

ON CAYLEY REPRESENTATIONS OF CENTRAL CAYLEY GRAPHS OVER ALMOST SIMPLE GROUPS

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ABSTRACT. A Cayley graph over a group G is said to be *central* if its connection set is a normal subset of G . We prove that every central Cayley graph over a simple group G has at most two pairwise nonequivalent Cayley representations over G associated with the subgroups of $\text{Sym}(G)$ induced by left and right multiplications of G . We also provide an algorithm which, given a central Cayley graph Γ over an almost simple group G whose socle is of a bounded index, finds the full set of pairwise nonequivalent Cayley representations of Γ over G in time polynomial in size of G .

Keywords: Cayley graph, Cayley representation, Cayley isomorphism, almost simple group.

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1. INTRODUCTION

Let G be a group (throughout the paper all groups are assumed to be finite) and X a subset of G not containing the identity element. By a *Cayley graph* $\text{Cay}(G, X)$ over G with *connection set* X , we mean a directed graph with vertex set G and arc set $\{(g, xg) : g \in G, x \in X\}$. A *Cayley representation* of an arbitrary directed graph Γ over a group G is defined to be a Cayley graph over G isomorphic to Γ . Two Cayley representations of Γ over G are *equivalent* if the corresponding Cayley graphs are *Cayley isomorphic*, i.e., there exists a group automorphism of G which is at the same time an isomorphism between the graphs. In general, a given graph may have many Cayley representations over a given group even up to equivalence, see, e.g., [12]. The problem of finding pairwise nonequivalent Cayley representations of a given graph is of substantial interest from a computational complexity point of view. For example, the isomorphism problem for Cayley graphs can be polynomially reduced to it for groups having a bounded number of generators. In the present paper, we are interested in a special case of this problem formed by the two conditions imposed on the input graphs and groups.

First, we assume that a Cayley graph $\Gamma = \text{Cay}(G, X)$ is *central*, i.e., the connection set X of Γ is normal: $X^g = X$ for every $g \in G$. Obviously, every Cayley graph over an abelian group is central, it explains why central graphs are also called quasiabelian in [22, 24]. The problem of finding pairwise nonequivalent Cayley representations was solved for cyclic groups [6] and for some abelian groups of small rank [18, 20]. We will show that such results are still possible even for groups that are very far from abelian, if we restrict ourselves to *central Cayley representations* that is Cayley representations of central Cayley graphs.

Namely, and this is our second condition, we suppose that G is a simple or almost simple group. Note that Cayley graphs over simple groups were a subject of intense interest for many

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years. It suffices to mention the positive solution of the Babai–Kantor–Lubotzky conjecture [2] stating that all nonabelian simple groups are expanders in a uniform way, see, e.g., [15, Conjecture 2.5]. Some general description of the full automorphism group of a connected Cayley graph over a simple group was given in [8]. A polynomial algorithm solving the isomorphism problem for central Cayley graphs over almost simple groups was proposed in [19].

Our first result concerns simple groups. It is worth noting that it is based (see details in Section 3) on the following fact: every nonabelian simple group G contains a class of involutions invariant with respect to $\text{Aut}(G)$ [9, Lemma 12.1].

Theorem 1. *Every directed graph Γ has at most two pairwise nonequivalent central Cayley representations over a simple group G . Moreover, if Γ is undirected, then all its central Cayley representation over G are equivalent.*

Remark. The second statement of the theorem has an equivalent reformulation: every undirected central Cayley graph over a simple group is a *CI-graph*, cf. [12, Definition 3.1].

As we will see in Section 4, the number of pairwise nonequivalent central Cayley representations of a graph over an almost simple group G cannot be bounded by any constant not depending on the size of G . For example, the *complete transposition graph* over the symmetric group $\text{Sym}(m)$ with connection set consisting of all transpositions of $\text{Sym}(m)$ has at least $\lfloor \frac{m}{4} \rfloor$ such representations (see Corollary 4.5 and the remark after it).

