Solvable Hypergroups and a Generalization of Hall's Theorems on Finite Solvable Groups to Association Schemes

Andrey Vasil'ev
Sobolev Institute of Mathematics
4 Acad. Koptyug avenue
Novosibirsk, 630090, Russia
and
Novosibirsk State University
1 Pirogova street
Novosibirsk, 630090, Russia

Paul-Hermann Zieschang
School of Mathematical and Statistical Sciences
University of Texas Rio Grande Valley
Edinburg, TX 78539, U. S. A.

Abstract

We generalize Philip Hall's celebrated theorems on finite solvable groups to scheme theory. Our result is based on a series of results on hypergroups.

1. Introduction

The concept of an association scheme provides a far-reaching and meaningful generalization of the concept of a group. A number of important results on finite groups have already found generalizations to association schemes. In [6], for instance, analogues of the homomorphism theorem, the isomorphism theorems, and the Jordan-Hölder Theorem for finite groups were proved for association schemes. In [4], a scheme theoretic generalization of Sylow's Theorems was proved, and in [1], a corresponding generalization of the Schur-Zassenhaus Theorem on finite groups was given.

It is the purpose of the present note to present a generalization of Philip Hall's celebrated theorem [2; Theorem] on finite solvable groups to scheme theory. We first introduce notation and terminology needed for the statement of our result. We begin with a review of the definition of an association scheme.

Let X be a finite set. We write 1_X to denote the set of all pairs (x, x) with $x \in X$. For each subset r of the cartesian product $X \times X$, we define r^* to be the set of all pairs (y, z) with $(z, y) \in r$. Whenever x is an element in X and r a subset of $X \times X$, we write xr to denote the set of all elements y in X with $(x, y) \in r$.

Let S be a partition of $X \times X$ with $1_X \in S$, and assume that $s^* \in S$ for each element s in S. The set S is called an *association scheme* or simply a *scheme on X* if, for any three elements p, q, and r in S, there exists an integer a_{pqr} such that $|yp \cap zq^*| = a_{pqr}$ for any two elements y in X and z in yr.[1]

Let S be an association scheme, and let P and Q be subsets of S. We define PQ to be the set of all elements s in S such that there exist elements p in P and q in Q with $a_{pqs} \neq 0$.

[1] We note that the definition of a scheme which we use in the present note differs from the more general definition given in [8]. In fact, the schemes which we consider in this note are exactly the schemes in the sense of [8] which are defined on finite sets.

The set PQ is called the *complex product* of P and Q, and the map which assigns to any two subsets of S its complex product will be called the *complex multiplication in* S. If P contains an element p with $P = \{p\}$, we write pQ instead of PQ. Similarly, if Q contains an element q with $Q = \{q\}$, we write Pq instead of PQ. Finally, if P contains an element p with $P = \{p\}$ and Q contains an element q with $Q = \{q\}$, we write pq instead of PQ.

A non-empty subset R of an association scheme is called *closed* if $p^*q \subseteq R$ for any two elements p and q in R.

Let S be a scheme on a set X. Instead of 1_X we write 1. For each element s in S, we abbreviate $n_s := a_{ss^*1}$, and we call n_s the *valency* of s. For each closed subset T of S, the sum of the positive integers n_t with $t \in T$ will be denoted by n_T, and we call n_T the *valency* of T.

A closed subset of an association scheme is called *thin* if all of its elements have valency 1.

There is an obvious way to identify groups with thin association schemes. This correspondence was established in [8; Section 5.5] and is called the *group correspondence*.

Let π be a set of prime numbers. The set of all prime numbers which do not belong to π will be denoted by π'. A positive integer n will be called a π-*number* if each prime divisor of n belongs to π.

Let S be an association scheme. An element of S is called π-*valenced* if its valency is a π-number. A closed subset of S is called π-*valenced* if each of its elements is π-valenced. A π-valenced closed subset T of S is called a closed π-*subset* if its valency is a π-number. A closed π-subset of S is called a *Hall* π-*subset* of S if its index in S is a π'-number; cf. [8; Section 4.4].

Let T and U be closed subsets of a scheme S. We say that T and U are *conjugate in* S if S contains an element s with $s^*Ts = U$. Assume that $T \subseteq U$. In [8; Lemma 2.3.6(ii)], it was shown that n_T divides n_U. The quotient n_U/n_T is called the *index* of T *in* U. The closed subset T is said to be *strongly normal* in U if $u^*Tu \subseteq T$ for each element u in U.

An association scheme S will be called *solvable* if it contains closed subsets T_0, …, T_n such that $T_0 = \{1\}$, $T_n = S$, and, for each element i in $\{1, \ldots, n\}$, T_{i-1} is strongly normal in T_i and has prime index in T_i.

It is easy to see that, via the group correspondence, a thin association scheme corresponds to a solvable group if and only if it is solvable, so that the notion of a solvable association scheme which we suggest here naturally generalizes the notion of a finite solvable group.

Now we are ready to state the main result of this note.

Theorem

Let π be a set of prime numbers, and let S be a solvable and π-valenced association scheme. Then we have the following.

