

WHAT DO FROBENIUS'S, SOLOMON'S, AND IWASAKI'S THEOREMS ON DIVISIBILITY IN GROUPS HAVE IN COMMON?

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Our result contains as special cases the Frobenius theorem (1895) on the number of solutions to the equation $x^n = 1$ in a group, the Solomon theorem (1969) on the number of solutions in a group to a system of equations having less equations than unknowns, and the Iwasaki theorem (1985) on roots of subgroups. There are other curious corollaries on groups and rings.

0. Introduction

The following result was proved in XIX century.

Frobenius theorem [Frob95] (see also [And16]). *The number of solutions to the equation $x^n = 1$ in a finite group is divisible by $\text{GCD}(n, |G|)$ for any integer n .*

This theorem was generalised in different directions, see, e.g., [Hall36], [Kula38], [Sehg62], [BrTh88], [Yosh93], [AsTa01], [ACNT13], and references therein. For example, Frobenius himself [Frob03] obtained the following generalisation in 1903:

for any positive integer n and any element g of a finite group G , the number of solutions to the equation $x^n = g$ in G is divisible by the greatest common divisor of n and the order of the centraliser of g ;

Ph. Hall ([Hall36], Theorem II) showed that

in any finite group, the number of solutions to a system of equations in one unknown is divisible by $\text{GCD}(|C|, n_1, n_2, \dots)$, where C is the centraliser of the set of all coefficients and n_i are exponent sums of the unknown in i th equation.

Here, as usual, an equation over a group G is an expression of the form $v(x_1, \dots, x_m) = 1$, where v is a word whose letters are unknowns, their inverses, and elements of G (called *coefficients*). In other terms, the left-hand side of an equation is an element of the free product $G * F(x_1, \dots, x_m)$ of G and the free group $F(x_1, \dots, x_m)$ of rank m (where m is the number of unknowns).

The following theorem is also about equations in groups and divisibility, but on the first view, it is not similar to the Frobenius theorem and its generalisations.

Solomon theorem [Solo69]. *In any group, the number of solutions to a system of coefficient-free equations is divisible by the order of the group provided the number of equations is less than the number of unknowns.*

This theorem was also generalised in different directions, see [Isaa70], [Stru95], [AmV11], [GRV12], [KM14], [KM17], and references therein. For instance, in [KM14], it was shown that

in any group, the number of solutions to a system of equations (with coefficients from this group) is divisible by the order of the intersection of centralisers of all coefficients provided the rank of the matrix composed of the exponent sums of j th unknown in i th equation is less than the number of unknowns.

Solomon himself wrote in [Solo69]:

"There seems to be no connection between this theorem and the Frobenius theorem on solutions of $x^k = 1$."

Nevertheless, a connection between the Frobenius and Solomon theorems exists.

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Theorem 1. *In any group (not necessary finite) the number of solutions to a (not necessary finite) system of equations in m unknowns is a multiple of the greatest common divisor of the centraliser of the set of coefficients and the number $\frac{\Delta_m}{\Delta_{m-1}}$, where Δ_i is the greatest common divisor of all minors of order i of the matrix of the system assuming the following conventions: $\Delta_i = 0$ if i larger than the number of equations, $\Delta_0 = 1$, and $\frac{0}{0} = 0$.*

The *greatest common divisor* $\text{GCD}(G, n)$ of a group G and an integer n is the least common multiple of orders of subgroups of G dividing n . The divisibility is always understood in the sense of cardinal arithmetic: each infinite cardinal is divisible by all smaller nonzero cardinals (and surely zero is divisible by all cardinals and divides only zero). This means that $\text{GCD}(G, 0) = |G|$ for any group G and, e.g., $\text{GCD}(\mathbf{SL}_2(\mathbb{Z}), 2018) = 2$. Although, the reader will not lose much by assuming all group to be finite; in this case, $\text{GCD}(G, n) = \text{GCD}(|G|, n)$ by Sylow's theorem (and because a finite p -group contains subgroups of all possible orders).

The *matrix of a system of equations over a group* is the integer matrix $A = (a_{ij})$, where a_{ij} is the exponent sum of j th unknown in i th equation. For example, the matrix of the system

$$\begin{cases} xay^2[x, y]^{2018}(xby)^3 = 1 \\ bx^3y[x, y]^{100}(xby)^4 = 1 \\ [x, y^5]x^{-2} = 1 \end{cases}$$

(where x and y are unknowns, and a and b are coefficients, i.e. some fixed elements group) has the form

$$\begin{pmatrix} 4 & 5 \\ 7 & 5 \\ -2 & 0 \end{pmatrix}.$$

As usual, the *minors of order i* are determinants of submatrices composed of entries at the intersections of some i rows and i columns. In the example above, there are three minors of order m (up to signs):

$$\det \begin{pmatrix} 4 & 5 \\ 7 & 5 \end{pmatrix} = -15, \quad \det \begin{pmatrix} 4 & 5 \\ -2 & 0 \end{pmatrix} = 10, \quad \det \begin{pmatrix} 7 & 5 \\ -2 & 0 \end{pmatrix} = 10,$$

and six minors of order $m - 1$: 4, 5, 7, 5, -2, 0. Thus, the theorem asserts that (in this example) the number of solutions is divisible by

$$\text{GCD} \left(\frac{\text{GCD}(-15, 10, 10)}{\text{GCD}(4, 5, 7, 5, -2, 0)}, |C(a) \cap C(b)| \right) = \text{GCD}(5, |C(a) \cap C(b)|).$$

Note that the agreements about boundary cases in Theorem 1 are natural. Indeed, we always can add a fictitious equation $1=1$ and to make the number of equations larger than m . We can also add a new variable z and the equation $z = 1$ (this does not affect the number of solutions and makes $m > 1$). As for the philosophical question on the interpretation of the fraction $\frac{0}{0}$, it can be understood arbitrarily, e.g., the reader may assume that $\frac{0}{0} = 2018$; in any case, theorem 1 remains valid (but weaker than under the suggested interpretation).