Nevertheless, we show that the number of pairwise nonequivalent central Cayley representations of a graph Γ over an almost simple group G of order n with the socle $\text{Soc}(G)$ of index at most c is bounded by $n^{f(c)}$, where f is a function not depending on n (Theorem 5.1). Moreover, the following holds.

Theorem 2. *Let \mathcal{K}_c be the class of almost simple groups G with $|G : \text{Soc}(G)| \leq c$. Given a central Cayley graph Γ over $G \in \mathcal{K}_c$ of order n , the full set of pairwise nonequivalent Cayley representations of Γ can be found in time polynomial in n .*

Remark. We do not know whether the number of pairwise nonequivalent Cayley representations of Γ is bounded by a polynomial in n of degree not depending on the index c . If it does, then this would be interesting to prove that such representations can be found within the same time.

Slightly generalizing Babai's argument from [1], one can prove (see Lemma 2.2 below) that the pairwise nonequivalent Cayley representations of a Cayley graph Γ over any group G are in one-to-one correspondence with the conjugacy classes of regular subgroups of the group $\text{Aut}(\Gamma)$ isomorphic to G . Thus, the following definition is a key to our arguments.

Definition 3. A G -base \mathcal{B} of a permutation group K is a maximal set of pairwise non-conjugate (in K) regular subgroups of K isomorphic to G . If Γ is a graph and $K = \text{Aut}(\Gamma)$, then \mathcal{B} is called a G -base of Γ .

Remark. According to our approach, in Definition 3 and further, by a graph we mean a directed graph.

The notion of a G -base was suggested in [7] as a generalization of the notion of a *cycle base* (see [6, 17]). One can check that all G -bases of K have the same size denoted by $b_G(K)$ (and $b_G(\Gamma)$ if $K = \text{Aut}(\Gamma)$).

In this language (cf. Corollary 2.3 below), Theorem 1 states that $b_G(\Gamma) \leq 2$ if Γ is a central Cayley graph over a simple group G . In fact, we prove that each regular subgroup of $\text{Aut}(\Gamma)$ is conjugate to either the group of right multiplications by elements of G or the group of left multiplications. While in the case of an almost simple group G with socle of index at most c , the number $b_G(\Gamma)$ is bounded by $n^{f(c)}$, and Theorem 2 claims that a G -base can be found within the same time.

2. REGULAR SUBGROUPS AND CAYLEY REPRESENTATIONS

Let Ω be a finite set of size n and let $K \leq \text{Sym}(\Omega)$ be a permutation group on Ω . Recall that K is called *semiregular* if each point stabilizer of K is trivial; and K is *regular* if K is transitive and semiregular. Clearly, K is semiregular if and only if $|K| = |\Delta|$ for every Δ from the set $\text{Orb}(K, \Omega)$ of orbits of K on Ω . In the proof of Theorem 2, we need the following elementary fact.

Lemma 2.1. *Let K and M be semiregular subgroups of $\text{Sym}(\Omega)$, and f an isomorphism from K to M . Then there exists $x \in \text{Sym}(\Omega)$ such that $g^f = g^x$ for every $g \in K$.*

Proof. Let $\text{Orb}(K, \Omega) = \{\Delta_1, \dots, \Delta_m\}$ and $\text{Orb}(M, \Omega) = \{\Lambda_1, \dots, \Lambda_m\}$, fix elements $\alpha_i \in \Delta_i$ and $\beta_i \in \Lambda_i$ for every $i \in \{1, \dots, m\}$. Define x as follows:

$$(\alpha_i^g)^x = \beta_i^{g^f}$$

for every $i \in \{1, \dots, m\}$ and every $g \in K$. Since K and M are semiregular, x is a well-defined bijection from Ω to itself.