(i) *The scheme S possesses at least one Hall π-subset.*

(ii) *Any two Hall π-subsets of S are conjugate in S.*

(iii) *Any closed π-subset of S is contained in a Hall π-subset of S.*

Via the group correspondence, the restriction of our theorem to the class of all thin association schemes results precisely in Hall's theorem [2; Theorem] on finite solvable groups, since thin association schemes are π-valenced for any set π of prime numbers.

It is perhaps not without interest to point out that both conditions under which we prove our theorem are needed, the solvability of S as well as the hypothesis that S is π-valenced. As for the necessity of the solvability we refer of course to Hall's note [2]. The necessity of the second hypothesis is shown by the association scheme $HM_{176}(28)$ in the list [3] of Akihide Hanaki and Izumi Miyamoto. This scheme is solvable and $\{2\}$-valenced, and it has valency 28. It has a closed subset of valency 4, but no closed subset of valency 7.

The key to the proof of our main result is Theorem 8.5. In this theorem, we show that, given a set π of prime numbers, each solvable π-valenced association scheme S contains a strongly normal closed π-subset U. Since U is strongly normal in S, we obtain from [8; Lemma 4.2.5(ii)] that the quotient scheme $S/\!/U$ is thin, and our main result is then obtained by an application of Hall's above-mentioned theorem to the group which, via the group correspondence, corresponds to $S/\!/U$.

The proof of Theorem 8.5 is quite involved and relies on a series of results on hypergroups. We define hypergroups in the beginning of Section 2. The reason why our hypergroup theoretic efforts (in Section 2 to Section 7) lead to a proof of Theorem 8.5 is the observation that each association scheme S is a hypergroup with respect to the hypermultiplication which assigns any pair (p, q) of elements of S to its complex product pq; cf. [8; Lemma 1.3.1], [8; Lemma 1.3.3(ii)], and [8; Lemma 1.3.3(i)]. The set S endowed with the above hypermultiplication will be called the *hypergroup defined by the complex multiplication in S*.

2. The definition of a hypergroup and some preliminaries on hypergroups

We define a *hypermultiplication* on a set S to be a map from $S \times S$ to the power set of S. The image of a pair (p, q) of elements of a set S under a hypermultiplication on S will be denoted by pq, and it will be called the *hyperproduct* or simply the *product* of p and q.

Let S be a set endowed with a hypermultiplication, and let P and Q be subsets of S. We write PQ for the union of the products pq with $p \in P$ and $q \in Q$, and we refer to PQ as the *hyperproduct* or simply as the *product* of P and Q. The map which takes any pair (P, Q) of subsets of S to the product PQ will be called the *extension* of the hypermultiplication on S.

If P contains an element p with $P = \{p\}$, we write pQ instead of PQ. Similarly, if Q contains an element q with $Q = \{q\}$, we write Pq instead of PQ. Note that $pq = PQ$ if P contains an element p with $P = \{p\}$ and Q contains an element q with $Q = \{q\}$.

Following (and slightly generalizing) Frédéric Marty's terminology in [5] we call a set S endowed with a hypermultiplication a *hypergroup* if it satisfies the following conditions.

- H1 For any three elements p, q, and r in S, we have $p(qr) = (pq)r$.
- H2 The set S contains an element e such that $se = \{s\}$ for each element s in S.
- H3 For each element s in S, there exists an element s^* in S such that $q \in p^*r$ and $p \in rq^*$ for any three elements p, q, and r in S satisfying $r \in pq$.

In Lemma 2.3, we will see that in a hypergroup H, the set ab is not empty for any two elements a and b in H.

An element e of a hypergroup H which satisfies $he = \{h\}$ for each element h in H is called a *neutral element* of H. It is easy to see and was shown in [9; Lemma 2.4] that hypergroups cannot have two distinct neutral elements. This allows us to speak about *the* neutral element of a hypergroup. Within this article, the neutral element of a hypergroup will always be denoted by 1.

A map * from a hypergroup H to itself which satisfies $b \in a^*c$ and $a \in cb^*$ for any three elements a, b, and c in H with $c \in ab$ is called an *inverse function* of H. It is easy to see and was shown in [9; Lemma 2.7] that hypergroups cannot have two different inverse functions. This allows us to speak about *the* inverse function of a hypergroup H. Within this article, the inverse of a hypergroup element h will always be denoted by h^*.

Lemma 2.1

Let H be a hypergroup, and let a and b be elements of H. Then $1 \in a^*b$ if and only if $a = b$.

Proof. Assume first that $1 \in a^*b$. Then, by Condition H3, $b \in a\cdot 1$. On the other hand, by Condition H2, $a\cdot 1 = \{a\}$. Thus, $a = b$.

Assume that $a = b$. Then, by Condition H2, $b \in a\cdot 1$. Thus, by Condition H3, $1 \in a^*b$. □

Lemma 2.2

Let H be a hypergroup, and let h be an element in H. Then $h^{**} = h$.

Proof. From Lemma 2.1 we know that $1 \in h^*h$. It follows that $h \in h^{**}\cdot 1$; cf. Condition H3. Now recall that, by Condition H2, $h^{**}\cdot 1 = \{h^{**}\}$. Thus, $h \in \{h^{**}\}$, and that implies that $h^{**} = h$. □

The following lemma was communicated to the second author by Chris French.

Lemma 2.3

For any two elements a and b in H, the set ab is not empty.

