_X \sigma$ and $\pi_Y(g) = \mathbb{F}^\times \cdot g_Y \sigma$. Note that these elements are defined correctly. Indeed, if $g = (g'_X \otimes g'_Y)\sigma'$, then $\sigma' = \sigma$ and $g'_X = \alpha \cdot g_X$ for some $\alpha \in \mathbb{F}^\times$. Hence $\mathbb{F}^\times \cdot g'_X \sigma' = \mathbb{F}^\times \cdot \alpha g_X \sigma = \mathbb{F}^\times \cdot g_X \sigma$, so $\pi_X(g)$ does not depend on the choice of the decomposition of g . Similar reasoning holds for π_Y , so $\pi_X : G \rightarrow \text{PTL}(X)$ and $\pi_Y : G \rightarrow \text{PTL}(Y)$ are correctly defined maps, and moreover, π_X and π_Y are homomorphisms.

Assume that $\mathbb{F}^\times = Z(\text{GL}(V)) \leq G$, so $\ker \pi_X \cap \ker \pi_Y = \mathbb{F}^\times$. Since $\pi_X(G) \leq \text{PTL}(X)$, we can consider the full preimage G_X of $\pi_X(G)$ in $\text{GL}(X)$; define G_Y accordingly. By construction, G_X and G_Y contain \mathbb{F}^\times . Now, by Remak's theorem, G/\mathbb{F}^\times embeds into a direct product $\pi_X(G) \times \pi_Y(G) \leq \text{PTL}(X) \times \text{PTL}(Y)$, so G/\mathbb{F}^\times embeds into $G_X/\mathbb{F}^\times \times G_Y/\mathbb{F}^\times$.

Proposition 3.1. *Let X and Y be vector spaces over a finite field \mathbb{F} , and let $V = X \otimes Y$. Let G and H be subgroups of $(\text{GL}(X) \otimes \text{GL}(Y)) \rtimes \text{Aut}(\mathbb{F}) \leq \Gamma L(V)$. Assume that $\mathbb{F}^\times \leq G$ and $\mathbb{F}^\times \leq H$, and G and H are k -equivalent for some $k \geq 1$. Then G_X is k -equivalent to H_X , and G_Y is k -equivalent to H_Y .*

Proof. We will show that $k\text{-rel}(H_X) \subseteq k\text{-rel}(G_X)$. By Lemma 2.1, it is sufficient to show that for all $x_1, \dots, x_k \in X$ and $\gamma_X \in G_X$ there exists $\beta_X \in H_X$ such that $x_i^{\gamma_X} = x_i^{\beta_X}$, $i = 1, \dots, k$. Notice that we may assume that the vector x_i is nonzero, $i = 1, \dots, k$.

Let $y \in Y$ be an arbitrary nonzero vector. Since γ_X lies in G_X , there exists $g \in G$ such that $g = (g_X \otimes g_Y)\sigma$, where $g_X \in \text{GL}(X)$, $g_Y \in \text{GL}(Y)$, $\sigma \in \text{Aut}(\mathbb{F})$

and $\gamma_X = g_X \sigma$. Let $\gamma_Y = g_Y \sigma$. As G is k -equivalent to H , there exists an element $h \in H$ such that $(x_i \otimes y)^g = (x_i \otimes y)^h$, $i = 1, \dots, k$.

Write $h = (h_X \otimes h_Y) \sigma'$, where $h_X \in \text{GL}(X)$, $h_Y \in \text{GL}(Y)$, $\sigma' \in \text{Aut}(\mathbb{F})$. Let $\gamma'_X = h_X \sigma'$ and $\gamma'_Y = h_Y \sigma'$. We have

$$x_i^{\gamma_X} \otimes y^{\gamma_Y} = (x_i \otimes y)^g = (x_i \otimes y)^h = x_i^{\gamma'_X} \otimes y^{\gamma'_Y},$$

for $i = 1, \dots, k$. It follows that there exist $\alpha_i \in \mathbb{F}^\times$ such that $x_i^{\gamma_X} = \alpha_i \cdot x_i^{\gamma'_X}$ and $y^{\gamma_Y} = \frac{1}{\alpha_i} \cdot y^{\gamma'_Y}$ for $i = 1, \dots, k$. As y is nonzero, α_i does not depend on i , i.e. $\alpha_i = \alpha$ for $i = 1, \dots, k$.

Since $\mathbb{F}^\times \leq H_X$, there exists $z \in H_X$ such that $x^z = \alpha \cdot x$ for all $x \in X$. Therefore $x_i^{\gamma_X} = x_i^{\gamma'_X z}$ for $i = 1, \dots, k$. Now $\beta_X = \gamma'_X \cdot z \in H_X$ is the required element.

By switching the roles of G and H , we obtain $k\text{-rel}(G_X) \subseteq k\text{-rel}(H_X)$. Thus $k\text{-rel}(H_X) = k\text{-rel}(G_X)$ and G_X is k -equivalent to H_X , as claimed. Similarly, $k\text{-rel}(H_Y) = k\text{-rel}(G_Y)$ and G_Y is k -equivalent to H_Y . \square

Corollary 3.2. *Let V be a vector space over a finite field \mathbb{F} , and let G be a primitive affine group with socle V . Set $G = V \rtimes G_0$, where G_0 is the zero stabilizer, and assume that $\mathbb{F}^\times \leq G_0$. Suppose that G_0 stabilizes a nontrivial tensor decomposition $V = X \otimes Y$ over \mathbb{F} , where $\dim X \neq \dim Y$. Finally, assume that $k \geq 4$ and the k -closures of $X \rtimes (G_0)_X$ and $Y \rtimes (G_0)_Y$ lie in the class \mathfrak{X} . Then $G^{(k)}$ lies in the class \mathfrak{X} .*

Proof. By Lemma 2.6, the k -closures of G , $X \rtimes (G_0)_X$ and $Y \rtimes (G_0)_Y$ are affine groups with socles V , X and Y , respectively. Set $H = G^{(k)}$ and $H = V \rtimes H_0$, so G_0 is $(k-1)$ -equivalent to H_0 by Proposition 2.2 (iii). By Proposition 2.8 (i), H_0 preserves some tensor decomposition of V , and by [30, Lemma 4.4.5 (1)] it preserves the decomposition $V = X \otimes Y$. By Proposition 3.1, $(G_0)_X$ is $(k-1)$ -equivalent to $(H_0)_X$, and $(G_0)_Y$ is $(k-1)$ -equivalent to $(H_0)_Y$. Therefore $(H_0)_X \leq (G_0)_X^{(k-1)}$ and $(H_0)_Y \leq (G_0)_Y^{(k-1)}$, hence

$$X \rtimes (H_0)_X \leq X \rtimes ((G_0)_X^{(k-1)} \cap \text{GL}(X)) = (X \rtimes (G_0)_X)^{(k)},$$

where the last equality follows from Lemma 2.5, and $\text{GL}(X)$ is the general linear group over the prime subfield of \mathbb{F} . Similarly, $Y \rtimes (H_0)_Y \leq (Y \rtimes (G_0)_Y)^{(k)}$. Therefore $(H_0)_X$ and $(H_0)_Y$ lie in \mathfrak{X} , and thus H lies in \mathfrak{X} as claimed. \square