Now take any $g \in K$ and $\alpha \in \Omega$. There exist the unique $i \in \{1, \dots, m\}$ and $t \in K$ such that $\alpha_i^t = \alpha^{x^{-1}}$. By definition of x ,

$$\alpha^{x^{-1}g^x} = (\alpha_i^{tg})^x = \beta_i^{(tg)^f} = ((\beta_i)^{t^f})^{g^f} = ((\alpha_i^t)^x)^{g^f} = ((\alpha^{x^{-1}})^x)^{g^f} = \alpha^{g^f}.$$

Thus, $g^x = g^f$ for every $g \in K$. □

Given a group G and its element g (its subgroup L), denote by g_l and g_r (L_l and L_r) the elements (subgroups) of $\text{Sym}(G)$ induced by left and right multiplications by g (by elements of L) respectively. Observe that L_l and L_r are always permutable, so $L^* = L_l L_r$ is a subgroup of $\text{Sym}(G)$. In this notation, G_l and G_r are regular subgroups of $\text{Sym}(G)$ isomorphic to G , the intersection $G_l \cap G_r$ is isomorphic to the center $Z(G)$ of G , and if $Z(G) = 1$ then $G^* = G_l \times G_r$ is the direct product of G_l and G_r . Further we refer to the set of all regular subgroups of $K \leq \text{Sym}(\Omega)$ isomorphic to G as $\text{Reg}(K, G)$ (this set can be empty).

Let Γ be a graph with vertex set Ω , and let $K = \text{Aut}(\Gamma)$. Then Γ has a Cayley representation over a group G if and only if $\text{Reg}(K, G) \neq \emptyset$. Indeed, if $\Gamma = \text{Cay}(G, X)$, then $G_r \in \text{Reg}(K, G)$. Vise versa, for every $H \in \text{Reg}(K, G)$ and a permutation group isomorphism $f : H \rightarrow G_r$, one can construct a Cayley representation of Γ over G as follows. Let $f_0 : \Omega \rightarrow G$ be the bijection induced by f , i.e.,

$$f_0(\alpha^h) = f_0(\alpha)^{f(h)}$$

for all $\alpha \in \Omega$ and $h \in H$. Denote by X_0 the set of all neighbors of $\alpha_0 = f_0^{-1}(e)$ in Γ , where e is the identity element of G . Then one can see that f_0 is an isomorphism from Γ onto the graph $\text{Cay}(G, X)$, where $X = f_0(X_0)$. Clearly, this graph is a Cayley representation of Γ over G ; we say that this representation is *associated* with H and f . It is easy to see that any

Cayley representation of Γ over G is associated with a suitable regular group $H \leq \text{Aut}(\Gamma)$ and an isomorphism $f : H \rightarrow G_r$.

Lemma 2.2. *Let Γ_i be a Cayley representation of a graph Γ , associated with a regular group $H_i \leq \text{Aut}(\Gamma)$ and an isomorphism $f_i : H_i \rightarrow G_r$, $i = 1, 2$. Then Γ_1 and Γ_2 are equivalent if and only if H_1 and H_2 are conjugate in $\text{Aut}(\Gamma)$.*

Proof. If $H_2 = (H_1)^\gamma$ for some $\gamma \in \text{Aut}(\Gamma)$, then $\sigma = f_1^{-1}\gamma f_2$ is an isomorphism from Γ_1 onto Γ_2 . Since σ takes G_r to itself preserving the identity element, it is a Cayley isomorphism. Thus, the Cayley representations Γ_1 and Γ_2 are equivalent. Conversely, if $\sigma \in \text{Aut}(G)$ is Cayley isomorphism from Γ_1 onto Γ_2 , then $\gamma = f_1\sigma f_2^{-1}$ is an automorphism of Γ such that $H_2 = (H_1)^\gamma$. \square

Corollary 2.3. *For every G -base \mathcal{B} of a graph Γ , the Cayley representations of Γ associated with the groups $H \in \mathcal{B}$, form the full system of pairwise nonequivalent Cayley representations of Γ over G . In particular, the size of this system is equal to $b_G(\Gamma)$.*

According to our definition of a Cayley graph $\Gamma = \text{Cay}(G, X)$, the automorphism group $K = \text{Aut}(\Gamma)$ always contains the group G_r of right multiplications. If Γ is central, we can say more.