Proof. Let a and b be elements in H. From Condition H2, Lemma 2.1, and Condition H1 we obtain that

$$a \in a\cdot 1 \subseteq a(b^{**}b^*) = a(bb^*) = (ab)b^*.$$

This shows that ab is not empty. □

Lemma 2.4

Let H be a hypergroup, and let a, b, and c be elements of H. Then the statements $c \in ab$, $b \in a^*c$, $a^* \in bc^*$, $c^* \in b^*a^*$, $b^* \in c^*a$, and $a \in cb^*$ are pairwise equivalent.

Proof. From $c \in ab$ one obtains that $b \in a^*c$; cf. Condition H3. Similarly, one obtains $a^* \in bc^*$ from $b \in a^*c$, $c^* \in b^*a^*$ from $a^* \in bc^*$, $b^* \in c^*a$ from $c^* \in b^*a^*$, $a \in cb^*$ from $b^* \in c^*a$, and $c \in ab$ from $a \in cb^*$. □

Recall from Lemma 2.1 that $1 \in h^*h$ for each hypergroup element h. A hypergroup element h is called *thin* if $h^*h = \{1\}$, and subsets of hypergroups will be called *thin* if all of their elements are thin.

Lemma 2.5

Let H be a hypergroup, and let a and b be elements in H. Assume that b is thin. Then $|ab| = 1$.

Proof. Let c be an element in ab. Then $a \in cb^*$. Thus, as b is thin, $ab \subseteq cb^*b = \{c\}$. □

For each subset A of a hypergroup, we set $A^* := \{a^* \mid a \in A\}$, and we notice that, by Lemma 2.2, $A^{**} = A$ for each subset A of a hypergroup.

Lemma 2.6

Let H be a hypergroup, and let A and B be subsets of H. Then the following hold.

(i) *Assume that $A \subseteq B$. Then $A^* \subseteq B^*$.*

(ii) *We have $(AB)^* = B^*A^*$.*

Proof. (i) Let h be an element in A^*. Then A contains an element a with $h = a^*$. Since $a \in A$, $a \in B$. Thus, $h = a^* \in B^*$.

(ii) Let h be an element in $(AB)^*$. Then AB contains an element c with $h = c^*$. Since $c \in AB$, there exist elements a in A and b in B such that $c \in ab$. Thus, by Lemma 2.4, $c^* \in b^*a^* \subseteq B^*A^*$. Since $h = c^*$ this implies that $h \in B^*A^*$. □

3. Closed subsets of hypergroups

A non-empty subset A of a hypergroup is called *closed* if $A^*A \subseteq A$.

Note that a closed subset F of a hypergroup H is a hypergroup with respect to the hypermultiplication which one obtains from the hypermultiplication of H if one restricts the domain of the hypermultiplication of H to $F \times F$ and the codomain of the hypermultiplication of H to the power set of F.

Lemma 3.1

A subset A of a hypergroup is closed if and only if $1 \in A$, $A^* = A$, and $AA = A$.

Proof. Assume first that A is closed. Then, by definition, A is not empty. Let a be an element in A. Then, by definition, $a^*a \subseteq A$. On the other hand, by Lemma 2.1, $1 \in a^*a$. Thus, $1 \in A$.

From $1 \in A$ we obtain that $a^* \in a^* \cdot 1 \subseteq A^*A \subseteq A$ for each element a in A. This shows that $A^* \subseteq A$. From this it follows that $A = A^{**} \subseteq A^*$; cf. Lemma 2.6(i). It follows that $A^* = A$.

From $1 \in A$ we obtain that $A = A \cdot 1 \subseteq AA$. From $A^* = A$ we obtain that $AA = A^*A \subseteq A$. Thus, $AA = A$.

Assume now, conversely, that $1 \in A$, $A^* = A$, and $AA = A$. From $1 \in A$ we obtain that A

is not empty. From $A^* \subseteq A$ and $AA \subseteq A$ we obtain that $A^*A \subseteq A$. $\square$

Lemma 3.2

Let H be a hypergroup, and let $\mathcal{F}$ be a non-empty set of closed subsets of H. Then the intersection of the closed subsets which belong to $\mathcal{F}$ is a closed subset of H.

Proof. Let A denote the intersection of the closed subsets of H which belong to $\mathcal{F}$. From Lemma 3.1 we know that $1 \in F$ for each element F in $\mathcal{F}$. Thus, $1 \in A$, and that shows that A is not empty.

Let F be an element in $\mathcal{F}$. Then $A \subseteq F$. Thus, by Lemma 2.6(i), $A^* \subseteq F^*$. Thus, as $A \subseteq F$, we obtain that $A^*A \subseteq F^*F \subseteq F$, and since F has been chosen arbitrarily from $\mathcal{F}$, this proves that $A^*A \subseteq A$. $\square$

Lemma 3.3

Let H be a hypergroup, let F be a closed subset of H, and let A and B be subsets of H. Then we have the following.

(i) *If $A \subseteq F$, $A(B \cap F) = AB \cap F$.*

(ii) *If $B \subseteq F$, $(A \cap F)B = AB \cap F$.*

Proof. (i) Assume that $A \subseteq F$. Then, by Lemma 3.1, $A(B \cap F) \subseteq F$. Thus, as $A(B \cap F) \subseteq AB$, $A(B \cap F) \subseteq AB \cap F$.

