ARITHMETIC INVARIANTS FOR FINITE SIMPLE AND RELATED GROUPS

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Abstract. In this short note we address the problems of characterization of simple and related groups by various arithmetic invariants. We come with some uniform way to think about such sort of questions and discuss what has been already done and what we still do not know but wish to know in this field.

Keywords: finite group, simple group, system of invariants, element order, conjugacy class size, character degree.

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1. Introduction

One may view finite group theory as a study of the interplay between the algebraic structure of a group and its arithmetic, i. e. expressed by numeric parameters, properties: well-known theorems of Lagrange, Sylow, and Feit and Thompson are among the numerous examples.

Here we are interested in the case when a set of arithmetic parameters of a finite group G induces a system of invariants that allow us to distinguish G from all other finite groups up to isomorphism. We focus our attention on the following three sets:

- $\mathrm{eo}(G) = \{|x| : x \in G\}$ the set of element orders;
- $\mathrm{cs}(G) = \{|x^G| : x \in G\}$ the set of conjugacy class sizes;
- $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ the set of the complex irreducible character degrees.

We also refer to the collections of the same parameters but counting their multiplicities as $\mathrm{eo}^*(G), \mathrm{cs}^*(G), \mathrm{cd}^*(G)$.

As examples show, one cannot expect that $\mathrm{eo}(G), \mathrm{cs}(G), \mathrm{cd}(G)$ or even $\mathrm{eo}^*(G), \mathrm{cs}^*(G), \mathrm{cd}^*(G)$ provide the full system of invariants in the class of all finite groups. However, the situation changes for finite simple groups and groups related to them: almost simple, quasisimple and so on.

The importance of simple groups in finite group theory is the same as the importance of prime numbers in number theory. Attempts to characterize them in the class of all finite groups became a popular field of investigations since the classification theorem was announced and the

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complete list of simple groups became available. Recall that according to their classification (CFSG), finite simple groups are exactly

(i) the groups of prime order;
(ii) the alternating groups of degree at least 5;
(iii) the simple classical groups;
(iv) the simple exceptional groups of Lie type;
(v) the 26 sporadic groups.

We begin by discussing three well-known conjectures. The first two of them appeared in 1987 in the private communication between Shi and Thompson (see, e.g., [24]), the third one was posed by Huppert in 2000, see [18]. All three of them were later added to the *Kourovka Notebook* [20, Problems 12.38, 12.39, 18.99].

Conjecture 1.1. (Shi's Conjecture) *Let L be a simple group and G be a group such that* $\mathrm{eo}(G) = \mathrm{eo}(L)$ *and* $|G| = |L|$*. Then* $G \simeq L$*.*

Conjecture 1.2. (Thompson's Conjecture) *Let L be a nonabelian simple group and G be a group such that* $\mathrm{cs}(G) = \mathrm{cs}(L)$ *and* $Z(G) = 1$*. Then* $G \simeq L$*.*

Conjecture 1.3. (Huppert's Conjecture) *Let L be a nonabelian simple group and G be a group such that* $\mathrm{cd}(G) = \mathrm{cd}(L)$*. Then* $G \simeq L \times A$*, where A is abelian.*

Remark. It is clear that the validity of the Huppert conjecture would imply that $G \simeq L$ if one adds any of the two conditions $|G| = |L|$ or $Z(G) = 1$ from the Shi and Thompson conjectures to the condition $\mathrm{cd}(G) = \mathrm{cd}(L)$. Moreover, the validity of the Thompson conjecture would imply that $G \simeq L$ if one replaces the condition $Z(G) = 1$ to the condition $|G| = |L|$ in the Thompson conjecture (see, [14, Lemma 2.4]). It is also clear that if we suppose that $|G| = |L|$, then we do not need to assume that a simple group L is nonabelian in all three statements.