Lemma 2.4. *Suppose that a Cayley graph $\Gamma = \text{Cay}(G, X)$ is central, and $K = \text{Aut}(\Gamma)$. Then the following hold:*

- (i) $G_l \leq K$; in particular, $G^* = G_l G_r \leq K$.
- (ii) $X = X^{-1}$ if and only if the permutation $\sigma : g \mapsto g^{-1}$ belongs to K ; in particular, if Γ is undirected, then G_l and G_r are conjugate in K .

Proof. (i) For every $g \in G$ and every $(h, xh) \in E(\Gamma)$, where $E(\Gamma)$ is the arc set of Γ ,

$$(h, xh)^{g^l} = (g^{-1}h, g^{-1}xh) = (g^{-1}h, x^g g^{-1}h) \in E(\Gamma),$$

because X is normal.

(ii) Obviously, if $\sigma \in K$, then $X = X^{-1}$, and $G_l = G_r^\sigma$ and G_r are conjugate in K . If $X = X^{-1}$, then

$$(h, xh)^\sigma = (h^{-1}, h^{-1}x^{-1}) = (h^{-1}, (x^{-1})^h h^{-1}) \in E(\Gamma),$$

for every $(h, xh) \in E(\Gamma)$. \square

Remark. It is easy to see that there are different ways to define central Cayley graphs and describe their basic properties, see, e.g., [13, Section 1.1 and Corollary 2.5], where such graphs were called *holomorph Cayley graphs*.

3. CENTRAL CAYLEY GRAPHS OVER SIMPLE GROUPS

It is clear that any two isomorphic regular subgroups of the symmetric group are conjugate in it, so Lemma 2.2 and Lemma 2.4(ii) yield that in order to prove Theorem 1, it suffices to establish the following assertion.

Theorem 3.1. *Suppose that Γ is a central Cayley graph over a simple group G , and $K = \text{Aut}(\Gamma)$. Then either $K = \text{Sym}(G)$ or $\text{Reg}(K, G) = \{G_l, G_r\}$; in particular, $b_G(\Gamma) \leq 2$.*

Proof. We start with a simple observation that the only 2-transitive group which is the full automorphism group of a graph with the vertex set Ω is $\text{Sym}(\Omega)$. So in the case when G is abelian, we are done by the Burnside theorem [3]. Indeed, if K is not 2-transitive, then

$K \leq \text{AGL}(1, p)$ for a prime p , so G_r being normal is the only subgroup of order p in K . Therefore, we may further assume that G is nonabelian simple and K is not 2-transitive.

Following [14], for a group G we define the holomorph $\text{Hol}(G)$ as the normalizer $N_{\text{Sym}(G)}(G_r)$ and set $D(2, G)$ to be the subgroup of $\text{Sym}(G)$ generated by $\text{Hol}(G)$ and the permutation $\sigma : g \mapsto g^{-1}$. Since G is nonabelian simple, groups G_l and G_r intersect trivially, hence

$$(1) \quad G^* = G_l \times G_r = \text{Inn}(G) \rtimes G_r \leq D(2, G) = \langle \sigma \rangle \rtimes (\text{Aut}(G) \rtimes G_r).$$

It can be deduced from the classification of finite simple groups (CFSG), see, e.g, Tables 5,6 in p. xvi [4, Introduction], or [11], that

$$(2) \quad |\text{Out}(G)| \leq \log n$$

(hereinafter, all the logarithms are binary and $|G| = n$). This fact and (1) directly yield

Lemma 3.2. $|D(2, G)| \leq 2n^2 \log n$.

The next lemma essentially proved in [19] exploits the classification of regular almost simple subgroups of a primitive group [14, Theorem 1.4].

Lemma 3.3. $G^* \leq K \leq D(2, G)$.