To show the reverse containment, let f be an element in $AB \cap F$. Since $f \in AB$, there exist elements a in A and b in B such that $f \in ab$. From $f \in ab$ we obtain that $b^* \in f^*a$; cf. Lemma 2.4.

Since F is a closed subset of H, we obtain from $f \in F$ and $a \in A \subseteq F$ that $f^*a \subseteq F$. Thus, as $b^* \in f^*a$, $b^* \in F$. Since F is closed, this implies that $b \in F$. Thus, as $b \in B$, $b \in B \cap F$. Now, as $f \in ab$, $f \in A(B \cap F)$.

(ii) Assume that $B \subseteq F$. Then $B^* \subseteq F$. Thus, by (i), $B^*(A^* \cap F) = B^*A^* \cap F$. Thus, as $F^* = F$, the claim follows from Lemma 2.6(ii). $\square$

Lemma 3.4

Let H be a hypergroup, and let D and E be closed subsets of H. Then the following hold.

(i) *For any two elements a and b in H with $a \in DbE$, we have $DbE \subseteq DaE$.*

(ii) *The set $\{DhE \mid h \in H\}$ is a partition of H.*

Proof. (i) Let a and b be elements in H, and assume that $a \in DbE$. Then there exist elements e in D and f in E such that $a \in ebf$. From $a \in ebf$ we obtain an element c in eb such that $a \in cf$. From $c \in eb$ we obtain that $b \in e^*c$, from $a \in cf$ we obtain that $c \in af^*$. Thus, $b \in e^*af^* \subseteq DaE$. Since D and E are assumed to be closed, this implies that $DbE \subseteq DaE$; cf. Lemma 3.1.

(ii) Since $1 \in D$ and $1 \in E$, we have $h \in DhE$ for each element h in H. Thus, H is equal to the union of the sets DhE with $h \in H$.

To show that $\{DhE \mid h \in H\}$ is a partition of H we now choose elements a and b in H, and we assume that $DaE \cap DbE$ is not empty. We have to show that $DaE = DbE$.

Since $DaE \cap DbE$ is not empty, we find an element c in $DaE \cap DbE$. Since $c \in DaE$, $DcE \subseteq DaE$. From $c \in DaE$ we also obtain that $DaE \subseteq DcE$; cf. (i). Thus, $DaE = DcE$. Similarly, $DbE = DcE$, so that $DaE = DbE$. □

Lemma 3.5

Let H be a hypergroup, and let D and E be closed subsets of H. Then DE is a closed subset of H if and only if $DE = ED$.

Proof. Assume first that DE is closed. Then, by Lemma 3.1, $(DE)^* = DE$. On the other hand, as D and E are closed, we also have $D^* = D$ and $E^* = E$. Thus, by Lemma 2.6(ii), $(DE)^* = ED$. From $(DE)^* = DE$ and $(DE)^* = ED$ we obtain that $DE = ED$.

Assume, conversely, that $DE = ED$. Then referring to Lemma 2.6(ii) we obtain that

$$(DE)^* DE = E^* D^* DE \subseteq E^* DE = E^* ED \subseteq ED = DE.$$

Thus, as DE is not empty, DE is closed. □

4. Normality

Let H be a hypergroup, and let D and E be closed subsets of H. We say that D *normalizes* E if $Ed \subseteq dE$ for each element d in D. If H normalizes a closed subset F of H, we say that F is *normal in H*.

Lemma 4.1

Let H be a hypergroup, and let D and E be closed subsets of H. Assume that D normalizes E. Then the following hold.

(i) *For each element d in D, we have $Ed = dE$.*

(ii) *The product ED is a closed subset of H.*

(iii) *The closed subset E of H is normal in ED.*

(iv) *The closed subset $E \cap D$ is normal in D.*

Proof. (i) Let d be an element in D. Then, $d^* \in D$. Thus, as D is assumed to normalize E, $Ed \subseteq dE$ and $Ed^* \subseteq d^*E$. Since E is closed, $Ed^* \subseteq d^*E$ implies that $dE \subseteq Ed$; cf. Lemma 2.6.

(ii) From (i) we obtain that $ED = DE$. Thus, by Lemma 3.5, ED is closed.

(iii) Let h be an element in ED. Then D contains an element d with $h \in Ed$. Since D normalizes E, we also have $Ed \subseteq dE$. Thus, $h \in dE$. Now, by Lemma 3.4(ii), $hE = dE$. Thus,

$$Eh \subseteq Ed \subseteq dE = hE,$$

and that means that h normalizes E. Since h has been chosen arbitrarily in ED, we have seen that ED normalizes E.

(iv) We are assuming that D normalizes E. Thus, by definition, $Ed \subseteq dE$ for each element d in D. It follows that

$$(E \cap D)d = Ed \cap D \subseteq dE \cap D = d(E \cap D)$$

for each element d in D; cf. Lemma 3.3. □

A closed subset E of a hypergroup H is said to be *subnormal in H* if H contains closed subsets F_0, ..., F_n such that $F_0 = E$, $F_n = H$, and, for each element i in $\{1, \ldots, n\}$, F_{i-1} is normal in F_i.

Lemma 4.2

Let H be a hypergroup, and let D and E be closed subsets of H. Assume that D is subnormal in H and that E is normal in H. Then ED is a closed subset of H and is subnormal in H.