3.2. Subfield stabilizers. Let V be a vector space over a finite field \mathbb{F}_q of order q , and let v_1, \dots, v_d be the \mathbb{F}_q -basis of V . Let \mathbb{F}_{q_0} be a proper subfield of \mathbb{F}_q , and let V_0 be the \mathbb{F}_{q_0} -span of v_1, \dots, v_d . We say that $G \leq \text{GL}(V)$ is a subfield subgroup, if it preserves the set $\mathbb{F}_q V_0 = \{\lambda v \mid \lambda \in \mathbb{F}_q, v \in V_0\}$ of lines passing through V_0 . The space V can be identified with a tensor product $V_0 \otimes \mathbb{F}_q$ over \mathbb{F}_{q_0} , and under such an identification the group G preserves this tensor product, see [30, Formula (4.5.2)]. As in the previous section, let $G_{V_0} \leq \text{GL}(V_0)$ and $G_{\mathbb{F}_q} \leq \text{GL}_1(\mathbb{F}_q)$ denote the groups induced by G on V_0 and \mathbb{F}_q respectively.

Proposition 3.3. *Let V be a vector space over a finite field \mathbb{F}_q of order q , and let G be a primitive affine group with socle V . Set $G = V \rtimes G_0$, where $G_0 \leq \text{GL}(V)$, and assume that $\mathbb{F}_q^\times \leq G_0$. Suppose that G_0 is a subfield subgroup, in particular, G_0 preserves the tensor decomposition $V = V_0 \otimes \mathbb{F}_q$ over the field \mathbb{F}_{q_0} . Assume that $k \geq 4$ and the k -closure of $V_0 \rtimes (G_0)_{V_0}$ lies in the class \mathfrak{X} . Then $G^{(k)}$ lies in the class \mathfrak{X} .*

Proof. By Lemma 2.6, the k -closures of G and $V_0 \rtimes (G_0)_{V_0}$ are affine groups with socles V and V_0 , respectively. By Proposition 2.8 (i), $G^{(k)}$ is also a subfield subgroup preserving $V = V_0 \otimes \mathbb{F}_q$ over \mathbb{F}_{q_0} . Set $H = G^{(k)}$ and $H = V \rtimes H_0$, $H_0 \leq \Gamma L(V)$. Since G_0 is $(k-1)$ -equivalent to H_0 , Proposition 3.1 implies that $(H_0)_{V_0}$ is $(k-1)$ -equivalent to $(G_0)_{V_0}$. By Lemma 2.5 and our assumptions, $(H_0)_{V_0}$ lies in \mathfrak{X} . Since $(H_0)_{\mathbb{F}_q} \leq \text{GL}_1(q)$, it follows that H_0 lies in class \mathfrak{X} , as required. \square

3.3. Stabilizers of $X \otimes \cdots \otimes X$. Let X be a vector space over a finite field \mathbb{F} , and let $V = X \otimes \cdots \otimes X$ be the tensor product of $m \geq 2$ copies of X . An element $g_1 \otimes \cdots \otimes g_m$ of the central product $\text{GL}(X) \otimes \cdots \otimes \text{GL}(X)$ acts on V naturally, indeed, if $x_1 \otimes \cdots \otimes x_m \in V$, where $x_i \in X$, $i = 1, \dots, m$, then $(x_1 \otimes \cdots \otimes x_m)^{g_1 \otimes \cdots \otimes g_m} = x_1^{g_1} \otimes \cdots \otimes x_m^{g_m}$.

A field automorphism $\sigma \in \text{Aut}(\mathbb{F})$ acts on V componentwise:

$$(x_1 \otimes \cdots \otimes x_m)^\sigma = x_1^\sigma \otimes \cdots \otimes x_m^\sigma.$$

Finally, a permutation $\tau \in \text{Sym}(m)$ acts on V by permuting components of simple tensors:

$$(x_1 \otimes \cdots \otimes x_m)^\tau = x_{\tau^{-1}(1)} \otimes \cdots \otimes x_{\tau^{-1}(m)},$$

and this action extends to the rest of V by linearity. Since the action of field automorphisms commutes with the action of $\text{Sym}(m)$, we obtain the action of $(\text{GL}(X) \otimes \cdots \otimes \text{GL}(X)) \rtimes (\text{Aut}(\mathbb{F}) \times \text{Sym}(m))$ on V . This is the stabilizer of the tensor decomposition $V = X \otimes \cdots \otimes X$, see [30, Formula (4.4.3)].

Set $L = (\text{GL}(X) \otimes \cdots \otimes \text{GL}(X)) \rtimes (\text{Aut}(\mathbb{F}) \times \text{Sym}(m))$ and as usual, identify \mathbb{F}^\times with the center of $\text{GL}(X)$. For $g \in L$ we can write $g = (g_1 \otimes \cdots \otimes g_m)\sigma\tau$, where $g_i \in \text{GL}(X)$, $i = 1, \dots, m$, $\sigma \in \text{Aut}(\mathbb{F})$ and $\tau \in \text{Sym}(m)$. For $i \in \{1, \dots, m\}$ set $\pi_i(g) = \mathbb{F}^\times \cdot g_i\sigma$, and notice that it gives a correctly defined map $\pi_i : L \rightarrow \text{PGL}(X)$. The restriction of π_i to $\text{GL}(X) \otimes \cdots \otimes \text{GL}(X)$ is a homomorphism into $\text{PGL}(X)$ with kernel $K_i = \text{GL}(X) \otimes \cdots \otimes \mathbb{F}^\times \otimes \cdots \otimes \text{GL}(X)$, where \mathbb{F}^\times is on the i -th position in the central product. We can also define a homomorphism $\pi : L \rightarrow \text{Sym}(m)$ by the rule $\pi(g) = \tau$. The kernel of π is equal to $(\text{GL}(X) \otimes \cdots \otimes \text{GL}(X)) \rtimes \text{Aut}(\mathbb{F})$.