The meaning of the value $\frac{\Delta_m}{\Delta_{m-1}}$ is as follows. It is well known (see, e.g., [Vin99]) that invertible integer elementary transformations of rows and columns can transform any integer matrix A into a diagonal matrix, where the diagonal entries divide each other (each diagonal entry divides the next one). This diagonal matrix is uniquely determined up to the signs of diagonal elements (and is sometimes called *Smith's form* of A); the diagonal elements of Smith's form (sometimes called the *invariant factors* of A) equal to the ratios $\frac{\Delta_i}{\Delta_{i-1}}$. Thus, in these terms, $\frac{\Delta_m}{\Delta_{m-1}}$ is the m th invariant factor of the matrix of the system of equations. One can also say that

the absolute value of $\frac{\Delta_m}{\Delta_{m-1}}$ is the period (exponent) of the quotient of the free abelian group \mathbb{Z}^m by the subgroup generated by the rows of the matrix of the system of equations

(with the stipulation that this ratio vanishes if and only if the period is infinite).

The Frobenius and Solomon theorems as well as their generalisations stated above are special cases of Theorem 1.

The following theorem is on the first view similar to neither the Frobenius theorem nor the Solomon theorem.

Iwasaki theorem [Iwa82]. *For any integer n , the number of elements of a finite group G whose n th powers lie in a subgroup $H \subseteq G$ is divisible by $|H|$.*

This beautiful theorem remains not widely known. In [SaAs07], it was noticed that the divisibility by $|H|$ still holds for the number of solutions to the “equation” $x^n \in HgH$, where HgH is any double coset of a subgroup H . Clearly, the Iwasaki theorem and its generalisations deals with predicates that are not equations in the usual sense.

Let us say that a *generalised equation* over a group G is an expression of the form $w(x_1, \dots, x_n) \in HgH$, where H is a subgroup of $G \ni g$, and $w(x_1, \dots, x_m)$ is an element of the free product $G * F(x_1, \dots, x_m)$ of G and a free group; in other terms, w is a word in the alphabet $G \sqcup \{x_1^{\pm 1}, \dots, x_m^{\pm 1}\}$. The elements of G occurred in this word are called the *coefficients* of the generalised equation. A system of generalised equations, a solution to this system, and a matrix of this system are defined in a natural way.

In [KM17], the following generalisation of the Iwasaki theorem was obtained:

the number of solutions to a system of generalised coefficient-free equations whose right-hand sides are double cosets of the same subgroup H (e.g., $\{x^{100}y^{2018}[x, y]^4 \in Hg_1H, [x^5, y^6]^7(xy)^8 \in Hg_2H, \dots\}$) is divisible by $|H|$.

The following theorem includes all results stated above.

Theorem 2. *Let S be a (not necessary finite) system of generalised equations in finitely many variables x_1, \dots, x_m over a group G and let P be its subsystem:*

$$S = \{u_i(x_1, \dots, x_m) \in H_i g_i H_i \mid i \in I\} \supseteq P = \{u_j(x_1, \dots, x_m) \in H_j g_j H_j \mid j \in J\},$$

(where $J \subseteq I$, $u_i \in G * F(x_1, \dots, x_m)$, $g_i \in G$, and H_i are subgroups of G). Then the number of solutions to S in G is divisible by the greatest common divisor of the subgroup

$$\tilde{H} = \left(\bigcap_{j \in J} N(H_j g_j H_j) \right) \cap \left(\bigcap_{i \in I \setminus J} H_i \right) \cap (\text{the centraliser of the set of coefficients of } S)$$

and the number $\frac{\Delta_m}{\Delta_{m-1}}$, where Δ_k is the greatest common divisor of all minors of order k of the matrix of the subsystem P . Henceforth, $N(A) \stackrel{\text{def}}{=} \{g \in G \mid g^{-1}Ag = A\}$ is the normaliser of a subset A in a group G .

To deduce Theorem 1 from Theorem 2, we rewrite the system of equations in the “generalised” form, i.e. we put $S = P = \{u_1(x_1, \dots, x_m) \in \{1\}1\{1\}, u_2(x_1, \dots, x_m) \in \{1\}1\{1\}, \dots\}$ and note that the normaliser of the trivial subgroup is the whole group.

On the other hand, setting

$$S = \{u_1(x_1, \dots, x_m) \in Hg_1H, u_2(x_1, \dots, x_m) \in Hg_2H, \dots\} \quad \text{and} \quad P = \emptyset \quad (\text{where } u_i \in F(x_1, \dots, x_m)),$$

we obtain the mentioned above generalisation (from [KM17]) of the Iwasaki theorem.