Proof. That ED is a closed subset of H follows from Lemma 4.1(ii).

We are assuming that D is subnormal in H. Thus, there exist closed subsets F_0, ..., F_n of H such that $F_0 = D$, $F_n = H$ and, for each element i in $\{1, \ldots, n\}$, F_{i-1} is normal in F_i. We will see that, for each element i in $\{1, \ldots, n\}$, EF_{i-1} is normal in EF_i.

Let h be an element in EF_i. Then F_i contains an element f such that $h \in Ef$. It follows that $Eh \subseteq Ef$. From $h \in Ef$ and $Ef \subseteq fE$ we also obtain that $h \in fE$, so that, by Lemma 3.4(ii), $hE = fE$. It follows that

$$EF_{i-1}h = F_{i-1}Eh \subseteq F_{i-1}Ef \subseteq F_{i-1}fE \subseteq fF_{i-1}E = fEF_{i-1} = hEF_{i-1}.$$

Since h has been chosen arbitrarily from EF_i, this shows that EF_{i-1} is normal in EF_i. □

A closed subset F of a hypergroup H is called *strongly normal in H* if $h^*Fh \subseteq F$ for each element h in H.

Lemma 4.3

Let H be a hypergroup, and let F be a closed subset of H. Assume that F is strongly normal in H. Then F is normal in H.

Proof. Since F is assumed to be strongly normal in H, we have $h^*Fh \subseteq F$ for each element h in H. It follows that $Fh \subseteq hh^*Fh \subseteq hF$ for each element h in H, and that means that F is normal in H. □

For each hypergroup H, we denote by $O^\vartheta(H)$ the intersection of all closed subsets of H which are strongly normal in H.

Lemma 4.4

Let H be a hypergroup. Then the following hold.

(i) *The set $O^\vartheta(H)$ is a closed subset of H and strongly normal in H.*

(ii) *For each element h in H, we have $h^*h \subseteq O^\vartheta(H)$.*

Proof. (i) From Lemma 3.2 we know that $O^\vartheta(H)$ is a closed subset of H.

Let $\mathcal{F}$ denote the set of all closed subsets of H which are strongly normal in H, and let F be an element in $\mathcal{F}$. Then we have

$$h^*O^\vartheta(H)h \subseteq h^*Fh \subseteq F$$

for each element h in H. Thus, $h^*O^\vartheta(H)h \subseteq O^\vartheta(H)$ for each element h in H, and that means that $O^\vartheta(H)$ is strongly normal in H.

(ii) For each element h in H, we have $h^*h \subseteq h^*O^\vartheta(H)h \subseteq O^\vartheta(H)$; cf. (i). □

The set of all thin elements of a hypergroup H will be denoted by $O_\vartheta(H)$, and a hypergroup H will be called *metathin* if $O^\vartheta(H) \subseteq O_\vartheta(H)$.

Lemma 4.5

Let H be a metathin hypergroup, and let h be an element in H. Then the following hold.

(i) *We have $\{h\} = hh^*h$.*

(ii) *The set h^*h is a closed subset of H.*

(iii) *The set h^*h is thin.*

(iv) *The set h^*h is normal in $O^\vartheta(H)$.*

Proof. (i) Let a be an element in hh^*h. Then h^*h contains an element b such that $a \in hb$. Since $b \in h^*h$, $b \in O^\vartheta(H)$; cf. Lemma 4.4(ii). Thus, as H is assumed to be metathin, b is thin, and that means that $b^*b = \{1\}$.

From $a \in hb$ we obtain that $h \in ab^*$, from $b \in h^*h$ we obtain that $h^* \in bh^*$. Thus, as $b^*b = \{1\}$, we obtain from Lemma 2.1 that $1 \in hh^* \subseteq ab^*bh^* = ah^*$. Thus, by Lemma 2.1, $a^* = h^*$, whence $a = h$.

Since a has been chosen arbitrarily from hh^*h, we have shown that $hh^*h \subseteq \{h\}$. On the other hand, hh^*h is not empty. Thus, $\{h\} = hh^*h$.

(ii) From Lemma 2.6(ii) we know that $(h^*h)^* = h^*h$. Thus, by (i), $(h^*h)^*h^*h = h^*hh^*h = h^*h$. Thus, as h^*h is not empty, h^*h is a closed subset of H.

(iii) We have $h^*h \subseteq O^\vartheta(H) \subseteq O_\vartheta(H)$. Thus, h^*h is thin.

(iv) In order to show that h^*h is normal in $O^\vartheta(H)$, we choose an element a in $O^\vartheta(H)$. We have to show that $a^*h^*ha \subseteq h^*h$.

Since H is assumed to be metathin, we have $O^\vartheta(H) \subseteq O_\vartheta(H)$. Thus, as $a \in O^\vartheta(H)$, a is thin. Thus, by Lemma 2.5, $|ha| = 1$. Now recall from Lemma 4.4(i) that $O^\vartheta(H)$ is strongly normal in H. Thus, $O^\vartheta(H)$ contains an element b such that $ha \subseteq bh$. Since $b \in O^\vartheta(H)$ and $O^\vartheta(H) \subseteq O_\vartheta(H)$, b is thin, so $b^*b = \{1\}$. It follows that $a^*h^*ha \subseteq h^*b^*bh = h^*h$; cf. Lemma 2.6. □

5. Quotients of hypergroups

Let H be a hypergroup, and let F be a closed subset of H.