Now define a map $\Pi : L \rightarrow \text{PGL}(X) \wr \text{Sym}(m)$ by

$$\Pi(g) = (\pi_1(g), \dots, \pi_m(g))\pi(g),$$

where $g \in L$. Clearly Π is a homomorphism with kernel $\bigcap_i K_i = \mathbb{F}^\times$.

We can define a faithful action of the abstract wreath product $\text{PGL}(X) \wr \text{Sym}(m)$ on

$$\mathcal{L} = \{\mathbb{F} \cdot (v_1 \otimes \cdots \otimes v_m) \mid v_i \in X, i = 1, \dots, m\}$$

which is the set of lines spanned by simple tensors. To do this, let $\text{PGL}(X)$ act on the lines of X , and let $\text{Sym}(m)$ permute the coordinates. The corresponding action is a product action, so we may identify $\text{PGL}(X) \wr \text{Sym}(m)$ with a wreath product in the product action $\text{PGL}(X) \uparrow \text{Sym}(m) \leq \text{Sym}(\mathcal{L})$. The group L also acts on \mathcal{L} naturally, and one can easily see that $g \in L$ and $\Pi(g)$ act on \mathcal{L} in the same way, i.e. they represent the same permutation from $\text{Sym}(\mathcal{L})$.

Let $G \leq L$ be a subgroup containing \mathbb{F}^\times , and assume that $\pi(G)$ is a transitive subgroup of $\text{Sym}(m)$. Let $G_X \leq \Gamma L(X)$ be the projection of G into the first component of the tensor product, i.e.

$$G_X = \{g_1\sigma \mid (g_1 \otimes \cdots \otimes g_m)\sigma\tau \in G, \text{ where } \sigma \in \text{Aut}(\mathbb{F}), \tau \in \text{Sym}(m), \tau(1) = 1\}.$$

Since $\pi(G)$ is transitive, the projection G_X does not depend on the labelling of points $\{1, \dots, m\}$ up to conjugation in $\Gamma L(X)$. One can similarly define the projection of $\Pi(G) \leq \text{P}\Gamma L(X) \uparrow \text{Sym}(m)$ into $\text{P}\Gamma L(X)$; notice that this projection is G_X/\mathbb{F}^\times .

By [25, Section 5.3] the group $\Pi(G) \leq \text{P}\Gamma L(X) \uparrow \text{Sym}(m)$ can be embedded into a wreath product $(G_X/\mathbb{F}^\times) \uparrow \pi(G)$, where G_X/\mathbb{F}^\times is the 1-component of $\Pi(G)$. In particular, G lies in the class \mathfrak{X} if and only if G_X and $\pi(G)$ lie in the class \mathfrak{X} .

Proposition 3.4. *Let X be a vector space over a finite field \mathbb{F} with $\dim X \geq 2$. Let G and H be subgroups of $(\text{GL}(X) \otimes \dots \otimes \text{GL}(X)) \rtimes (\text{Aut}(\mathbb{F}) \times \text{Sym}(m))$, $m \geq 2$, preserving the m -fold tensor power $X \otimes \dots \otimes X$, and assume that G, H contain \mathbb{F}^\times and $\pi(G), \pi(H)$ are transitive subgroups of $\text{Sym}(m)$. Suppose that G and H are k -equivalent for some $k \geq 3$. Then G_X and H_X are k -equivalent, and $\pi(G)^{[r]} = \pi(H)^{[r]}$ for $r = \min\{|\text{Orb}_k(G_X/\mathbb{F}^\times)|, m\}$, where G_X/\mathbb{F}^\times acts on the set of lines of X and $\text{Orb}_k(G_X/\mathbb{F}^\times)$ is the set of orbits on k -tuples in this action.*

Proof. We will use the following notation throughout the proof. Given s simple tensors $v_i = v_{i1} \otimes \dots \otimes v_{im}$, $i = 1, \dots, s$, let $T(v_1, \dots, v_s)$ denote the $s \times m$ matrix:

$$T(v_1, \dots, v_s) = \begin{pmatrix} \mathbb{F} \cdot v_{11} & \dots & \mathbb{F} \cdot v_{1m} \\ \vdots & & \vdots \\ \mathbb{F} \cdot v_{s1} & \dots & \mathbb{F} \cdot v_{sm} \end{pmatrix}$$

Since each entry of this matrix is a line in X , the matrix $T(v_1, \dots, v_s)$ is defined correctly and does not depend on the choice of v_{i1}, \dots, v_{im} in the decomposition of v_i , $i = 1, \dots, s$.

Claim: G_X is k -equivalent to H_X . We will prove that $k\text{-rel}(H_X) \subseteq k\text{-rel}(G_X)$. Let $x_1, \dots, x_k \in X$ be arbitrary nonzero vectors, and let $g_X \in G_X$ be an arbitrary element. Choose $g \in G$ such that $g = (g_1 \otimes \dots \otimes g_m)\sigma\tau$, where $\sigma \in \text{Aut}(\mathbb{F})$, $\tau \in \text{Sym}(m)$, $\tau(1) = 1$ and $g_X = g_1\sigma$. Assume first that x_1, \dots, x_k lie on the same line, i.e. for some nonzero $u \in X$ we have $x_i \in \mathbb{F}^\times \cdot u$ for all $i = 1, \dots, k$.

Let ω be a cyclic generator of \mathbb{F}^\times . Since $\dim X \geq 2$ we can find a vector $w \in X$ which is linearly independent from u . Setting $v = u^{g_X}$ we have $(\omega \cdot u)^{g_X} = \omega^\sigma \cdot v$. Now consider three simple tensors: $a = u \otimes w \otimes \dots \otimes w$, $b = \omega \cdot u \otimes w \otimes \dots \otimes w$ and $c = w \otimes w \otimes \dots \otimes w$, which differ only in the first component. By the definition of G_X , the element g_X stabilizes the first component, therefore

$$\begin{aligned} a^g &= v \otimes w^{g_{\tau^{-1}(2)}^\sigma} \otimes \dots \otimes w^{g_{\tau^{-1}(m)}^\sigma}, \\ b^g &= \omega^\sigma \cdot v \otimes w^{g_{\tau^{-1}(2)}^\sigma} \otimes \dots \otimes w^{g_{\tau^{-1}(m)}^\sigma}, \\ c^g &= w^{g_X} \otimes w^{g_{\tau^{-1}(2)}^\sigma} \otimes \dots \otimes w^{g_{\tau^{-1}(m)}^\sigma}. \end{aligned}$$

It is readily seen that each column of the matrix $T(a^g, b^g, c^g)$ has all coordinates equal to each other with the exception of the first column. The first column does not have all coordinates equal to each other, as vectors v , $\omega^\sigma \cdot v$ and w^{g_X} do not lie on the same line.