Proof. The first inclusion follows from Lemma 2.4(i). If K is imprimitive, then G has a non-trivial proper normal subgroup by [19, Lemma 4.2] that in our case contradicts to simplicity of G . So K must be uniprimitive (primitive but not 2-transitive). The rest follows from [19, Lemma 3.2]. \square

The key lemma below is a reformulation of [9, Lemma 12.1].

Lemma 3.4. *If G is a nonabelian simple group, then there exists an involution $t \in G$ such that $\text{Aut}(G) = C_{\text{Aut}(G)}(t) \text{Inn}(G)$.*

The next lemma completes the proof of the theorem.

Lemma 3.5. $\text{Reg}(K, G) = \{G_l, G_r\}$.

Proof. Let $H \in \text{Reg}(K, G)$. The group G^* is clearly normal in $D(2, G)$, so $H \cap G^*$ is normal in H . Since H is simple, $H \cap G^* = H$ or $|H \cap G^*| = 1$. In the latter case, $|HG^*| = n^3 \leq |K| \leq |D(2, G)|$, a contradiction to Lemma 3.2. Thus, $H \leq G^*$.

Assume to the contrary that H is neither G_l nor G_r . Then, due to simplicity of G , the projections of $H \leq G_l \times G_r$ to each of the two factors are isomorphic to G . It follows that there exists an automorphism $\tau \in \text{Aut}(G)$ such that

$$H = G_\tau = \{g_l(g^\tau)_r : g \in G\}.$$

Let us choose an involution $t \in G$ as in Lemma 3.4. Since $\tau \in \text{Aut}(G)$, there exist $\varphi \in C_{\text{Aut}(G)}(t)$ and $\psi \in \text{Inn}(G)$ such that $\tau = \varphi\psi$. Fix any $x \in G$ such that ψ is a conjugation by x . Consider the permutation $h = t_l(t^\tau)_r$ from H . Clearly, h is nontrivial.

Now, if $t \neq x$, then

$$(tx)^h = (tx)^{t_l(t^\tau)_r} = t(tx)t^\tau = xt^{\varphi\psi} = x(x^{-1}t^\varphi x) = tx.$$

If $t = x$, then $t^\tau = (t^\varphi)^t = t^3 = t$ and hence $t^h = t$. Thus in the both cases, the element $h \in G_\tau$ leaves at least one point fixed. Since h is nontrivial, the group $H = G_\tau$ is not regular, a contradiction. \square

Remark. The proof of Lemma 3.5 is valid for any permutation group K with $G^* \leq K \leq D(2, G)$. □

4. BASE OF COMPLETE TRANSPOSITION GRAPH

A lower bound on the number of the pairwise nonequivalent central Cayley representation of the complete transposition graph mentioned in Introduction is a consequence of the following assertion.

Theorem 4.1. *Suppose that $G = \text{Sym}(m)$ and $K = D(2, G) = \langle \sigma \rangle \times (\text{Aut}(G) \times G_r)$, where $\sigma : g \mapsto g^{-1}$. Then $b_G(K) \geq \lfloor \frac{m}{4} \rfloor$.*

Proof. If $m \leq 7$, then $\lfloor \frac{m}{4} \rfloor \leq 1$ and the theorem is obvious. Further we assume that $m \geq 8$. This condition provides that $\text{Aut}(G) = \text{Inn}(G)$. Therefore, $K = \langle \sigma \rangle \times G^*$, where $G^* = G_l \times G_r$; every $k \in K$ is uniquely represented as

$$(3) \quad k = \sigma^\varepsilon u_l z_r, \text{ where } \varepsilon \in \{0, 1\}, \text{ and } u, z \in G;$$

and $z_r^\sigma = z_l$ for every $z \in G$.

Set $A = \text{Soc}(G) = \text{Alt}(m)$ and fix an element $x \in G \setminus A$. For every $y \in G$ set $\tau_y = y_l x_r \in G^*$ and $H_y = \langle \tau_y \rangle A_r$. Since A_r is normal in G^* , H_y is a subgroup of G^* , and $H_y = G_r$ if and only if $y = 1$.