As a matter of fact, a relation between Solomon’s and Iwasaki’s theorems was established in [KM14] and [KM17]; our achievement consists only in adding “Frobeniusness”. The main theorem of [KM17] says that, if we have a group F with a fixed epimorphism onto \mathbb{Z} and some set of homomorphisms from F into another group G , and this set is invariant with respect to some natural transformations (depending on the epimorphism $F \rightarrow \mathbb{Z}$ and a subgroup H of G), then the number of these homomorphisms $F \rightarrow G$ is divisible by $|H|$. Choosing suitable sets of homomorphisms, the authors of [KM17] obtained Solomon’s and Iwasaki’s theorem as special cases of their main theorem.

Our main theorem (see Section 1) is a modular analogue of the main theorem of [KM17]: we take an epimorphism $F \rightarrow \mathbb{Z}/n\mathbb{Z}$ instead of $F \rightarrow \mathbb{Z}$. One can say that the main theorem of this paper is related to the main theorem of [KM17] in the same way as Theorem 1 to the generalisation (from [KM14]) of the Solomon theorem mentioned in the beginning of this paper. An important role in our argument is played by an elementary (but nontrivial) lemma due to Brauer [Bra69]. Actually, we need this lemma not to prove the main theorem but rather to explain that its statement *per se* makes some sense. For readers’ convenience, we give a proof of the Brauer lemma in the last section. Section 5 contains the proof of the main theorem.

In Section 2, we deduce Theorem 2 from the main theorem. As another corollary, we obtain a theorem on equations in rings (Theorem 3 in Section 3) that implies, e.g., the following fact, which can be considered as a generalisation of the Frobenius theorem in another direction:

for any representation $\rho: G \rightarrow \mathbf{GL}(V)$ of a group G and any words $u_i(x_1, \dots, x_m) \in F(x_1, \dots, x_m)$,

the number of solutions to the equation $\sum_{i=1}^n \left(\rho(u_i(x_1, \dots, x_m)) \right)^{l_i} = \text{id}$ is divisible by $\begin{cases} \text{GCD}(G, \text{GCD}(\{l_i\})) & \text{always;} \\ \text{GCD}(G, \text{LCM}(\{l_i\})) & \text{if } n \leq m; \\ |G| & \text{if } n < m. \end{cases}$

In Section 4, we show that the main theorem implies some fact about the number of crossed homomorphisms, generalising earlier known results.

In the next to last section, we discuss open questions.

Notation and conventions we use are mainly standard. Note only that, if $k \in \mathbb{Z}$ and x and y are elements of a group, then x^y , x^{ky} , and x^{-y} denote $y^{-1}xy$, $y^{-1}x^ky$ and $y^{-1}x^{-1}y$, respectively. The commutator subgroup of a group G is denoted by G' or $[G, G]$. If X is a subset of a group, then $|X|$, $\langle X \rangle$, $\langle\langle X \rangle\rangle$, $C(X)$, and $N(X)$ are the cardinality of X , subgroup generated by X , normal closure of X , centraliser of X , and normaliser of X . The index of a subgroup H of a group G is denoted by $|G : H|$. The letter \mathbb{Z} denotes the set of integers. If R is an associative ring with unity, then R^* denotes the group of units of this ring. GCD and LCM are the greatest common divisor and least common multiple. The symbol $\exp(G)$ denotes the period (exponent) of a group G if this period is finite; we assume $\exp(G) = 0$ if the period is infinite. The symbol $\langle g \rangle_n$ denotes the cyclic group of order n generated by an element g . The free group of rank n is denoted by $F(x_1, \dots, x_n)$ or F_n . The symbol $A * B$ denotes the free product of groups A and B .

Let us recall once again that the finiteness of groups is not assumed by default; the divisibility is always understood in the sense of cardinal arithmetics (an infinite cardinal is divisible by all nonzero cardinals not exceeding it), and $\text{GCD}(G, n) \stackrel{\text{def}}{=} \text{LCM}(\{|H| \mid H \text{ is a subgroup of } G, \text{ and } |H| \text{ divides } n\})$.

1. Main theorem

A group F equipped with an epimorphism $F \rightarrow \mathbb{Z}/n\mathbb{Z}$ (where $n \in \mathbb{Z}$) is called an n -indexed group. This epimorphism $F \rightarrow \mathbb{Z}/n\mathbb{Z}$ is called *degree* and denoted \deg . Thus, to any element f of an indexed group F , an element $\deg f \in \mathbb{Z}/n\mathbb{Z}$ is assigned; the group F contains elements of all degrees and $\deg(fg) = \deg f + \deg g$ for any $f, g \in F$.

Suppose that $\varphi: F \rightarrow G$ is a homomorphism from an n -indexed group F to a group G and H is a subgroup of G . The subgroup

$$H_\varphi = \bigcap_{f \in F} H^{\varphi(f)} \cap C(\varphi(\ker \deg))$$

is called the φ -core of H [KM17]. In other words, the φ -core H_φ of H consists of elements h such that $h^{\varphi(f)} \in H$ for all f , and $h^{\varphi(f)} = h$ if $\deg f = 0$.

Main theorem. Suppose that an integer n is a multiple of the order of a subgroup H of group G and a set Φ of homomorphisms from an n -indexed group F to G satisfies the following conditions.

I. Φ is invariant with respect to conjugation by elements of H :

if $h \in H$ and $\varphi \in \Phi$, then the homomorphism $\psi: f \mapsto \varphi(f)^h$ lies in Φ .

II. For any $\varphi \in \Phi$ and any element h of the φ -core H_φ of H , the homomorphism ψ defined by

$$\psi(f) = \begin{cases} \varphi(f) & \text{for all elements } f \in F \text{ of degree zero;} \\ \varphi(f)h & \text{for some element } f \in F \text{ of degree one (and, hence, and for all degree-one elements)} \end{cases}$$

belongs to Φ too.