Now, groups G and H are k -equivalent, in particular, they are 3-equivalent, so there exists an element $h \in H$ such that $a^g = a^h$, $b^g = b^h$ and $c^g = c^h$. Write

$h = (h_1 \otimes \cdots \otimes h_m)\phi\mu$, where $\phi \in \text{Aut}(\mathbb{F})$ and $\mu \in \text{Sym}(m)$. It follows that

$$\begin{aligned} a^h &= w^{h_{\mu^{-1}(1)}\phi} \otimes \cdots \otimes u^{h_1\phi} \otimes \cdots \otimes w^{h_{\mu^{-1}(m)}\phi}, \\ b^h &= w^{h_{\mu^{-1}(1)}\phi} \otimes \cdots \otimes \omega^\phi \cdot u^{h_1\phi} \otimes \cdots \otimes w^{h_{\mu^{-1}(m)}\phi}, \\ c^h &= w^{h_{\mu^{-1}(1)}\phi} \otimes \cdots \otimes w^{h_1\phi} \otimes \cdots \otimes w^{h_{\mu^{-1}(m)}\phi}, \end{aligned}$$

where the “central” component of each tensor is on position $\mu(1)$. Since $u^{h_1\phi}, \omega^\phi \cdot u^{h_1\phi}$ and $w^{h_1\phi}$ do not lie on the same line, each column of the matrix $T(a^h, b^h, c^h)$ has equal coordinates with the exception of column number $\mu(1)$. Since $T(a^g, b^g, c^g) = T(a^h, b^h, c^h)$, that must be the first column and hence $\mu(1) = 1$.

Equality $a^g = a^h$ of simple tensors implies that there exist scalars $\alpha_i \in \mathbb{F}^\times$, $i = 1, \dots, m$, such that $v = \alpha_1 \cdot u^{h_1\phi}$ and $w^{g_{\tau^{-1}(i)}\sigma} = \alpha_i \cdot w^{h_{\mu^{-1}(i)}\phi}$ for $i = 2, \dots, m$, and $\alpha_1 \cdot \alpha_2 \cdots \alpha_m = 1$. Similarly, $b^g = b^h$ implies that there exist scalars $\beta_i \in \mathbb{F}^\times$, $i = 1, \dots, m$, with $\omega^\sigma \cdot v = \beta_1 \omega^\phi \cdot u^{h_1\phi}$ and $w^{g_{\tau^{-1}(i)}\sigma} = \beta_i \cdot w^{h_{\mu^{-1}(i)}\phi}$ for $i = 2, \dots, m$, and $\beta_1 \cdots \beta_m = 1$. Since the vector w is nonzero, equalities

$$\begin{aligned} w^{g_{\tau^{-1}(i)}\sigma} &= \alpha_i \cdot w^{h_{\mu^{-1}(i)}\phi}, \\ w^{g_{\tau^{-1}(i)}\sigma} &= \beta_i \cdot w^{h_{\mu^{-1}(i)}\phi} \end{aligned}$$

imply $\alpha_i = \beta_i$ for $i = 2, \dots, m$. It follows that $\alpha_1 = \beta_1 = \alpha$ and hence $v = \alpha \cdot u^{h_1\phi}$ and $\omega^\sigma \cdot v = \alpha \omega^\phi \cdot u^{h_1\phi}$. We can choose an element $z \in \mathbb{F}^\times$ such that $v = u^{zh_1\phi}$ and $\omega^\sigma \cdot v = \omega^\phi \cdot u^{zh_1\phi}$. We derive $\omega^\sigma = \omega^\phi$, and since ω is a cyclic generator of \mathbb{F}^\times , we have $\sigma = \phi$. Finally, as $\mathbb{F}^\times \leq H_X$ and $\mu(1) = 1$, the element $h'_X = zh_1\sigma$ lies in H_X . To sum up, $u^{g_X} = v = u^{h'_X}$ and $(\alpha \cdot u)^{g_X} = \alpha^\sigma \cdot v = (\alpha \cdot u)^{h'_X}$ for any $\alpha \in \mathbb{F}^\times$.

Recall that $x_i \in \mathbb{F}^\times \cdot u$, $i = 1, \dots, k$. There exist scalars $\alpha_i \in \mathbb{F}^\times$ such that $x_i = \alpha_i \cdot u$, $i = 1, \dots, k$. It follows that for any $i \in \{1, \dots, m\}$

$$x_i^{g_X} = (\alpha_i \cdot u)^{g_X} = (\alpha_i \cdot u)^{h'_X} = x_i^{h'_X},$$

and this finishes the proof of the case when all x_i lie on the same line.

Now assume $x_1, \dots, x_k \in X$ do not lie on the same line. Choose some nonzero vector $w \in X$ and define simple tensors $v_i = x_i \otimes w \otimes \cdots \otimes w$, $i = 1, \dots, k$, which differ only in the first component. Arguing as before, we see that each column of the matrix $T(v_1^g, \dots, v_k^g)$ has equal coordinates, with the exception of the first column. The first column does not have all coordinates equal to each other as $x_i^{g_X}$, $i = 1, \dots, k$, do not lie on the same line.

Since G and H are k -equivalent, there exists an element $h \in H$ such that $v_i^g = v_i^h$ for all $i = 1, \dots, k$. Write $h = (h_1 \otimes \cdots \otimes h_m)\phi\mu$, where $\phi \in \text{Aut}(\mathbb{F})$, $\mu \in \text{Sym}(m)$. All columns of the matrix $T(v_1^h, \dots, v_k^h)$ have equal coordinates with the exception of column number $\mu(1)$, which has coordinates $x_1^{h_1\phi}, \dots, x_k^{h_1\phi}$. Since $T(v_1^g, \dots, v_k^g) = T(v_1^h, \dots, v_k^h)$, we derive $\mu(1) = 1$.

Equality $v_i^g = v_i^h$, $i = 1, \dots, k$, of simple tensors implies that there exist scalars $\alpha_{ij} \in \mathbb{F}^\times$ such that for every $i \in \{1, \dots, k\}$ we have $x_i^{g_X} = \alpha_{i1} \cdot x_i^{h_1\phi}$ and $w^{g_j\sigma} = \alpha_{ij} \cdot w^{h_j\phi}$ for $j = 2, \dots, m$, and $\prod_{j=1}^m \alpha_{ij} = 1$. Since the vector w is nonzero, scalars α_{ij} , $j = 2, \dots, m$, do not depend on i . It follows that α_{i1} does not depend on i and we can set $\alpha = \alpha_{i1}$ for all $i = 1, \dots, k$.