Lemma 4.2. *If t is an involution from A , then H_t is a regular subgroup of K , isomorphic to G .*

Proof. Assume that $g = g^{\tau_t z_r} = t g x z$ for some $g \in G$ and $z \in A$. Since t and z are even permutations, while x is an odd one, the permutations g and $t g x z$ are of distinct parities, a contradiction. Therefore, H_t is a semiregular subgroup. In fact, it is a regular subgroup due to $|H_t| = |G|$. Indeed, $H_t > A_r$, because $t_l \neq 1$, and $|H_t : A_r| \leq 2$, because $\tau_t^2 \in A_r$.

Let N be a nontrivial normal subgroup of H_t distinct from A_r . Since A_r is simple and $|H_t : A_r| = 2$, it follows that $H_t = N \times A_r$, where $|N| = 2$. Let $n = t_l x_r z_r$ be the involution in N . It follows that $u_r n = n u_r$ and, consequently, $u x z = x z u$ for every $u \in A$. Then $x z \in C_G(A)$, where $C_G(A)$ is trivial due to simplicity of A . Thus, $x z = 1$, which is impossible, because x and z are the permutations of distinct parities. Thus, $A_r = \text{Soc}(H_t)$, hence $A_r \trianglelefteq H_t \leq \text{Aut}(A_r) \simeq G$. Finally, $|H_t| = |G|$ implies $H_t \simeq G$. □

Lemma 4.3. *Let t be an involution from A and $y \in G$. If $y \neq t$, then $y_l z_r \notin H_t$ for every $z \in G$. In particular, $H_y = H_t$ if and only if $y = t$.*

Proof. Assume that $y_l z_r \in H_t$ for some $z \in G$. Then $y_l z_r = t_l x_r u_r$ for some $u \in A$ due to Lemma 4.2. Since $G^* = G_l \times G_r$, we conclude that $y = t$, a contradiction. □

Lemma 4.4. *Let t and y be involutions from A . If H_t and H_y are conjugate in K , then t and y are conjugate in G .*

Proof. Suppose that H_t and H_y are conjugate by $k = \sigma^\varepsilon u_l z_r \in K$, cf. (3). Straightforward computations yield

$$(\tau_t)^k = \begin{cases} (u t u^{-1})_l (z^{-1} x z)_r & \text{for } \varepsilon = 0, \\ (u x^{-1} u^{-1})_l (z^{-1} t^{-1} z)_r & \text{for } \varepsilon = 1. \end{cases}$$

Since $(\tau_t)^k \in H_y$, Lemma 4.3 implies that either $y = t^{u^{-1}}$ or $y = (x^{-1})^{u^{-1}}$. The latter case is impossible, because y is an even permutation, while x^{-1} is odd. □

Let T be the set of involutions from $\text{Soc}(G) = \text{Alt}(m)$ pairwise non-conjugate in G . Lemmas 4.2 and 4.4 yield $b_G(K) \geq |T| \geq \lfloor \frac{m}{4} \rfloor$. \square

Corollary 4.5. *Let $G = \text{Sym}(m)$, $m \geq 5$, $X = X^{-1}$ a proper normal subset of $G \setminus \text{Soc}(G)$, and $\Gamma = \text{Cay}(G, X)$. Then the number of pairwise nonequivalent central Cayley representations of Γ is at least $\lfloor \frac{m}{4} \rfloor$.*

Proof. By [19, Theorem 1.3], $\text{Aut}(\Gamma) = D(2, G)$. The rest follows from Corollary 2.3 and Theorem 4.1. \square

Remark. The complete transposition graph over $G = \text{Sym}(m)$ mentioned in Introduction is a particular case of graphs from the corollary (if $m \geq 5$). If $m < 5$, then the conclusion of the corollary remains obviously true for such a graph. It is worth noting that the equality $\text{Aut}(\Gamma) = D(2, G)$ for the complete transportation graph was obtained in [10, Theorem 1.1].