The validity of the Shi conjecture was established in [35]. The Thompson conjecture was proven to be right for all simple group except the alternating groups [11], and in the case of the alternating groups, it was done modulo the binary Goldbach conjecture [10]. The validity of the Huppert conjecture was verified for all finite simple groups except classical groups, see [33] for references. Similar problems are also worth to consider for groups related to simple groups.

The purpose of this note is to come with some uniform way to think about such sort of questions. Recall that an *invariant* of an equivalence relation on a set M is a function

$$f : M \to Q,$$

where Q is some set (usually, a field or the ring of integers), such that f is a constant on each equivalence class. Since $f(x) \neq f(y)$ implies that $x, y \in M$ are non-equivalent, invariants allow to distinguish the equivalence classes. The system of invariants $\mathcal{F}$ is *full*, if $\mathcal{F}$ distinguishes every two classes, i. e., for every non-equivalent $x, y \in M$ there exists $f \in \mathcal{F}$ with $f(x) \neq f(y)$.

Given a subset N of M closed with respect to the equivalence, we say that a system of invariants $\mathcal{F}$ is *full for N*, if for every $z \in N$ and an arbitrary $x \in M$ non-equivalent to z, there is $f \in \mathcal{F}$ with $f(x) \neq f(z)$.

In our case, the equivalence is the isomorphism relation on the set of finite groups and we are looking for systems of invariants full for simple groups and for groups related to them.

2. Invariants for simple groups

First, we fix $n = n(G) = |G|$, the order of a group G, which is definitely the invariant of the isomorphism relation on the set of finite groups.

Let $D(G) = D(n)$ be the set of the divisors of n and $\tau(n) = |D(n)|$. It is well known that for $\mathrm{inv} \in \{\mathrm{eo}, \mathrm{cs}, \mathrm{cd}\}$, $\mathrm{inv}(G) \subseteq D(G) = D(n)$. Put the elements d_i, $i = 1, \dots, \tau(n)$ of $D(n)$ in the ascending order. For $\mathrm{inv} \in \{\mathrm{eo}, \mathrm{cs}, \mathrm{cd}\}$ and every $d_i \in D(G)$, set

$$\delta_i^{\mathrm{inv}}(G) := \begin{cases} 1, & \text{if } d_i \in \mathrm{inv}(G) \\ 0, & \text{otherwise.} \end{cases}$$

For $\mathrm{inv}^* \in \{\mathrm{eo}^*, \mathrm{cs}^*, \mathrm{cd}^*\}$, and every $d_i \in D(G)$, set

$$\delta_i^{\mathrm{inv}^*}(G) := \text{multiplicity of } d_i \text{ in } \mathrm{inv}^*(G).$$

It is clear that functions $\delta_i^{\mathrm{inv}}(G)$ and $\delta_i^{\mathrm{inv}^*}(G)$ are invariants of G for every $i = 1, \dots, \tau(n)$. Let $\delta^{\mathrm{inv}}(G) = (\dots \delta_i^{\mathrm{inv}}(G) \dots)$ and $\delta^{\mathrm{inv}^*}(G) = (\dots \delta_i^{\mathrm{inv}^*}(G) \dots)$ be the tuples of length $\tau(n)$.

In this language, the validity of Shi's conjecture means that the following statement is true.

Theorem 2.1. *The system of invariants $(n(G), \delta^{\mathrm{eo}}(G))$ is full for the simple groups.*

Recently in [14], we proved

Theorem 2.2. *The system of invariants $(n(G), \delta^{\mathrm{cs}}(G))$ is full for the alternating and symmetric groups.*

From the validity of Thompson's conjecture for all simple groups except the alternating groups and [14, Lemma 2.4], it follows that the system of invariants is full for all simple groups (see also [14, Corollary 1.3]).

Theorem 2.3. *The system of invariants $(n(G), \delta^{\mathrm{cs}}(G))$ is full for the simple groups.*

In view of these results, it would be quite interesting to answer the following question.