Then $|\Phi|$ is divisible by $|H|$.

Note that the mapping ψ from Condition I is a homomorphism for any $h \in G$, and the formula for ψ from Condition II defines a homomorphism for any $h \in H_\varphi$ (as explained below). Thus, Conditions I and II only require these homomorphisms to belong to Φ .

Lemma 0. Suppose that $\varphi: F \rightarrow G$ is a homomorphism from an n -indexed group F to a group G , $f_1 \in F$ is an element of degree one and $g \in G$. Then the homomorphism $\psi: F \rightarrow G$ such that $\psi(f) = \varphi(f)$ for all $f \in F$ of degree zero and $\psi(f_1) = \varphi(f_1)g$ exists if and only if $g \in C(\varphi(\ker \deg))$ and $(\varphi(f_1)g)^n = (\varphi(f_1))^n$.

Proof. The group F can be presented in the form

$$F \simeq (F_0 * \langle x \rangle_\infty) / \langle\langle \{u^x u^{-f_1} \mid u \in F_0\} \cup \{x^n f_1^{-n}\} \rangle\rangle, \quad \text{where } F_0 = \ker \deg.$$

Therefore, the mapping $\psi: F_0 \cup \{x\} \rightarrow G$ can be extended to a homomorphism if and only if its restriction to F_0 is homomorphism and the relations $u^x = u^{f_1}$ (for $u \in F_0$) and $x^n = f_1^n$ are mapped to true equalities in G :

$$\psi(u)^{\psi(x)} = \psi(u^{f_1}) \quad \text{and} \quad \psi(x)^n = \psi(f_1^n). \quad (*)$$

If the restrictions of ψ and φ to F_0 coincide and $\psi(x) = \varphi(f_1)g$, then the first equality $(*)$ says that g commutes with $\varphi(u)$ (for all $u \in F_0$), while the second equalities $(*)$ takes the form $(\varphi(f_1)g)^n = (\varphi(f_1))^n$. This completes the proof.

Recall also the following beautiful (but not widely known) fact.

Brauer lemma [Bra69]. *If U is a finite normal subgroup of a group V , then, for all $v \in V$ and $u \in U$, the elements $v^{|U|}$ and $(vu)^{|U|}$ are conjugate by an element of U .*

These two lemmata imply immediately that the mapping ψ from Condition II is a homomorphism for any $h \in H_\varphi$ because $(\varphi(f)h)^n = (\varphi(f))^n$ by the Brauer lemma applied to $U = H_\varphi \subset V = H_\varphi \cdot \langle \varphi(f_1) \rangle \ni \varphi(f_1) = v$. Indeed, we obtain the equality $(\varphi(f_1)h)^{|H_\varphi|} = (\varphi(f_1))^{|H_\varphi|}u$ for some $u \in H_\varphi$ and, hence, $(\varphi(f_1)h)^n = (\varphi(f_1))^{nu} = (\varphi(f_1^n))^u$ (because $|H_\varphi|$ divides n). It remains to note that $u \in H_\varphi$ commutes with $\varphi(f_1^n)$ because $\deg f_1^n = n = 0 \in \mathbb{Z}/n\mathbb{Z}$. Thus, we obtain the equality $(\varphi(f_1)h)^n = (\varphi(f_1))^n$. It remains to refer to Lemma 0.

In the case $n = 0$ the main theorem was proved in [KM17]. So, our theorem is a “modular analogue” of the main result of [KM17]. On the other hand, our main theorem is deduced (in Section 5) from this special case $n = 0$.

Lemma 1. *In Condition II of the main theorem, $\psi(f) \in \varphi(f)H_\varphi$ for all $f \in F$.*

Proof. Indeed, if $\deg f = d$, then $f = f_1^d f_0$, where f_1 is the (fixed) element of degree one (from Condition II) and f_0 is an element of degree zero. Then

$$\psi(f) = \psi(f_1)^d \psi(f_0) = (\varphi(f_1)h)^d \varphi(f_0) = \varphi(f_1)^d \varphi(f_0)h' = \varphi(f_1^d f_0)h' = \varphi(f)h',$$

where the equality \equiv is valid for some $h' \in H_\varphi$ because $h \in H_\varphi$ and $\varphi(F)$ normalises H_φ .

2. Proof of Theorem 2

Let $L \subseteq G$ be the subgroup generated by all coefficients of the system S . Take as H any subgroup of the group \tilde{H} whose order divides $n \stackrel{\text{def}}{=} \frac{\Delta_m}{\Delta_{m-1}}$, and put

$$F = L * F(x_1, \dots, x_m) \quad \text{and} \quad \Phi = \left\{ \varphi: F \rightarrow G \mid \varphi(f) = f \text{ for } f \in L \quad \text{and} \quad \varphi(u_i) \in H_i g_i H_i \text{ for } i \in I \right\}.$$

As the indexing $\deg: F \rightarrow \mathbb{Z}/n\mathbb{Z}$, take an epimorphism whose kernel contains L and all u_j , where $j \in J$. Such an epimorphism exists because n is the period of the finitely generated abelian group $F/([F, F] \cdot L \cdot \langle \{u_j \mid j \in J\} \rangle)$.

Let us verify that the conditions of the main theorem hold. Condition I holds obviously for all $h \in H$ (and even for all $h \in \tilde{H}$) because (by definition) \tilde{H} centralises L and normalises double cosets $H_i g_i H_i$.