We have $x_i^{g_X} = \alpha \cdot x_i^{h_1\phi}$ for $i = 1, \dots, k$. Choose an element $z \in \mathbb{F}^\times$ such that $x_i^{g_X} = x_i^{zh_1\phi}$ for $i = 1, \dots, k$. Since $\mathbb{F}^\times \leq H_X$ and $\mu(1) = 1$, the element $h'_X = zh_1\phi$

lies in H_X and we have $x_i^{g_X} = x_i^{h'_X}$, $i = 1, \dots, k$. This finishes the proof of the case when not all x_i , $i = 1, \dots, k$, lie on the same line.

It follows that $k\text{-rel}(H_X) \subseteq k\text{-rel}(G_X)$, and by repeating the same argument for G and H with interchanged roles, we derive $k\text{-rel}(H_X) = k\text{-rel}(G_X)$, i.e. G_X and H_X are k -equivalent. The first claim is proved.

Claim: $\pi(G)^{[r]} = \pi(H)^{[r]}$. By Proposition 2.4 (iii) it is sufficient to prove that $\Pi(G)$ is k -equivalent to $\Pi(H)$. Let $(\mathbb{F}v_1, \dots, \mathbb{F}v_k)$ be a k -tuple of lines for some $v_1, \dots, v_k \in X \otimes \dots \otimes X$. Take some $g \in \Pi(G)$ and let $g' \in G$ be such that $g = \Pi(g')$. Define $u_i = v_i^{g'}$ for $i = 1, \dots, k$. It follows that

$$(\mathbb{F}v_1, \dots, \mathbb{F}v_k)^g = (\mathbb{F}v_1, \dots, \mathbb{F}v_k)^{g'} = (\mathbb{F}u_1, \dots, \mathbb{F}u_k).$$

Since G and H are k -equivalent, there exists $h' \in H$ such that $u_i = v_i^{h'}$, $i = 1, \dots, k$. Setting $h = \Pi(h') \in \Pi(H)$ we have

$$(\mathbb{F}v_1, \dots, \mathbb{F}v_k)^h = (\mathbb{F}v_1, \dots, \mathbb{F}v_k)^{h'} = (\mathbb{F}u_1, \dots, \mathbb{F}u_k) = (\mathbb{F}v_1, \dots, \mathbb{F}v_k)^g.$$

Hence $\Pi(G)$ and $\Pi(H)$ are k -equivalent by Lemma 2.1 and the claim is proved. \square

Corollary 3.5. *Let V be a vector space over a finite field \mathbb{F} , and let G be a primitive affine permutation group with socle V . Set $G = V \rtimes G_0$, where $G_0 \leq \text{GL}(V)$, and assume that $\mathbb{F}^\times \leq G_0$. Suppose that G_0 preserves a tensor decomposition $V = X \otimes \dots \otimes X$, where X is a vector space over \mathbb{F} and there are $m \geq 2$ tensor factors in the decomposition of V . Assume that $k \geq 4$ and the k -closures of $X \rtimes (G_0)_X$ and $\pi(G_0)$ lie in the class \mathfrak{X} . Then $G^{(k)}$ lies in the class \mathfrak{X} .*

Proof. By Lemma 2.6, the k -closures of G and $X \rtimes (G_0)_X$ are affine groups with socles V and X , respectively. Set $H = G^{(k)}$ and $H = V \rtimes H_0$, so G_0 is $(k-1)$ -equivalent to H_0 . By Proposition 2.8 (i) and [30, Lemma 4.4.5 (2)], H_0 preserves the tensor decomposition $V = X \otimes \dots \otimes X$, and by Proposition 3.4, the group $(G_0)_X$ is $(k-1)$ -equivalent to $(H_0)_X$, and $\pi(G_0)^{[r]} = \pi(H_0)^{[r]}$, where $r = \min\{|\text{Orb}_{k-1}((G_0)_X/\mathbb{F}^\times)|, m\}$. By Lemma 2.5 and our assumptions, $(H_0)_X$ lies in the class \mathfrak{X} .

Now, as $k-1 \geq 3$, we have $|\text{Orb}_{k-1}((G_0)_X/\mathbb{F}^\times)| \geq k+1$ since X contains at least 3 lines. If $m < k+1$, then $r = m$ and $\pi(G_0)^{[r]} = \pi(G_0)^{[m]} = \pi(G_0)$, where the last equality follows from the definition of the closure with respect to partitions. If $m \geq k+1$, then $r \geq k+1$ and $\pi(G_0)^{[r]} \leq \pi(G_0)^{[k+1]} \leq \pi(G_0)^{(k)}$, where the last inclusion follows from Proposition 2.3. In either case $\pi(H_0) \leq \pi(G_0)^{[r]}$ lies in the class \mathfrak{X} , so the claim is proved. \square

4. PROOF OF THEOREM 1.1

Set $H = G^{(4)}$. By Proposition 2.2 (i) it suffices to prove that $H \in \mathfrak{X}$. Suppose that G is a counterexample to Theorem 1.1 of minimal degree n . Note that $n > 24$, because otherwise H is $\text{Alt}(25)$ -free and hence lies in \mathfrak{X} . By [23, Theorem 3.1 (i)] (see also Proposition 2.4), we may assume that G is a primitive basic permutation group.

Since G is primitive, [24, Theorem 2] implies that

$$(4.1) \quad \text{Soc}(G) = \text{Soc}(H)$$

unless G is 4-transitive. By [5, Tables 7.3 and 7.4]), if G is 4-transitive and $n > 24$, then $G \geq \text{Alt}(n)$, so $\text{Alt}(n) \in \mathfrak{X}$ and hence $H = \text{Sym}(n) \in \mathfrak{X}$, a contradiction. The case $n \leq 24$ is not possible, so we may assume that equality (4.1) holds.

Since G is basic, applying the O’Nan-Scott theorem [16], we arrive to the following three cases:

- (i) G is almost simple,
- (ii) G is in a *diagonal action*,
- (iii) G is an affine group.

In the case (i), equality (4.1) implies that the group $H/\text{Soc}(G)$ is solvable and hence $H \in \mathfrak{X}$, a contradiction.