Problem 2.4. *Is the system of invariants $(n(G), \delta^{\mathrm{cd}}(G))$ full for the simple groups?*

Remark. It is clear that the validity of Huppert's conjecture implies the positive answer to Problem 2.4, but not vice versa. Since Huppert's conjecture has been verified for all simple groups except classical groups of dimension greater than 4, it suffices to solve the problem for the latter groups.

Since

$$n(G) = \sum_{d_i \in D(G)} \delta_i^{\mathrm{eo}^*}(G) = \sum_{d_i \in D(G)} \delta_i^{\mathrm{cs}^*}(G) = \sum_{d_i \in D(G)} (\delta_i^{\mathrm{cd}^*}(G))^2,$$

the tuple $\delta^{\mathrm{inv}^*}(G)$ determines $(n(G), \delta^{\mathrm{inv}}(G))$. Though we do not know whether the system of invariants $(n(G), \delta^{\mathrm{cd}}(G))$ is full for all simple groups, it follows from [29,31,32] that $\delta^{\mathrm{cd}^*}(G)$ is. Thus, the following general assertion on the characterization of simple groups holds true.

Theorem 2.5. *For $\mathrm{inv}^* \in \{\mathrm{eo}^*, \mathrm{cs}^*, \mathrm{cd}^*\}$, the system of invariants $\delta^{\mathrm{inv}^*}(G)$ is full for the simple groups.*

3. Groups related to simple groups

In many cases, one need to deal with groups which are not simple, but are close to them. We recall that

- G is *almost simple*, if $S \leq G \leq \operatorname{Aut} S$ for a nonabelian simple group S (e.g., $G = S_m$ with $S = A_m$),
- G is *quasisimple*, if $G = [G, G]$ and $G/Z(G) = S$ for a nonabelian simple group S (e.g., $G = \mathrm{SL}_m(q)$ with $S = \mathrm{PSL}_m(q)$),
- G is *almost quasisimple*, if G includes a quasisimple normal subgroup H with $C_G(H) \leq H$ (e.g., $G = \mathrm{GL}_m(q)$ or $\Gamma\mathrm{L}_m(q)$ with $H = \mathrm{SL}_m(q)$).

In what follows, G is said to be *related* to S in all these cases (in the last case, $S = H/Z(H)$).

The following general problem arise.

Problem 3.1. *For which groups related to simple and which* inv $\in \{\mathrm{eo}, \mathrm{cs}, \mathrm{cd}\}$*, the system of invariants* $(n(G), \delta^{\mathrm{inv}}(G))$ *or* $\delta^{\mathrm{inv}^*}(G)$ *is full in the class of all finite groups*?

We begin to discuss this problem with the case of the symmetric groups. We know from Theorem 2.2 that the system of invariants $(n(G), \delta^{\mathrm{cs}}(G))$ is full for them. The same holds true in the case inv $=$ eo.

Theorem 3.2. [2, Theorems 6,7] *The system of invariants* $(n(G), \delta^{\mathrm{eo}}(G))$ *is full for the symmetric groups.*

Remark. It is worth noting that the main result of the article [2] (available only in Chinese) can be deduced now from much stronger assertion. Namely, it follows from [9, 13] that the symmetric group S_m is uniquely characterized by the set $\mathrm{eo}(S_m)$ in the class of all finite groups for all positive integers $m \notin \{2, 3, 4, 5, 6, 8, 10\}$. In the exceptional cases, one can verify that the equality $(n(G), \delta^{\mathrm{eo}}(G)) = (n(S_m), \delta^{\mathrm{eo}}(S_m))$ yields $G \simeq S_m$.

The case inv $=$ cd is open.

Problem 3.3. *Is the system of invariants* $(n(G), \delta^{\mathrm{cd}}(G))$ *full for the symmetric groups?*

Nevertheless, as it follows from the main result of [30], if we replace $(n(G), \delta^{\mathrm{cd}}(G))$ by $\delta^{\mathrm{cd}^*}(G)$, we obtain a system of invariants full for the symmetric groups. Therefore, the following holds.