Condition II holds also for all $h \in H_\varphi$ because

- on L , the homomorphism ψ coincides with φ as L consists of zero-degree elements;
- $\psi(u_j) = \varphi(u_j)$ for $j \in J$ because again $\deg u_j = 0$;
- for $i \in I \setminus J$, we have $\psi(u_i) \in \varphi(u_i)H_\varphi \subseteq \varphi(u_i)H_i$ (where the inclusion \subseteq follows from Lemma 1).

Thus, the main theorem implies that $|\Phi|$ is divisible by the order of any subgroup $H \subseteq \tilde{H}$ whose order divides n , i.e. $|\Phi|$ is divisible by $\text{GCD}(\tilde{H}, n)$. It remains to note that $|\Phi|$ is the number of solutions to S .

3. Rings and representations

A *generalised homogeneous modulo n* equation with a set of unknowns X over an associative unital ring R is a finite expression of the form

$$\sum_i \prod_j c_{ij} x_{ij}^{k_{ij}} = 0, \quad \text{where coefficients } c_{ij} \in R, \text{ unknown } x_{ij} \in X, \text{ and exponents } k_{ij} \in \mathbb{Z},$$

such that, for some mapping $\deg: X \rightarrow \mathbb{Z}/n\mathbb{Z}$, the value $\sum_j k_{ij} \deg(x_{ij})$ (called the *degree of the equation*) does not depend on i (i.e. the “polynomial” in the left-hand side of the equation is homogeneous with respect to some assigning of degrees to variables), and $\langle \{\deg x \mid x \in X\} \rangle = \mathbb{Z}/n\mathbb{Z}$.

A system of equations is called *generalised homogeneous modulo n* if all equations of this system are generalised homogeneous modulo n (of possibly different degrees) with respect to the same function $\deg: X \rightarrow \mathbb{Z}/n\mathbb{Z}$.

As we explain below, the set $M = \{n \in \mathbb{Z} \mid \text{a given system is generalised homogeneous modulo } n\}$ consists of all divisors of a number n_0 , called the *homogeneity modulus* of the system. In other words, the homogeneity modulus is the maximal number from M or zero if M is infinite.

To find the homogeneity modulus, consider a homogenous system of linear equations, where unknowns are degrees of variables and also (the negations of) degrees of equations. The matrix of this system (called the *homogeneity matrix* of the initial system of equations) has the following form. Suppose that $X = \{x_1, \dots, x_m\}$. The *homogeneity matrix of p th equation* is the integer matrix $A_p = (a_{kl})$ of size

$$(\text{the total number of monomials in the system}) \times (m + (\text{the number of equations})),$$

where, for $l \leq m$, the (k, l) th entry is the exponent sum of l th unknown in k th monomial, the $(m + p)$ th column consists of ones, and the remaining columns are zero for $l > m$. The homogeneity matrix of the system of equations is composed from the matrices A_p written one under another: $A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \end{pmatrix}$. For example, the system of equations

$\{ax^3y^2 + y^7bx - 1 = 0, \quad xy^2x + y^7x^5 = 0\}$ (where x and y are unknowns and $a, b \in R$ are coefficients) has the following homogeneity matrix:

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 7 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \\ 5 & 7 & 0 & 1 \end{pmatrix}, \quad \text{composed of matrices} \quad A_1 = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 7 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 5 & 7 & 0 & 1 \end{pmatrix}.$$

Homogeneity-modulus lemma. *The homogeneity modulus of a system of s equations in m unknowns over an associative unital ring is $\frac{\Delta_{m+s}}{\Delta_{m+s-1}}$, where Δ_i is the greatest common divisor of all minors of order i of the homogeneity matrix of the system. As always, the following conventions are assumed: $\Delta_k = 0$ if the total number of monomials in all equations is less than k ; $\Delta_0 = 1$; $\frac{0}{0} = 0$.*

Proof. Let A be the homogeneity matrix. We have to find the maximal number n such that the system of linear homogeneous equations $AX = 0$ (in $m + s$ variables) has a solution in $\mathbb{Z}/n\mathbb{Z}$ whose components generate $\mathbb{Z}/n\mathbb{Z}$ as an additive group (or, equivalently, the first m components of the solution generate $\mathbb{Z}/n\mathbb{Z}$, because the equations say that the last s components are combinations of the first m ones). In other words, n is the largest order of cyclic quotient of the finitely generated group \mathbb{Z}^{m+s}/N , where N is the subgroup generated by rows of A . As noted already, the largest cyclic quotient n of \mathbb{Z}^{m+s}/N is $\frac{\Delta_{m+s}}{\Delta_{m+s-1}}$, as required.

Theorem 3. *Let R be an associative ring with unity and let G be a subgroup of the multiplicative group of this ring. Then, for each system of equations over R in m unknowns, the number of its solutions lying in G^m is divisible by the greatest common divisor of the homogeneity modulus of the system and the intersection of G with the centraliser of the set of coefficients of the system.*

Proof. Let G_0 be the intersection of G and the centraliser of the set of coefficients and let n be the homogeneity modulus. Consider the free group $F = F(X)$ (where X is the set of unknowns) and an epimorphism $\deg: F \rightarrow \mathbb{Z}/n\mathbb{Z}$.