Suppose that G is in a diagonal action as in the case (ii). Then $S = \text{Soc}(G) = T^m$, where T is a nonabelian simple group, and G/S is a subgroup of $\text{Out}(T) \times L$, where L is the symmetric group of degree m acting faithfully on the set simple factors of the socle by conjugation. By (4.1), $\text{Soc}(H) = S$, so H/S is also a subgroup of $\text{Out}(T) \times L$. If $m \leq 4$ or G/S includes the alternating subgroup of L , then the nonabelian composition factors of G and H are the same, and we are done. Otherwise, G has a base of size 2 in view of [8, Theorem 1.1], and $G^{(3)} = G$ by Proposition 2.2 (vi). Hence $H = G \in \mathfrak{X}$, a contradiction.

Thus, we may assume that G is an affine group. Observe that in this case H is also an affine group by (4.1), and a primitive basic group by Proposition 2.2 (ii) and Proposition 2.4 (iii).

By [19], we may assume that G is not solvable. Let V be the socle of G and let $G_0 \leq \text{GL}(V)$ be the zero stabilizer, so $G = V \rtimes G_0$. We may write $H = V \rtimes H_0$ for the zero stabilizer $H_0 \leq \text{GL}(V)$. Moreover, $H_0 = G_0^{(3)} \cap \text{GL}(V)$ by Lemma 2.5.

We wish to apply Aschbacher’s theorem [1] to G_0 and H_0 . By Proposition 2.8, G_0 and H_0 lie in the same Aschbacher classes. Set $|V| = p^s$, where p is a prime. Choose the minimal $a \geq 1$ dividing s such that $G_0 \leq \Gamma L_a(q)$ where $q = p^{s/a}$. Note that $a \geq 2$ since G is not solvable. By Proposition 2.2 (iv), we may also assume that $\mathbb{F}_q^\times \leq G_0$, since $\mathbb{F}_q^\times \cdot G_0$ still lies in the class \mathfrak{X} . Note that the corresponding affine group $V \rtimes (\mathbb{F}_q^\times \cdot G_0)$ is a permutation group of degree $n = |V|$, so it is still a counterexample of minimal degree.

Since G_0 preserves the structure of the extension field \mathbb{F}_q of \mathbb{F}_p (class \mathcal{C}_3 for Aschbacher’s theorem in $\text{GL}(V)$), the group H_0 also preserves this field and hence $H_0 \leq \Gamma L_a(q)$. Now, we use Aschbacher’s classification inside $\Gamma L_a(q)$. The group G_0 is irreducible (since G is primitive) and primitive as a linear group (since G is basic), hence the same is true for H_0 . As a is minimal, G_0 does not preserve any extension field of \mathbb{F}_q , hence G_0 and thus H_0 do not lie in class \mathcal{C}_3 . If G_0 preserves a tensor product decomposition (class \mathcal{C}_4), a subfield (class \mathcal{C}_5), or a tensor power (class \mathcal{C}_7), then from the minimality of the counterexample G and Corollary 3.2, Proposition 3.3 or Corollary 3.5 respectively, it follows that $H \in \mathfrak{X}$, a contradiction.

If $\text{SL}_a(q) \leq G_0$, then $\text{SL}_a(q) \leq H_0 \leq \Gamma L_a(q)$. Hence G_0 and H_0 have the same nonabelian composition factors, and $H \in \mathfrak{X}$, a contradiction. Therefore, Aschbacher’s theorem in $\Gamma L_a(q)$ applies, and we deduce that either G_0 and H_0 normalize a subgroup of symplectic type (class \mathcal{C}_6), or they stabilize a symplectic, unitary or orthogonal form (class \mathcal{C}_8), or they lie in class \mathcal{C}_9 . If G_0 stabilizes a form, then by Proposition 2.8 (i), H_0 preserves the same form and hence we can apply Aschbacher’s theorem inside a symplectic, unitary or orthogonal group. Note that

if a group preserving some form also stabilizes a tensor decomposition or a subfield, then we have classified that group as belonging to \mathcal{C}_4 , \mathcal{C}_7 or \mathcal{C}_5 on the previous step. Thus, either G_0 and H_0 lie in \mathcal{C}_6 or \mathcal{C}_9 , or G_0 contains $\mathrm{Sp}_a(q)$, $\mathrm{SU}_a(q^{1/2})$ or $\Omega_a^\pm(q)$, depending on the type of the form G_0 stabilizes. In the latter case H_0 stabilizes the same form and contains the same classical group as G_0 . Hence H_0 and G_0 have the same nonabelian composition factors, which implies $H_0 \in \mathfrak{X}$, a contradiction.

Therefore, to finish the proof we are left to consider the cases when G_0 and H_0 lie in class \mathcal{C}_6 or \mathcal{C}_9 .

4.1. The class \mathcal{C}_6 . The following lemma is probably well-known.

Lemma 4.1. *Let R be a nilpotent group of class at most two acting faithfully and irreducibly on a finite vector space V . If $|R| \leq |V|^{1/2}$, then R has a regular orbit on V .*

Proof. Let $x \in R$ be a nontrivial element. If x is an element of the center $Z(R)$, then $C_V(x)$ is an R -invariant subspace of V , and since R is faithful and irreducible, $C_V(x) = 0$. If $x \notin Z(R)$, then there exists some $g \in R$ such that $[x, g] = x^{-1}x^g$ is nontrivial. It follows that $C_V([x, g]) \supseteq C_V(x^{-1}) \cap C_V(x^g)$, and since $[x, g] \in [R, R] \leq Z(R)$, we have $C_V(x^{-1}) \cap C_V(x^g) = 0$. Observe that $|C_V(x^{-1})| = |C_V(x)| = |C_V(x^g)|$, hence $|C_V(x)| \leq |V|^{1/2}$.

Now, suppose that R does not have a regular orbit on V . Then

$$|V| \leq \sum_{x \in R \setminus 1} |C_V(x)| < |R| \cdot |V|^{1/2},$$

hence $|V|^{1/2} < |R|$, a contradiction. \square

If G_0 lies in class \mathcal{C}_6 , then it contains a normal r -subgroup R for some prime $r \neq p$ with the following properties (see [12, Section 4.6] for details). We have $|R/Z(R)| = r^{2m}$, $m \geq 1$, and if r is odd, then $|Z(R)| = r$ and R is an extraspecial group, while if $r = 2$, then $|Z(R)| \leq 4$ and R is either an extraspecial group or a central product of a cyclic group of order 4 and an extraspecial group. The dimension of V over \mathbb{F}_q is $a = r^m$, and R acts absolutely irreducibly on V . Moreover, G_0 is contained in a subgroup $N \leq \Gamma\mathrm{L}_a(q)$, such that R is normal in N and N/R is isomorphic to $\mathrm{Sp}_{2m}(r)$ or to $O_{2m}^\pm(2)$ if $r = 2$. By Proposition 2.8 (i), the group H_0 also lies in the class \mathcal{C}_6 . Note that the order of the subgroup of symplectic type R can be determined from a , so it is the same for both G_0 and H_0 .