Theorem 3.4. *For* $\mathrm{inv}^* \in \{\mathrm{eo}^*, \mathrm{cs}^*, \mathrm{cd}^*\}$*, the system of invariants* $\delta^{\mathrm{inv}^*}(G)$ *is full for the symmetric groups.*

Further it is more convenient to discuss each of the cases inv $=$ eo, cs, or cd separately. We start with the case inv $=$ cd.

The irreducible character degrees of a group G counting with the multiplicities form the first column of the character table of G and uniquely determine the complex group algebra of G. This makes the characterization of groups related to simple groups by $\mathrm{cd}^*(G)$ a very active and popular research field. Turning to our language, the following was proved in [4].

Theorem 3.5. *The system of invariants* $\delta^{\mathrm{cd}^*}(G)$ *is full for all quasisimple groups.*

In the case of almost simple groups, only partial results are available. Besides the symmetric groups mentioned above, we know that the system of invariants $\delta^{\mathrm{cd}^*}(G)$ is full for almost simple groups with socle $PSL_2(q)$ [17], $PGU_3(q)$ [25]; and for groups G with $PSL_n(q) < G \leq PGL_n(q)$ [26], or $PSU_n(q) < G \leq PGU_n(q)$, where $q+1$ divides neither n nor $n-1$ [27].

Turning to the case, when the irreducible character degrees of a group are considered without counting multiplicities, we observe many attempts to formulate analogues of Huppert's conjecture for groups related to simple, see, e.g., [1, 19, 28], with some partial results on them. In the framework of our approach, we suggest to unify them by adding $n(G)$ as an additional invariant. Though we do not know whether the system of invariants $(n(G), \delta^{\mathrm{cd}}(G))$ is full for the simple groups, we also do not know the answer to the following question.

Problem 3.6. *Does there exist a quasisimple group G such that the system of invariants $(n(G), \delta^{\mathrm{cd}(G)})$ is not full for G?*

Very little is known about the characterization of groups related to simple groups by the conjugacy class sizes. Except Theorem 2.2 on the symmetric groups, it is known (see [12]) that $\mathrm{cs}(G) = \mathrm{cs}(\mathrm{SL}_2(5))$ implies that $G \simeq \mathrm{SL}_2(5) \times A$ for an abelian group A. In particular, the system of invariants $(n(G), \delta^{\mathrm{cs}}(G))$ is full for $\mathrm{SL}_2(5)$. As in the previous case, we do not know an answer to the following question.

Problem 3.7. *Does there exist a quasisimple group G such that the system of invariants $(n(G), \delta^{\mathrm{cs}(G)})$ is not full for G?*

Let us turn to the case inv = eo. Though the system of invariants $(n(G), \delta^{\mathrm{eo}}(G))$ is full for the symmetric groups S_m, the same does not hold even for all almost simple and quasisimple groups related to A_m. Recall that $A_6 \simeq \mathrm{PSL}_2(9)$, $|\operatorname{Aut} A_6| = 4 \cdot |A_6|$, and as one can easily verify using the *Atlas of Finite Groups* [7]:

$$\mathrm{eo}(\operatorname{Aut} A_6) = \mathrm{eo}(C_2 \times \mathrm{PGL}_2(9)) = \mathrm{eo}(C_2 \times \mathrm{SL}_2(9)).$$

Remark. The case $m = 6$ is the only exception for groups related to A_m, because $\operatorname{Aut} A_m = S_m$ for $m \neq 2, 6$.

There are infinitely many examples of groups related to simple classical groups such that the system of invariants $(n(G), \delta^{\mathrm{eo}}(G))$ is not full for them. For example, according to [6, Proposition 2.6], for groups related to $\mathrm{PSL}_m(q)$, where m is odd and $m \neq p^k + 1$, we have

$$\mathrm{eo}(\mathrm{PGL}_m(p^k)) = \mathrm{eo}(\mathrm{SL}_m(p^k)) \text{ and } |\mathrm{PGL}_m(p^k)| = |\mathrm{SL}_m(p^k)|.$$

The similar examples exist for other classical groups.

