ON p-INDEX EXTREMAL GROUPS

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Abstract. The question on connection between the structure of a finite group G and the properties of the indices of elements of G has been a popular research topic for many years. The *p-index* $|x^G|_p$ of an element x of a group G is the p-part of its index $|x^G| = |G : C_G(x)|$. The presented short note describes some new results and open problems in this direction, united by the concept of the p-index of a group element.

Key words: finite group, conjugacy class, index, p-index of a group element.
MSC: 20E45, 20D60.

The size of the conjugacy class of an element of a finite group, which is known to be equal to the index of its centralizer, is one of main numerical characteristics of a finite group. The question on connection between the structure of a finite group and the properties of the size set of its conjugacy classes has been a popular research topic for many years. Thus, at the very beginning of the 20th century, W. Burnside in [1] showed that a group of composite order, having a conjugacy class whose size is a power of a prime number, cannot be simple. That assertion was the key to his famous theorem on the solvability of finite $\{p, q\}$-groups. The presented short note describes some new results and open problems in this direction, united by the concept of the *p-index* of a group element.

First, we introduce necessary definitions and notation. Let G be a finite group and $x \in G$. As usual, $Z(G)$ denotes the center of G, $C_G(x)$ denotes the centralizer of x in G, and x^G denotes the conjugacy class in G containing the element x, in other words, x^G is the orbit of x with respect to the action by conjugacy of G on itself. The size of the given orbit (class) $|x^G| = |G : C_G(x)|$ is called the *index* of the element x in G. The set of all indices of the elements of G is denoted by $N(G)$ (the notation $\mathrm{cl}(G)$ is also very common, see, e. g., [2]).

If p is a prime (π is some set of primes), then for a positive integer n, we refer to n_p as the p-part of n, i. e. the largest power of p dividing n (respectively, to n_π as the π-part of n). The *p-index* $|x^G|_p$ of an element x of a group G is the p-part of $|x^G|$. For a group G, we set $N(G)_p = \{|x^G|_p : x \in G\}$ and call the *p-index* $|G||_p$ of G the largest element in the set $N(G)_p$, i. e. $|G||_p = \max\{|x^G|_p : x \in G\}$.

Following [2], we denote by $e_p(G)$ the number of nontrivial elements in $N(G)_p$. We are interested in groups G with $e_p(G) = 1$; we call them *extremal with respect to p-index* or, simply, *p-index extremal* and write $G \in R(p)$ (in the recent paper by the second author [3] these groups were called *$R(p)$-groups*).

Although the class $R(p)$ has been singled out recently, results on groups in this class (with various additional restrictions) are well known. Already in 1953, N. Ito proved that in a group G with $N(G) = \{1, n\}$ the number n is a power of some prime p, and the group itself is nilpotent [4]. More precisely, G is the direct product of a p-group A and an abelian group B of order coprime to p. It was later established in [5] that the nilpotency class of the group A does not exceed 3. In [6], A. Kamina showed that for any primes p and q the group G with $N(G) = \{1, p^\alpha, q^\beta, p^\alpha q^\beta\}$ is the direct product of a p-group A and q-group B and, in

A. V. Vasil'ev was supported by National Natural Science Foundation of China (No. 12171126), I. B. Gorshkov was supported by the Russian Science Foundation, project 18-71-10007.

particular, is nilpotent, while Beltran and Felipe [7, 8] generalized this result for the case of two arbitrary coprime positive integers m and n. Obviously, all the above groups will be extremal with respect to any prime p (later we will return to groups with this property).

Recall that a group is called p-solvable if it contains a normal series whose factors are either p-groups or p'-groups. In [2], Casolo and Tombari considered the p-index extremal p-solvable groups. It was shown that the p-length of every such group must be equal to 1. Moreover, if a Sylow p-subgroup P of such group is nonabelian, then G has a normal p-complement [2, Theorem 2.3] and $Z(P) \leq Z(G)$ [2, Lemma 2.4]. It turns out that the last two statements are true in a much more general situation. Note first that the class $R(p)$ splits naturally into two non-overlapping subclasses:

1) $G \in R(p)^*$, if there is a p-element in G with a nontrivial p-index;

2) $G \in R(p)^{**}$, otherwise.

Recently, the second author obtained the following results.

Theorem 1. [3, Theorem 1.3] *If $G \in R(p)^*$, then G has the normal p-complement.*

Corollary 1. [3, Corollary 1.4] *If $G \in R(p)^*$ and P is a Sylow p-subgroup of G, then $Z(P) \leq Z(G)$.*

Certainly, if there is a p-element in G with nontrivial p-index, then a Sylow p-subgroup P is nonabelian. The converse is not true in general [9, Theorem B]. Nevertheless, it is easy to deduce from [10, Lemma 7] that in the case of a p-index extremal group, noncommutativity of a Sylow p-subgroup of G yields $G \in R(p)^*$.

In [11], the first author noted that in a group G with trivial center and $|G||_p = p$, a Sylow p-subgroup must be abelian (and even elementary abelian). Using Corollary 1 in the case of $G \in R(p)^*$ and [12, Theorem B] in the case of $G \in R(p)^{**}$, it is easy to establish a similar assertion for an arbitrary p-index extremal group G with trivial center.