Proposition 4.2. *If $m < 12$, then G and H are $\mathrm{Alt}(25)$ -free. If $m \geq 12$, then G has a base of size at most 3, in particular, $G = H$.*

Proof. Recall that G_0/R embeds into $\mathrm{GL}_{2m}(r)$. If $\mathrm{Alt}(25)$ is a section of G , then it is a section of $\mathrm{GL}_{2m}(r)$ as well. By Proposition 2.7, we have $2m \geq 23$. This implies that if $m < 12$ then G (and similarly H) is $\mathrm{Alt}(25)$ -free. So we may assume $m \geq 12$.

Suppose that $|R| > |V|^{1/2}$. Recall that $|V| = q^{r^m}$, $r \leq q - 1$ and $|R| \leq r^{2m+2}$, therefore

$$q^{r^m/2} = |V|^{1/2} < |R| \leq r^{2m+2} \leq q^{2m+2}.$$

This is impossible for $m \geq 12$, so $|R| \leq |V|^{1/2}$ and R has a regular orbit on V by Lemma 4.1.

Let $v \in V$ be such that the stabilizer R_v is trivial and set $J = (G_0)_v$. We will now prove that J has a regular orbit on V , our argument following the argument in [17, Lemma 3.6] (see also [10, Proposition 5.6]).

Since $R \cap J = 1$, the group J is isomorphic to a subgroup of $\mathrm{Sp}_{2m}(r)$ or $O_{2m}^\pm(2)$. Suppose that J has no regular orbit and therefore

$$(4.2) \quad |V| \leq \sum_{h \in J \setminus 1} |C_V(h)|.$$

For $h \in J \setminus 1$, [9, Theorem 4.1] implies that there are $2m + 2$ conjugates of h in N generating N . Hence

$$\dim C_V(h) \leq \left(1 - \frac{1}{2m+2}\right) \dim V$$

and $|C_V(h)| \leq |V|^{1-1/(2m+2)}$. The obtained inequality and (4.2) imply

$$|V| \leq |J| \cdot |V|^{1-1/(2m+2)}.$$

We derive that

$$(4.3) \quad q^{\frac{r^m}{2m+2}} = |V|^{1/(2m+2)} \leq |J| \leq |N/R| \leq |\mathrm{GL}_{2m}(r)| < r^{4m^2}.$$

Since $r \leq q - 1$, we obtain $r^m < 4m^2(2m + 2)$. The only integer solutions for that inequality for $m \geq 12$ and $r \geq 2$ are $r = 2$ and $m = 12, 13, 14$.

Note that $q \geq 3$. Inequalities (4.3) imply

$$q^{\frac{2^m}{2m+2}} \leq |N/R| \leq \max\{|\mathrm{Sp}_{2m}(2)|, |O_{2m}^\pm(2)|\}.$$

A direct computation shows that this is possible only for $q = 3$ and $m = 12$.

We are left with the case $r = 2$, $q = 3$, $m = 12$. By [12, Table 4.6.B], we have $N/R \simeq O_{24}^\varepsilon(2)$, where $\varepsilon \in \{+, -\}$. Recall that J is isomorphic to a subgroup of $O_{24}^\varepsilon(2)$, and consider an arbitrary element $h \in J \setminus 1$. By [9, Theorem 4.4], there are $m + 3 = 15$ conjugates of h which generate $O_{24}^\varepsilon(2)$ unless h is a transvection, in which case there are $2m = 24$ conjugates. There are at most 2^{24} transvections in $O_{24}^\varepsilon(2)$, so at most 2^{24} elements $h \in J \setminus 1$ corresponding to a transvection; denote this set of elements by T .

Now, if $h \in T$, there are 25 conjugates of h generating N , hence

$$\dim C_V(h) \leq \left(1 - \frac{1}{25}\right) \dim V$$

and $|C_V(h)| \leq |V|^{1-1/25}$. If $h \notin T$, there are 16 conjugates of h generating N , so

$$\dim C_V(h) \leq \left(1 - \frac{1}{16}\right) \dim V$$

and $|C_V(h)| \leq |V|^{1-1/16}$. We apply our estimates to inequality (4.2) and obtain

$$|V| \leq |T| \cdot |V|^{1-1/25} + (|J| - |T|) \cdot |V|^{1-1/16}.$$

Recall that $|T| \leq 2^{24}$, $|J| \leq |O_{24}^-(2)|$ and $|V| = 3^{2^{12}}$. Thus

$$1 \leq 2^{24} \cdot 3^{-\frac{2^{12}}{25}} + |O_{24}^-(2)| \cdot 3^{-\frac{2^{12}}{16}},$$

which is a contradiction. Hence for any $m \geq 12$ the group J has a regular orbit and so G has a base of size at most 3. Now $G = H$ by Proposition 2.2 (vi). \square

The proposition above contradicts G being a counterexample, so we can move to the next class.

4.2. The class \mathcal{C}_9 . We need the following general statement about permutation groups.

Proposition 4.3. *Let P and Q be permutation groups on the same set V , and suppose that they are 3-equivalent. If $P/Z(P)$ and $Q/Z(Q)$ are almost simple groups, then either $\text{Soc}(P/Z(P)) = \text{Soc}(Q/Z(Q))$ or P and Q are $\text{Alt}(25)$ -free.*

Proof. Recall that since P and Q are 3-equivalent, they have the same orbits and systems of imprimitivity by Proposition 2.2 (ii). There exists an orbit $\Delta \subset V$ such that P^Δ is nonsolvable. Groups P^Δ and Q^Δ are 3-equivalent by Proposition 2.4 (i), hence Q^Δ is also nonsolvable (otherwise $P^\Delta \leq (Q^\Delta)^{(3)}$ would be solvable by [19]). The group $P^\Delta/Z(P^\Delta)$ is almost simple with the same socle as P , and similar properties hold for Q^Δ . We may thus replace P and Q by P^Δ and Q^Δ , and assume that P is transitive on Δ .

If P is imprimitive on Δ , then let $\Pi \subset \Delta$ be the minimal nontrivial block of imprimitivity. If P^Π is nonsolvable, then $P^\Pi/Z(P^\Pi)$ is almost simple with the same socle as P , and hence again, we can replace P and Q by P^Π and Q^Π by Proposition 2.4 (ii). If P^Π is solvable, then the action of P on the imprimitivity system Δ/Π defined by Π is nonsolvable and almost simple, therefore we can replace our groups by $P^{\Delta/\Pi}$ and $Q^{\Delta/\Pi}$ by Proposition 2.4 (ii). We can thus assume that P acts primitively on Δ .