Let us apply the main theorem taking Φ to be the set of all homomorphisms $\varphi: F \rightarrow G$ such that the tuple $(\varphi(x_1), \dots, \varphi(x_m))$ is a solution to the system of equations. Thus, the number of solutions is $|\Phi|$. Take H to be any subgroup of G_0 of order dividing n . Condition I of the main theorem obviously holds. To verify Condition II, choose an element $t \in F$ of degree one and write each variable x_i in form $x_i = t^{\deg x_i} y_i$, where $y_i = t^{-\deg x_i} x_i$ has degree zero. In new notation, each equation $w(x_1, \dots, x_m) = 0$ takes the form $v(t, y_1, \dots, y_m) = 0$ and the exponent sum of t in each term of this equation is the same (modulo n). Now, note that, if $v(\varphi(t), \varphi(y_1), \dots, \varphi(y_m)) = 0$ and $h \in H_\varphi$, then $v(\varphi(t)h, \varphi(y_1), \dots, \varphi(y_m)) = 0$. This follows from the (right) divisibility of $v(\varphi(t)h, \varphi(y_1), \dots, \varphi(y_m))$ by $v(\varphi(t), \varphi(y_1), \dots, \varphi(y_m))$ by virtue of the following fact.

Fact ([CM17], Lemma 1). *If M is a monoid, $b_i, a, h \in M$, elements a and h are invertible, and the elements $a^{-s}ha^s$, where $s \in \mathbb{Z}$, commute with all b_i , then, for any expression of the form $u(t) = b_0 t^{m_1} b_1 \dots t^{m_l} b_l$, where $m_i \in \mathbb{Z}$, we*

$$\text{have } u(ah) = \begin{cases} h^{a^{-1}} h^{a^{-2}} \dots h^{a^{-k}} u(a) & \text{if } k = \sum m_i > 0; \\ h^{-1} h^{-a} \dots h^{-a^{-1-k}} u(a) & \text{if } k = \sum m_i < 0; \\ u(a), & \text{if } k = \sum m_i = 0. \end{cases}$$

This fact is applied to each term of v . For nonzero n , we use also that t^n has degree zero and $(\varphi(t)h)^n = (\varphi(t))^n$ according to Lemma 0.

Thus, the main theorem implies that $|\Phi|$ (i.e. the number of solutions to the system of equations) is divisible by $|H|$ as required (because H is an arbitrary subgroup of G_0 whose order divides the homogeneity modulus).

Example. *If $\rho: G \rightarrow R^*$ is a homomorphism from a finite group G to the multiplicative group of an associative ring R with unity (e.g., $\rho: G \rightarrow \mathbf{GL}(V)$ is a linear representation of G), then, for any words $u_i(x_1, \dots, x_m) \in F(x_1, \dots, x_m)$,*

the number of solutions to the equation $\prod_{i=1}^n (\rho(u_i(x_1, \dots, x_m)))^{l_i} = 1$ is divisible by $\begin{cases} \text{GCD}(G, \text{GCD}(\{l_i\})) & \text{always;} \\ \text{GCD}(G, \text{LCM}(\{l_i\})) & \text{if } n \leq m; \\ |G| & \text{if } n < m. \end{cases}$

To show this, it suffices to apply Theorem 4 to the subgroup $\rho(G) \subseteq R^*$. The homogeneity matrix of this equation has the form $B = \begin{pmatrix} A & 1 \\ \vdots & \vdots \\ 0 & \dots 0 & 1 \end{pmatrix}$, where the last row corresponds to 1 in the right-hand side of the equation, and the

i th row of the matrix A corresponds to the i th term in the left-hand side of the equation and, therefore, all elements of this row are divisible by l_i . It remains to note that the k th invariant factor of the matrix B coincides with the $(k-1)$ th invariant factor of A and use the following fact, which we leave to readers as an easy exercise:

if the i th row of an integer matrix $n \times m$ is divisible by l_i ,

$$\text{then the } m\text{th invariant factor of this matrix} \begin{cases} \text{is divisible by } \text{GCD}(\{l_i\}) & \text{always;} \\ \text{is divisible by } \text{LCM}(\{l_i\}) & \text{for } n = m; \\ \text{vanishes} & \text{for } n < m. \end{cases}$$

Note that Theorem 1 can be obtained as a corollary of Theorem 3. Indeed, take $R = \mathbb{Z}G$; the group ring contains G as a subgroup of the multiplicative group. Any system of equations over G can be rewritten in “ring” form: $\{w_i(x_1, \dots) - 1 = 0\}$. It remains to note that the value $\frac{\Delta_m}{\Delta_{m-1}}$ from Theorem 1 become exactly the homogeneity modulus from the homogeneity-modulus lemma.

4. Crossed homomorphisms

Suppose that a group F acts (on the right) on a group B by automorphisms: $(f, b) \mapsto b^f$. Recall that a *crossed homomorphism* from F to B with respect to this action is a mapping $\alpha: F \rightarrow B$ such that $\alpha(ff') = \alpha(f)^{f'}\alpha(f')$ for all $f, f' \in F$. The following theorem was proved in [ACNT13] (using character theory) for finite groups F and B .

Theorem 4. *If a group F admitting an epimorphism onto $\mathbb{Z}/n\mathbb{Z}$ acts by automorphisms on a group B , then the number of crossed homomorphisms $F \rightarrow B$ is divisible by $\text{GCD}(B, n)$.*

Proof. The set of crossed homomorphisms is in one-to-one correspondence with the set Φ of (usual) homomorphisms from F to the semidirect product $G = F \ltimes B$ (with respect to the given action) such that their compositions with the projection $\pi: F \ltimes B \rightarrow F$ is the identity mapping $F \rightarrow F$. We have to show that $|\Phi|$ is a multiple of $|H|$ for any subgroup $H \subseteq B$ whose order divides n (by definition of $\text{GCD}(B, n)$).