Moreover, even the system of invariants $\delta^{\mathrm{eo}^*}(G)$ is not enough to distinguish some almost simple group G in the class of all finite groups. As was observed by Thompson (see, e.g., [34, Section 4.3]), there are two maximal subgroups $G_1 \simeq L_3(4).2$ and $G_2 \simeq 2^4 \rtimes A_7$ of the Mathieu group M_{23} such that $\mathrm{eo}^*(G_1) = \mathrm{eo}^*(G_2)$.

Despite all these examples, we believe that it is 'almost enough' to have $n(G)$ and $\mathrm{eo}(G)$ to characterize 'the most' of groups related to simple groups. We propose the following conjecture.

Conjecture 3.8. *The order $n(G)$ of G, the number $i(G)$ of involutions in G, and $\delta^{\mathrm{eo}}(G)$ form a system of invariants full for all but finitely many almost quasisimple groups.*

As was noted in the introduction, even $\mathrm{eo}^*(G)$, $\mathrm{cs}^*(G)$, $\mathrm{cd}^*(G)$ are very far from being full systems of invariants on the set of all finite groups. Nevertheless, it is interesting what mutual restrictions the equality $\mathrm{inv}^*(G) = \mathrm{inv}^*(H)$ or its weaker versions (up to $\mathrm{inv}(G) = \mathrm{inv}(H)$) impose on the internal structures of groups G and H in general?

Say, we know that the equality $\mathrm{eo}^*(G) = \mathrm{eo}^*(H)$ does not guarantee neither mutual solvability, nor mutual semisimplicity of groups G and H. Indeed, on the one hand, recently in [23] (see also [22] for smaller examples), Piwek found a solvable group G and nonsolvable group H with $\mathrm{eo}^*(G) = \mathrm{eo}^*(H)$, thus solving old Thompson's question, see [20, Problem 12.37]. On the other hand, the subgroup $G_1 \simeq L_3(4).2$ of the Mathieu group M_{23} is semisimple, i.e., G_1 has the trivial solvable radical, while the subgroup $G_2 \simeq 2^4 \rtimes A_7$ of the same group has obviously not, though $\mathrm{eo}^*(G_1) = \mathrm{eo}^*(G_2)$. We are wondering are there similar examples for the case $\mathrm{inv} = \mathrm{cs}$ or $\mathrm{inv} = \mathrm{cd}$.

Problem 3.9. *Let* $\mathrm{inv} = \mathrm{cs}$ *or* $\mathrm{inv} = \mathrm{cd}$*, and let* $\mathrm{inv}^*(G) = \mathrm{inv}^*(H)$.

(i) *If G is sovbale, must H be solvable?*
(ii) *If G has trivial solvable radical, must H have trivial solvable radical?*

4. Concluding Remarks

The results on arithmetic characterizations of simple and related groups find applications in algebra and representation theory. As we noted above, the set $\mathrm{cd}^*(G)$ uniquely determines the complex group algebra $\mathbb{C}G$, thus the fact that the set of invariants $\mathrm{cd}^*(G)$ is full for all simple and even for all quasisimple groups (Theorem 3.5 above) implies the answer to Question 2 of seminal Brauer paper [3] for quasisimple groups, see also the introduction in [4]. The validity of the Shi conjecture (Theorem 2.1) implies in view of [21, Corollary 5.2] that a finite simple group and a finite group with the same Burnside rings are isomorphic, thus answering Yoshida's question in [36, Problem 2] for the case of the simple groups. Let us also mention that Theorem 2.1 recently appeared in research on the so-called *group isomorphism problem* from computational complexity [5, 16] and even in machine learning [15].

It is worth noting that in all these applications the order of a group is included in a set of input invariants. In some way, this justifies our approach when we consider $(n(G), \mathrm{inv}(G))$ as a *minimal* possible set of invariants for a group G, while $\mathrm{inv}^*(G)$ is considered as a *maximal* one.

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