For each element h of H, we set

$$h^F := FhF,$$

and we define $H//F$ to be the set of all hyperproducts h^F with $h \in H$. In [7; Lemma 2.2], it was shown that we obtain a hypermultiplication on $H//F$ if we define

$$a^F b^F := \{h^F \mid h \in aFb\}$$

for any two elements a and b in H. We call this hypermultiplication on $H//F$ the *hypermultiplication* on $H//F$ *defined by* F.

We notice that the hypermultiplication on $H//F$ defined by F carries the danger of ambiguity. In fact, the hypermultiplication on $H//F$ defined by F is defined on the same set as the restriction to $H//F \times H//F$ of the extension of the hypermultiplication of H. However, while the codomain of the restriction to $H//F \times H//F$ of the extension of the hypermultiplication of H is the power set of $H//F$, the codomain of the hypermultiplication on $H//F$ defined by F is the power set of H. To say it differently, for any two elements a and b in H, $(FaF)(FbF)$ stands for the set union of the products FhF with $h \in aFb$, whereas $a^F b^F$ is defined to be the of all products h^F with $h \in aFb$, although $FaF = a^F$ and $FbF = b^F$. In the following, we will take care that no misunderstanding will arise in this regard.

In [7; Lemma 2.3], it was shown that $H//F$ is a hypergroup with respect to the hypermultiplication on $H//F$ defined by F (and with inverse function $h^F \mapsto (h^*)^F$). We refer to $H//F$ as the *quotient* of H *over* F.

Lemma 5.1

Let H be a hypergroup, let F be a closed subset of H, let h be an element in H, and let $A_1, \ldots, A_n$ be subsets of H. Then we have $h^F \in (A_1//F) \cdots (A_n//F)$ if and only if $h \in (FA_1F) \cdots (FA_nF)$.

Proof. We first prove the statement for $n = 1$. By definition, we have $h^F \in A_1//F$ if and only if there exists an element a in A_1 such that $h^F = a^F$. By definition, we also have $h^F = a^F$ if and only if $FhF = FaF$. Thus, $h^F \in A_1//F$ if and only if $h \in FA_1F$; cf. Lemma 3.4(ii).

Now we assume that $2 \leq n$, and we first suppose that $h^F \in (A_1//F) \cdots (A_n//F)$. Then, by definition, there exist elements b and c in H such that $b^F \in (A_1//F) \cdots (A_{n-1}//F)$, $c^F \in A_n//F$, and $h^F \in b^F c^F$. From $h^F \in b^F c^F$ we obtain an element d in bFc such that $h^F = d^F$. It follows that $h \in FdF \subseteq FbFcF$.

Since $b^F \in (A_1//F) \cdots (A_{n-1}//F)$, induction yields $b \in (FA_1F) \cdots (FA_{n-1}F)$. In the first paragraph of this proof, we also saw that $c^F \in A_n//F$ implies that $c \in FA_nF$. Thus, we have

$$FbFcF \subseteq (FA_1F) \cdots (FA_nF).$$

Since $h \in FbFcF$, this implies that $h \in (FA_1F) \cdots (FA_nF)$.

Suppose, conversely, that $h \in (FA_1F) \cdots (FA_nF)$. Then, by definition, there exist elements b in $(FA_1F) \cdots (FA_{n-1}F)$ and c in FA_nF such that $h \in bc$. From $h \in bc$ (together with $bc \subseteq bFc$) we obtain that $h^F \in b^F c^F$.

Since $b \in (FA_1F) \cdots (FA_{n-1}F)$, induction yields $b^F \in (A_1//F) \cdots (A_{n-1}//F)$. In the first part of this proof, we also saw that $c \in FA_nF$ implies that $c^F \in A_n//F$. Thus, we have

$$b^F c^F \subseteq (A_1//F) \cdots (A_n//F).$$

Since $h^F \in b^F c^F$, this implies that $h^F \in (A_1//F)\cdots(A_n//F)$. □

Theorem 5.2

Let H be a hypergroup, and let D be a closed subset of H. Then $E \mapsto E//D$ is a bijective map from the set of all closed subsets of H containing D to the set of all closed subsets of $H//D$.

Proof. Let E be a closed subset of H, and assume that $D \subseteq E$. We first show that $E//D$ is a closed subset of $H//D$.

Since E is closed, we have $1 \in E$; cf. Lemma 3.1. Thus, by definition, $1^D \in E//D$.

Since E is closed, we also have $e^* \in E$ for each each element e in E; cf. Lemma 3.1. Thus, by definition, $(e^*)^D \in E//D$. Since $(e^D)^* = (e^*)^D$, this implies that $(e^D)^* \in E//D$.

Let a and b be elements in E, and let h be an element in H with $h^D \in a^D b^D$. From $h^D \in a^D b^D$ we obtain that $h \in DaDbD$; cf. Lemma 5.1. Thus, as $DaDbD \subseteq E$, $h \in E$, so that $h^D \in E//D$. Since h has been chosen arbitrarily from H with $h^D \in a^D b^D$, this shows that $a^D b^D \subseteq E//D$.