The group F is n -indexed by the hypothesis of Theorem 4. Therefore, the assertion follows immediately from the main theorem. Conditions of the main theorem hold by trivial reasons: Condition I is fulfilled because $\pi(h^{-1}gh) = \pi(g)$; Condition II follows immediately from Lemma 1 because $\pi(gh) = \pi(g)$ (for $g \in G$ and $h \in H$).

5. Proof of the main theorem

Take an element $f_1 \in F$ of degree one, put $F_0 = \ker \deg \subset F$, and consider the semidirect product $\tilde{F} = \langle a \rangle_\infty \ltimes F_0$, where a acts on F_0 as f_1 does: $u^a = u^{f_1}$ for $u \in F_0$. The group \tilde{F} admits a natural indexing (0-indexing) $\deg: \tilde{F} \rightarrow \mathbb{Z}$ (denoted by the same symbol \deg). The kernel of this map is F_0 and $\deg a = 1$. Moreover, there is a natural epimorphism $\alpha: \tilde{F} \rightarrow F$ mapping a to f_1 and identity on F_0 . Let us verify that the conditions of the main theorem hold for the set $\tilde{\Phi} = \{\varphi \circ \alpha \mid \varphi \in \Phi\}$ of homomorphisms from \tilde{F} to G .

Condition I holds obviously. To verify Condition II, take the degree-one element $f = a \in \tilde{F}$ and some homomorphism $\tilde{\varphi} = \varphi \circ \alpha \in \tilde{\Phi}$ (where $\varphi \in \Phi$). Then the homomorphism $\tilde{\psi}$ from Condition II has the form

$$\tilde{\psi}(\tilde{f}) = \begin{cases} \varphi(\tilde{f}) & \text{for all elements } \tilde{f} \in F_0; \\ \varphi(f_1)h & \text{for } \tilde{f} = a; \end{cases} \quad \text{where } \varphi \in \Phi \text{ and } h \in H_{\tilde{\varphi}}. \quad (1)$$

We have to show that $\tilde{\psi}$ lies in $\tilde{\Phi}$, i.e. has the form $\tilde{\psi} = \varphi' \circ \alpha$, where $\varphi' \in \Phi$. Note that $H_{\tilde{\varphi}} = H_\varphi$, because the images of $\tilde{\varphi} = \varphi \circ \alpha$ and φ coincide, and the images of zero-degree elements for these homomorphisms coincide: $\tilde{\varphi}(\ker \deg) = \tilde{\varphi}(F_0) = \varphi(F_0)$. Formula (1) takes the form

$$\tilde{\psi}(\tilde{f}) = \begin{cases} \varphi(\tilde{f}) & \text{for } \tilde{f} \in F_0; \\ \varphi(f_1)h & \text{for } \tilde{f} = a; \end{cases} \quad \text{where } \varphi \in \Phi \text{ and } h \in H_\varphi.$$

This means that $\tilde{\psi} = \psi \circ \alpha$, where

$$\psi(f) = \begin{cases} \varphi(f) & \text{for } f \in F_0; \\ \varphi(f_1)h & \text{for } f = f_1; \end{cases} \quad \text{where } \varphi \in \Phi \text{ and } h \in H_\varphi.$$

The homomorphism $\psi: F \rightarrow G$ lies in Φ by Condition II of the theorem we are proving. Therefore, $\tilde{\psi} \in \tilde{\Phi}$. Thus, the conditions of the main theorem hold for the set $\tilde{\Phi}$ of homomorphisms from the 0-indexed group \tilde{F} to G . Therefore, $|\tilde{\Phi}|$ is divisible on $|H|$ by virtue of the main theorem of [KM17]. It remains to note that $|\Phi| = |\tilde{\Phi}|$ since α is surjective. This completes the proof.

6. Open questions

Theorems 1,2,3,4 assert that some numbers are multiples of the ratios of two integers. Oddly, we do not know *whether these ratios can be replaced by their numerators*?

Questions 1 and 2. *Is it possible to replace the ratio Δ_m/Δ_{m-1} by its numerator Δ_m in Theorems 1 and 2?*

For coefficient-free systems of equations, Question 1 is equivalent to the following question posed in [AsYo93] (for finite groups F and G):

is the number of homomorphisms from a finitely generated group F to a group G divisible by $\text{GCD}(|F/F'|, G)$?

This problem remains unsolved even for finite groups (as far as we know). A survey of some results can be found in [AsTa01]; e.g., the answer is positive if F is abelian [Yosh93].

Theorem 3 suggests a similar question.

Question 3. *Is it possible, in Theorem 3, to replace the homogeneity modulus by its numerator Δ_{n+s} (see the homogeneity-modulus lemma)?*

As for Theorem 4, it also leads us to a similar question. Indeed, Theorem 4 implies, in particular, that *if a finitely generated group F acts by automorphisms on a group B , then the number of crossed homomorphisms $F \rightarrow B$ is divisible by $\text{GCD}(\exp(F/F'), B)$.*

Question 4. *Is it possible, in the proposition above, to replace the period $\exp(F/F')$ by the order of this quotient group?*

This question was posed for the first time in [AsYo93] (for finite groups F and B). To show the similarity of Questions 4 and 1, we recall that the absolute value of the ratio Δ_m/Δ_{m-1} in Question 1 is the period of the quotient group of the free abelian group \mathbb{Z}^m by the subgroup generated by the rows of the matrix of the system of equations, while the absolute value of the numerator Δ_m is the order of this quotient group.