5. CENTRAL CAYLEY GRAPH OVER ALMOST SIMPLE GROUPS

In order to present our arguments we need some notation concerning imprimitive permutation groups. First, if $\Delta \subseteq \Omega$ is a block of $K \leq \text{Sym}(\Omega)$, then the group $\{f^\Delta : f \in K, \Delta^f = \Delta\}$ is denoted by K^Δ , where f^Δ is the induced bijection from Δ to Δ^f . For an imprimitivity system \mathcal{L} of a transitive group K , set $K_{\mathcal{L}}$ and $K^{\mathcal{L}}$ to be, respectively, the intersection of the setwise stabilizers $K_{\{\Delta\}}$ for all $\Delta \in \mathcal{L}$ and the permutation group induced by the action of K on \mathcal{L} .

Let G be a finite group and $K \leq \text{Sym}(G)$ such that $K \geq G^* = G_l G_r$. If X is a block of K containing the identity element e of G , then X is a subgroup of G because $K \geq G_r$ (cf. [23, Theorem 24.12]); moreover, X is normal in G because $K \geq G_l$. Denote by $L = L(K)$ the intersection of all non-singleton blocks of K containing e . Following [19], we say that L is the *minimal block* of K . Denote the block system of K containing L by $\mathcal{L} = \mathcal{L}(K)$. Clearly, \mathcal{L} coincides with the set of all L -cosets, $L^* \leq K_{\mathcal{L}}$, and $\text{Orb}(L^*, G) = \mathcal{L}$.

The following assertion is our key argument in proving Theorem 2. Recall that \mathcal{K}_c is the class of almost simple groups G with $|G : \text{Soc}(G)| \leq c$.

Theorem 5.1. *If Γ is a central Cayley graph over $G \in \mathcal{K}_c$, then $b_G(\Gamma)$ is polynomial in n .*

Proof. Let $K = \text{Aut}(\Gamma)$. It suffices to point out a set of subgroups of K , which includes some G -base of K and has the size polynomial in n . The lemma below summarizes some properties of K proved in [19, Lemma 4.2, Proposition 6.1, and Theorem 6.2].

Lemma 5.2. *Let $L = L(K)$ be the minimal block of K , and $\mathcal{L} = \mathcal{L}(K)$. Then L is an almost simple group containing the socle S of G and one of the following holds:*

- (i) $L = S$ and $K^L \leq D(2, L)$;
- (ii) $K = (K_{\mathcal{L}}) \wr K^{\mathcal{L}}$ is the full wreath product and $(K_{\mathcal{L}})^\Delta = \text{Sym}(\Delta)$ for every $\Delta \in \mathcal{L}$.

Following [19], we say that K is of *normal type* provided Item (i) of Lemma 5.2 holds, and of *symmetric type* otherwise.

Let K be of normal type. Since $K \leq K^L \wr K^{\mathcal{L}}$ and $|\mathcal{L}| \leq c$, Item (i) of Lemma 5.2 and Lemma 3.2 yield that

$$|K| \leq |K^L|^c |\text{Sym}(\mathcal{L})| \leq (2n^2 \log n)^c \cdot c! = \text{poly}(n).$$

By [5], every almost simple group is 3-generated. Hence the number of the subgroups of K isomorphic to G is polynomial in the size of K . It follows that $|\text{Reg}(K, G)|$ is polynomial in n , and so is any G -base of K .

Therefore, we may suppose that K is of symmetric type. By Lemma 5.2, the minimal block L is an almost simple group with the socle S , hence every $H \in \text{Reg}(K, G)$ includes a normal subgroup isomorphic to L . If $T = \text{Soc}(H)$ does not stabilize each $\Delta \in \mathcal{L}$, then T acts faithfully on the set G/L of size at most c . In this case, the order of G is bounded by a function depending only on c , and we are obviously done. Therefore, T is a semiregular subgroup of $K_{\mathcal{L}}$, in particular, the same is true for $S_r = \text{Soc}(G_r)$. Lemma 2.1 implies that T^{Δ} and S_r^{Δ} are conjugate in $(K_{\mathcal{L}})^{\Delta} = \text{Sym}(\Delta)$ for each $\Delta \in \mathcal{L}$. However, $K_{\mathcal{L}}$ is the direct product of the symmetric groups $\text{Sym}(\Delta)$, $\Delta \in \mathcal{L}$, by Lemma 5.2(ii), so there is $k \in K_{\mathcal{L}}$ such that $T = S_r^k$.