What we have seen so far is that $E//D$ is closed; cf. Lemma 3.1.

Since E has been chosen arbitrarily among the closed subset of H containing D, this shows that $E \mapsto E//D$ is a map from the set of all closed subsets of H containing D to the set of all closed subsets of $H//D$.

Let B and C be closed subsets of H with $D \subseteq B$, $D \subseteq C$, and $B//D = C//D$. Let b be an element in B. Then $b^D \in C//D$. Thus, C contains an element c with $b^D = c^D$. It follows that $b \in DcD \subseteq C$. We, thus, have shown that $B \subseteq C$. Similarly, one shows that $C \subseteq B$, so that we have $B = C$.

Let A be a subset of H, and assume that $A//D$ is a closed subset of $H//D$. From Lemma 5.1 one obtains that $A//D = DAD//D$. Thus, we shall be done if we succeed in showing that DAD is a closed subset of H.

Let h be an element in DAD. Then, by Lemma 5.1, $h^D \in A//D$. Thus, as $A//D$ is assumed to be closed, $(h^D)^* \in A//D$. Thus, as $(h^D)^* = (h^*)^D$, $(h^*)^D \in A//D$. Now, by definition, $(h^*)^D = a^D$ for some element a in A. It follows that $h^* \in DaD \subseteq DAD$.

Let b and c be elements in DAD, and let h be an element in bc. Then $h \in (DAD)(DAD)$. Thus, by Lemma 5.1, $h^D \in (A//D)(A//D)$. Since $A//D$ is assumed to be closed, this implies that $h^D \in A//D$. Thus, A contains an element a with $h^D = a^D$. It follows that $h \in DaD \subseteq DAD$. □

Lemma 5.3

Let H be a hypergroup, let D and E be closed subsets of H, and assume that $D \subseteq E$. Then we have the following.

(i) *Assume that E is normal in H. Then $E//D$ is a normal closed subset of $H//D$.*

(ii) *The closed subset E of H is strongly normal in H if and only if $E//D$ is strongly normal in $H//D$.*

Proof. (i) Let h be an element in H. Then, as E is assumed to be normal in H, we have $Eh \subseteq hE$. It follows that $EhD \subseteq hED = hE \subseteq DhE$. Thus, by Lemma 5.1, $(E//D)h^D \subseteq h^D(E//D)$.

(ii) Let h be an element in H. We first assume that $h^*Eh \subseteq E$ and will show that $(h^D)^*(E//D)h^D \subseteq E//D$.

Let a be an element in H with $a^D \in (h^D)^*(E//D)h^D$. Since $(h^D)^* = (h^*)^D$, we then have $a^D \in (h^*)^D(E//D)h^D$, so that, by Lemma 5.1, $a \in Dh^*EhD$. Thus, as $h^*Eh \subseteq E$ and $D \subseteq E$, we conclude that $a \in E$. It follows that $a^D \in E//D$.

Now we assume that $(h^D)^*(E//D)h^D \subseteq E//D$ and will show that $h^*Eh \subseteq E$.

Let a be an element in h^*Eh. From $a \in h^*Eh$ (together with $(h^*)^D = (h^D)^*$) we obtain that $a^D \in (h^D)^*(E//D)h^D$; cf. Lemma 5.1. Thus, as $(h^D)^*(E//D)h^D \subseteq E//D$, we have $a^D \in E//D$. Since $D \subseteq E$, this implies that $a \in E$; cf. Lemma 5.1. $\square$

Lemma 5.4

Let H be a hypergroup, and let F be a closed subset of H. Then F is strongly normal in H if and only if $H//F$ is thin.

Proof. Let h be an element in H. Since F is closed, we have $h^*Fh \subseteq F$ if and only if $(Fh^*F)(FhF) \subseteq F$. From Lemma 5.1 we also know that $(Fh^*F)(FhF) \subseteq F$ if and only if $(h^*)^F h^F \subseteq \{1^F\}$. Thus, as $(h^*)^F = (h^F)^*$, we also have that $(h^*)^F h^F \subseteq \{1^F\}$ if and only if h^F is thin. $\square$

6. Homomorphisms of hypergroups

Let H and H' be hypergroups.

As it is customary, we set $\phi(A) := \{\phi(a) \mid a \in A\}$ whenever ϕ is a map from H to H' and A a subset of H.

A map ϕ from H to H' is called a *homomorphism* if

$$\phi(ab) = \phi(a)\phi(b)$$

for any two elements a and b in H and $\phi(1) = 1$.

Lemma 6.1

Let H and H' be hypergroups, let ϕ be a homomorphism from H to H', and let h be an element in H. Then $\phi(h^*) = \phi(h)^*$.

Proof. By definition, $\phi(1) = 1$. On the other hand, by Lemma 2.1, $1 \in h^*h$. Thus,

$$1 = \phi(1) \in \phi(h^*h) = \phi(h^*)\phi(h),$$

so that, by Lemma 2.1, $\phi(h^*) = \phi(h)^*$. $\square$

Let H and H' be hypergroups, and let ϕ be a homomorphism from H to H'. We define

$$\ker(\phi) := \{h \in H \mid \phi(h) = 1\}$$

and call this set the *kernel* of ϕ.