7. Proof of the Brauer lemma

We follow the original proof from [Bra69] but translate it into a more convenient (in our view) language.

Brauer Lemma [Bra69]. *If U is a finite normal subgroup of a group V , then, for all $v \in V$ and $u \in U$, the elements $v^{|U|}$ and $(vu)^{|U|}$ are conjugate by an element of U .*

Proof. The group \mathbb{Z} acts by permutations on the subgroup U :

$$a \circ i = v^{-i} a (vu)^i, \quad (\text{where } i \in \mathbb{Z} \text{ and } a \in U).$$

Let m be the minimum length of an orbit. In other words, m is the minimum length of a cycle in the decomposition of the permutation $a \mapsto v^{-1} a v u$ (of U) into the product of independent cycles. The set $X = \{a \in U \mid a \circ m = a\}$ is the union of all orbits of length m ; therefore, $|X|$ is divisible by m . On the other hand, (by definition of the action) $X = \{a \in U \mid v^{-m} a (vu)^m = a\} = \{a \in U \mid a^{-1} v^m a = (vu)^m\}$ and, hence, $|X|$ is the order of the centraliser of v^m in U (because, in any group, a nonempty set of the form $\{x \mid x^{-1} y x = z\}$ is a coset of the centraliser of y). Thus, $|X|$ divides $|U|$ and, therefore, m divides $|U|$ and $a \circ |U| = a$ (if a lies in an orbit of length m). This completes the proof.

REFERENCES

- [Vin99] E. B. Vinberg, A Course in Algebra (Issue 56 of Graduate studies in mathematics), American Mathematical Soc., 2003.
- [Stru95] S. P. Strunkov, On the theory of equations in finite groups, *Izvestiya: Math.*, 59:6 (1995), 1273-1282.
- [AmV11] A. Amit, U. Vishne, Characters and solutions to equations in finite groups, *J. Pure Appl. Algebra*, 10:4 (2011), 675-686.
- [And16] R. Andreev, A translation of “Verallgemeinerung des Sylow’schen Satzes” by F. G. Frobenius. arXiv:1608.08813.
- [ACNT13] T. Asai, N. Chigira, T. Niwasaki, Yu. Takegahara, On a theorem of P. Hall, *Journal of Group Theory*, 16:1 (2013), 69-80.
- [AsTa01] T. Asai, Yu. Takegahara, $|\text{Hom}(A, G)|$, IV, *J. Algebra*, 246 (2001), 543-563.
- [AsYo93] T. Asai, T. Yoshida, $|\text{Hom}(A, G)|$, II, *J. Algebra*, 160 (1993), 273-285.
- [Bra69] R. Brauer, On A Theorem of Frobenius, *The American Mathematical Monthly*, 76:1 (1969), 12-15.
- [BrTh88] K. Brown, J. Thévenaz, A generalization of Sylow’s third theorem, *J. Algebra*, 115 (1988), 414-430.
- [Frob95] F. G. Frobenius, Verallgemeinerung des Sylow’schen Satzes, *Sitzungsberichte der Königl. Preuß. Akad. der Wissenschaften (Berlin)* (1895), 981-993.

- [Frob03] F. G. Frobenius, Über einen Fundamentalsatz der Gruppentheorie, Sitzungsberichte der Königl. Preuß. Akad. der Wissenschaften (Berlin) (1903), 987-991.
- [Hall36] Ph. Hall, On a theorem of Frobenius, Proc. London Math. Soc. 40 (1936), 468-531.
- [GRV12] C. Gordon, F. Rodriguez-Villegas, On the divisibility of $\#\text{Hom}(\Gamma, G)$ by $|G|$, J. Algebra, 350:1 (2012), 300-307. See also arXiv:1105.6066.
- [Isaa70] I. M. Isaacs, Systems of equations and generalised characters in groups, Canad. J. Math., 22 (1970), 1040-1046.
- [Iwa82] S. Iwasaki, A note on the n th roots ratio of a subgroup of a finite group, J. Algebra, 78:2 (1982), 460-474.
- [KM14] A. A. Klyachko, A. A. Mkrtchyan, How many tuples of group elements have a given property? With an appendix by Dmitrii V. Trushin, Intern. J. of Algebra and Comp. 24:4 (2014), 413-428.
See also arXiv:1205.2824.
- [KM17] A. A. Klyachko, A. A. Mkrtchyan, Strange divisibility in groups and rings, Arch. Math. 108:5 (2017), 441-451.
See also arXiv:1506.08967.
- [Kula38] A. Kulakoff, Einige Bemerkungen zur Arbeit: "On a theorem of Frobenius" von P. Hall, Mat. Sb., 3(45):2 (1938), 403-405.
- [SaAs07] J. Sato, T. Asai, On the n -th roots of a double coset of a finite group, J. School Sci. Eng., Kinki Univ., 43 (2007), 1-4.
- [Sehg62] S. K. Sehgal, On P. Hall's generalisation of a theorem of Frobenius, Proc. Glasgow Math. Assoc., 5 (1962), 97-100.
- [Solo69] L. Solomon, The solutions of equations in groups, Arch. Math., 20:3 (1969), 241-247.
- [Yosh93] T. Yoshida, $|\text{Hom}(A, G)|$, Journal of Algebra, 156:1 (1993), 125-156.