Corollary 2. [3, Corollary 1.5] *If $G \in R(p)$ and $Z(G) = 1$, then a Sylow p-subgroup of G is abelian.*

Although an $R(p)$-group G with trivial center can have a noncyclic Sylow p-subgroup (for example, the group $A_5 \in R(2)^{**}$ has this property), it seems that such groups can be fully classified.

Problem 1. *Describe the structure of the p-index extremal groups with trivial center and noncyclic Sylow p-subgroups.*

Let us return to Theorem 1. According to it, the group G is a semidirect product of the normal p'-subgroup of G and a p-subgroup P. The structure of P is generally clear (see [5, 13]), it would be interesting to have an answer to the following question.

Problem 2. *Describe the structure of the normal p-complement of $G \in R(p)^*$.*

It is easy to see that groups with the $R(p)^{**}$ property are quite common. For example, using the well-known properties of centralizers, one can show that for any finite nonabelian simple group G there exists a prime p such that a Sylow p-subgroup S of G is cyclic and $C_G(S) = C_G(x)$ for any nontrivial element $x \in S$, and hence $G \in R(p)^{**}$. It turns out that any group from the class $R(p)^{**}$ has the following property.

Theorem 2. [10, Lemma 8] *If $G \in R(p)^{**}$, then G contains at most one nonabelian composition factor S whose order is divisible by p.*

We believe that more can be said in this situation.

Problem 3. *Let $G \in R(p)^{**}$ be a nonsolvable group. Is it true that $G = P \times H$ is the direct product of a p-group P and a group H such that $O^{p'}(H/O_{p'}(H))$ is a nonabelian simple group?*

The aforementioned results of Ito, Kamina and others dealt with groups that are extremal with respect to any p-index, we call them *index extremal* ($\mathcal{D}$*-groups* in the terminology from [2]). These groups can also be described as groups with the following property:

$$N(G) \subseteq \prod_{p \in \pi(N(G))} N(G)_p, \text{ where } N(G)_p = \{1, p^{\alpha(p)}\} \ \forall p \in \pi(N(G)). \tag{1}$$

In the case when equality takes place in (1) instead of inclusion, the group G will be nilpotent. This fact was established in [2, Theorem 1]. Moreover, in this paper, an explicit description was given for the index extremal groups satisfying the additional condition $p^{\alpha(p)} \in N(G)$ for any $p \in \pi(N(G))$ [2, Theorem 2]. In particular, all such groups turn out to be solvable. In the general case, the latter is not true — the nonabelian simple group A_5 is obviously index extremal. Therefore, the following problem appears.

Problem 4. *Describe the structure of the finite index extremal groups.*

Let p and q be distinct primes. A finite group is said to *have property* $\{p,q\}^*$ if $G \in R(p) \cap R(q)$ and $a_{\{p,q\}} > 1$ for any $a \in N(G) \setminus \{1\}$. In [14], Ito studied the simplest nontrivial subclass of $\{p,q\}^*$-groups, showing that if $N(G) = \{1, m, n\}$, where m and n are coprime numbers, then G is solvable. In the general case, however, a $\{p,q\}^*$-group need not be solvable. Moreover, it can be shown (see, for example, [15, Lemma 3.4(v)] and [16, Lemma 3.2(v)] for classical groups) that, with some exceptions, for a simple group G of Lie type, there are primes p and q such that G is a $\{p,q\}^*$-group.

Recall that $|G||_p$ denotes the p-index of the group G. If π is a set of primes, then we put $|G||_\pi = \prod_{p\in\pi} |G||_p$. It is obvious that $|G||_\pi$ divides $|G|_\pi$ for each π. In [17], the following properties of $\{p,q\}^*$-groups with trivial center were obtained.

Theorem 3. *Let p and q be distinct primes greater than 5. If G is a $\{p,q\}^*$-group with trivial center, then $|G|_{\{p,q\}} = |G||_{\{p,q\}}$.*

Corollary 3. *Let p and q be distinct primes greater than 5. If G is a $\{p,q\}^*$-group with trivial center, then $C_G(g) \cap C_G(h) = 1$ for each nontrivial p-element g and each nontrivial q-element h of G.*

The last corollary turned out to be the key one in the proof of an assertion about orthogonal groups of dimensions 8 and 16 [18, Theorem 1], which became the final step (see details in [18]) in proving the validity of the well-known conjecture stated by J. G. Thompson in a letter to W. Shi in 1987, and which in 1992 was puted by A. S. Kondratiev in the *Kourovka notebook*.

Thompson's Conjecture [19, Problem 12.38]. *If L is a finite nonabelian simple group, G is a finite group with trivial center, and $N(G) = N(L)$, then G and L are isomorphic.*

Certainly, the assertion of Thompson's conjecture cannot be extended to the case when L is an arbitrary finite group with trivial center; in particular, there are a nonsolvable group L and solvable group G, both with trivial center, satisfying $N(L) = N(G)$ [20]. However, the following problem remains open.

Problem 5. [19, Problem 20.29] *Is it true that for any natural number n and any nonabelian simple group L, if G is a finite group with trivial center and $N(G) = N(P)$, where $P = L^n$ is the nth direct power of L, then G and P are isomorphic?*

In the case $n = 1$, the positive answer to this question follows from the validity of Thompson's conjecture. If $n > 1$, then at the moment there are only a few very particular results [21–23].

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