Since P is a nonsolvable primitive group, $Z(P) = 1$. Hence P is almost simple and by [15, Theorem 2], either $\text{Soc}(P) = \text{Soc}(P^{(3)})$ or P is 3-transitive. The same is true for Q , and as P and Q are 3-equivalent, P is 3-transitive if and only if Q is 3-transitive. Since $P^{(3)} = Q^{(3)}$, we either have $\text{Soc}(P) = \text{Soc}(P^{(3)}) = \text{Soc}(Q^{(3)}) = \text{Soc}(Q)$, or P and Q are 3-transitive. In the latter case, [5, Table 7.4] implies that $|\Delta| \leq 24$ or $\text{Soc}(Q) = \text{PSL}_2(u)$ for some prime power u , which means that P and Q are $\text{Alt}(25)$ -free. \square

Let G_0 lie in \mathcal{C}_9 . Since $n > 16$, the group H_0 also lies in \mathcal{C}_9 by Proposition 2.8 (ii). Both $G_0/Z(G_0)$ and $H_0/Z(H_0)$ are almost simple groups. By applying Proposition 4.3 to $P = G_0$ and $Q = H_0$ we derive that $H_0 \in \mathfrak{X}$, which is the final contradiction.

5. CONCLUDING REMARKS

5.1. As mentioned in the introduction, it is important from the computational point of view to have an upper bound on the size of primitive sections of a permutation group from the class under consideration. Combining the main result of [3] with our Corollary 1.2, we obtain the following.

Corollary 5.1. *Suppose that G is an $\text{Alt}(d)$ -free primitive permutation group with $d \geq 25$. Then the order of $G^{(k)}$ is bounded from above by n^c for every $k \geq 4$, where c is a constant depending only on d .*

5.2. Example (ii) from the introduction shows that for $k = 4$ (and even $k = 5$) the best possible value of d providing that the k -closure of every $\text{Alt}(d)$ -free group is again $\text{Alt}(d)$ -free is $d = 25$. It is clear that for greater value of k the value of d can be reduced. It would be interesting to find the minimal possible d as a function on

k , or at least to find the minimal possible k such that the k -closure of $\text{Alt}(d)$ -free group is $\text{Alt}(d)$ -free for all $d \geq 5$.

5.3. The concept of the $\text{Alt}(d)$ -free groups provides a natural way to restrict the composition factors of a group. Namely, only nonabelian composition factors that do not have linear representations of bounded dimension must be excluded. For example, if \mathfrak{X} is the class of $\text{Alt}(25)$ -free groups, then the list of simple groups of \mathfrak{X} includes the groups of order p for all primes p , all the sporadic groups and exceptional groups of Lie type, as well as the classical groups of bounded dimension (cf. Proposition 2.7), and the alternating groups of degree less than 25.

It is worth noting that historically there were other ways to restrict composition factors of a group, see the classical papers by Luks [18] and Babai–Luks [4]. Namely, let Γ_d with $d \geq 4$ denote the class of finite groups every nonabelian composition factor of which can be embedded in $\text{Alt}(d)$. In fact, the restriction in the Luks paper was even stronger: it requires that every (not necessarily nonabelian) composition factor can be embedded in $\text{Alt}(d)$. In our definition of Γ_d we follow the Babai–Luks paper, since it provides a wider class of groups.

Obviously, every group from Γ_d is $\text{Alt}(d+1)$ -free. Therefore, if $k \geq 4$ and $d \geq 24$, then $G^{(k)}$ is an $\text{Alt}(d+1)$ -free group for every $G \in \Gamma_d$ by virtue of Corollary 1.2. The question is whether $G^{(k)} \in \Gamma_d$?

Inspecting the proof of Theorem 1.1, one can see that applying our arguments to a group $G \in \Gamma_d$ for $d \geq 24$, it follows that for $k \geq 4$, either $G^{(k)}$ belongs to Γ_d , or G is a basic affine group such that the zero stabilizer G_0 lies in one of the Aschbacher classes \mathcal{C}_6 and \mathcal{C}_9 (as in Subsections 4.1 and 4.2).

For $G_0 \in \mathcal{C}_9$ it is not hard to show that $G^{(4)} \in \Gamma_d$. Here the only problem is (cf. the proof of Proposition 4.3) the case when G_0 is 3-transitive. Since $d \geq 24$, we may assume that $\text{Soc}(G_0) = \text{PSL}_2(u)$. Applying the results from [13, 14] that describe the irreducible quasisimple linear groups acting without regular orbits, we conclude that for a finite number of exceptions $b(G_0) \leq 2$, and we are done in view of Proposition 2.2 (vi).

For $G_0 \in \mathcal{C}_6$ the situation is more complicated. Let us keep the notation of Proposition 4.2. If $m \geq 12$, then $G^{(4)} = H^{(4)}$ and everything is fine. However, the inequality $m < 12$ on the dimension of G_0 can only guarantee that $m < 12$ for H_0 , that is, the dimension (but not the size) of H_0 is bounded. We do not know how to overcome this problem for $k = 4$. Nevertheless, it can be done for $k \geq 15$. Indeed, according to [10, Proposition 5.6], in our situation for each m (even $m < 12$) the base number of G_0 is at most 13, so the base number of G is at most 14. Now Proposition 2.2 (vi) implies that $G^{(15)} = H^{(15)}$, as required. Thus, the following is true.

Corollary 5.2. *If $G \in \Gamma_d$ with $d \geq 24$, then $G^{(k)} \in \Gamma_d$ for $k \geq 15$.*

It would certainly be interesting to prove the similar statement for $k < 15$ if it is possible.

5.4. Let us finish by returning to the computational problem from which we started the introduction. In the context of the present paper it can be formulated as follows: find the k -closure of a given $\text{Alt}(d)$ -free permutation group in polynomial time in its degree for fixed k and d . Using [23, Theorem 3.1], this problem can be reduced in polynomial time to finding (a) the k -closure of every primitive basic $\text{Alt}(d)$ -free

group, and (b) the intersection of the k -closure of an $\text{Alt}(d)$ -free group with any other $\text{Alt}(d)$ -free group. Problem (b) can be solved by the Babai–Luks algorithm for all k and d . Problem (a) is open for now. We believe that the results of this paper could be used to solve problem (a).

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