Thus, T is conjugate to S_r in K . It follows that H is conjugate (in K) to a regular subgroup from the normalizer $N = N_K(S_r) = N_{\text{Sym}(G)}(S_r) \cap K$ of S_r in K . It is easy to see that $|N| = \text{poly}(n)$. Indeed, $|N_{\text{Sym}(G)}(S_r)| \leq |\text{Aut}(S_r)||C|$, where $C = C_{\text{Sym}(G)}(S_r)$ is the centralizer of S_r . Since S_r is a simple group, $|\text{Aut}(S_r)| \leq n \log n$, see (2). On the other hand, the set $\text{Orb}(S_r, G)$ is an imprimitivity system for C , so C is embedded in the corresponding imprimitive wreath product. It follows that $|C| \leq |C_{\text{Sym}(\Delta)}(S_r)|^c |\text{Sym}(c)|$ for $\Delta \in \text{Orb}(S_r, G)$. However, by semiregularity of S_r , $|C_{\text{Sym}(\Delta)}(S_r)| = |S_r| = n$, and we are done. Thus, if K is of symmetric type, then the size of $\text{Reg}(N, G)$ is polynomial in n , and so is the size of a G -base in K . \square

Proof of Theorem 2. Let Γ be a central Cayley graph over $G \in \mathcal{K}_c$. According to Corollary 2.3, in order to find the full system of pairwise nonequivalent central Cayley representations of Γ , it suffices to find a G -base \mathcal{B} of Γ , that is a maximal set of pairwise non-conjugate regular subgroups of $K = \text{Aut}(\Gamma)$, isomorphic to G . Theorem 5.1 implies that the size of \mathcal{B} is polynomial in n . Here we show that \mathcal{B} can be found efficiently.

We begin with the observation that a generating set of K can be found in time $\text{poly}(n)$ by [19, Corollary 1.2]. Applying standard algorithms for permutation groups (here our main source is [21]), one can find the socle $S_r = \text{Soc}(G_r)$, the minimal block $L = L(K)$ and $\mathcal{L} = \mathcal{L}(K)$ in time $\text{poly}(n)$, and check whether K is of symmetric type or normal type.

If K is of symmetric type, then put $M = N_K(S_r) = N_{\text{Sym}(G)}(S_r) \cap K$. Note that $N_{\text{Sym}(G)}(S_r)$ can be found in time $\text{poly}(n)$ due to [16, Corollary 3.24]. If K is of normal type, we simply put $M = K$. In both cases, arguments from the proof of Theorem 5.1 show that $|M| = \text{poly}(n)$ and the set $\text{Reg}(M, G)$ contains a G -base of K .

The set $\text{Reg}(M, G)$ can be found in time polynomial in $|M|$ by standard procedures. Indeed, it suffices to iterate all 3-generated subgroups of M and for each of them check whether it is regular and isomorphic to G . In particular, $|\text{Reg}(M, G)| = \text{poly}(n)$.

It remains to choose subgroups in $\text{Reg}(M, G)$ pairwise non-conjugate in K . Subgroups $H_1, H_2 \in \text{Reg}(K, G)$ are conjugate in K , if and only if the corresponding Cayley graphs Γ_1 and Γ_2 are Cayley isomorphic (Lemma 2.2). However, testing Cayley isomorphism of two Cayley graphs over G can be easily done in time polynomial in the order of $\text{Aut}(G)$, which, in our case, is polynomial in n , because G is almost simple (see (2)). This completes the proof of Theorem 2.

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