Lemma 6.2

Let H and H' be hypergroups, let ϕ be a homomorphism from H to H', and set $F := \ker(\phi)$. Let a and b be elements in F. Then we have $\phi(a) = \phi(b)$ if and only if $aF = bF$.

Proof. From Lemma 6.1 we obtain that

$$\phi(a)^*\phi(b) = \phi(a^*)\phi(b) = \phi(a^*b).$$

Thus, by Lemma 2.1, $\phi(a) = \phi(b)$ if and only if $1 \in \phi(a^*b)$.

On the other hand, we have $1 \in \phi(a^*b)$ if and only if a^*b contains an element f with $\phi(f) = 1$, and this means that $f \in F$.

To conclude the proof we notice that $f \in a^*b$ if and only if $b \in af$; cf. Lemma 2.2. Thus, we have $1 \in \phi(a^*b)$ if and only if $aF = bF$; cf. Lemma 3.4(ii). □

Bijective hypergroup homomorphisms are called *isomorphisms*.

If there exists an isomorphism from H to H' (or from H' to H), we say that H and H' are *isomorphic*, and we indicate this by writing $H \cong H'$.

Let ϕ be a homomorphism from H to H'. As is customary, we define

$$\mathrm{im}(\phi) := \phi(H)$$

and call this set the *image* of ϕ.

The following two theorems are [7; Theorem 3.3] and [7; Theorem 3.4(ii)]. For the sake of completeness of this article, we include a proof.

Theorem 6.3

Let H and H' be hypergroups, and let ϕ be a homomorphism from H to H'. Then $\ker(\phi)$ is a normal closed subset of H, $\mathrm{im}(\phi)$ is a closed subset of H', and $H/\!/\ker(\phi) \cong \mathrm{im}(\phi)$.

Proof. By definition, $\phi(1) = 1$. Thus, $1 \in \ker(\phi)$, so $\ker(\phi)$ is not empty.

Let a and b be elements in $\ker(\phi)$. Then $\phi(a) = 1$ and $\phi(b) = 1$. Thus, referring to Lemma 6.1 we obtain that

$$\phi(a^*b) = \phi(a^*)\phi(b) = \phi(a)^*\phi(b) = 1^*1 = \{1\},$$

and that means that $a^*b \subseteq \ker(\phi)$.

So far, we have seen that $\ker(\phi)$ is a closed subset of H. In order to show that $\ker(\phi)$ is normal in H, we set $F := \ker(\phi)$.

Let f be an element in F, and let h be an element in H. We have to show that $fh \subseteq hF$.

Let a be an element in fh. Then

$$\phi(a) \in \phi(fh) = \phi(f)\phi(h) = 1 \cdot \phi(h) = \{\phi(h)\}.$$

It follows that $\phi(a) = \phi(h)$, so that, by Lemma 6.2, $a \in hF$. We, thus, have seen that $\ker(\phi)$ is a normal closed subset of H.

Since $1 = \phi(1) \in \mathrm{im}(\phi)$, $\mathrm{im}(\phi)$ is not empty.

Now let c and d be elements in $\mathrm{im}(\phi)$. Then H contains elements a and b such that $\phi(a) = c$ and $\phi(b) = d$. Thus, by Lemma 6.1,

$$c^*d = \phi(a)^*\phi(b) = \phi(a^*)\phi(b) = \phi(a^*b) \subseteq \mathrm{im}(\phi),$$

and, since c and d have been chosen arbitrarily from $\mathrm{im}(\phi)$, we have shown that $\mathrm{im}(\phi)$ is a closed subset of H'. It remains to show that $H/\!/\ker(\phi) \cong \mathrm{im}(\phi)$.

We set $F := \ker(\phi)$. From (i) we know that F is normal in H. Thus, by Lemma 6.2, $\phi(a) = \phi(b)$ if and only if $a^F = b^F$ for any two elements a and b in H.

For each element h in H, define $\psi(h^F) := \phi(h)$. Since $\phi(a) = \phi(b)$ if and only if $a^F = b^F$ for any two elements a and b in H, ψ is an injective map from $H/\!/F$ to $\mathrm{im}(\phi)$. The definition of ψ also implies that $\mathrm{im}(\psi) = \mathrm{im}(\phi)$.

In order to show that ψ is a homomorphism, we choose elements a and b in H. Then

$$\psi(a^F b^F) = \psi(\{h^F \mid h \in aFb\}) = \{\psi(h^F) \mid h \in aFb\} = \{\phi(h) \mid h \in aFb\} = \phi(aFb).$$

On the other hand, we have

$$\phi(afb) = \phi(a)\phi(f)\phi(b) = \phi(a) \cdot 1 \cdot \phi(b) = \phi(a)\phi(b) = \phi(ab)$$

for each element f in F. Thus,

$$\psi(a^F b^F) = \phi(ab) = \phi(a)\phi(b) = \psi(a^F)\psi(b^F),$$

and we are done. $\square$

Theorem 6.4

Let H be a hypergroup, let D and E be closed subsets of H, and assume that $D \subseteq E$. Assume that E is normal in H. Then $E/\!/D$ is a normal closed subset of $H/\!/D$ and $(H/\!/D)/\!/(E/\!/D) \cong H/\!/E$.

Proof. That $E/\!/D$ is a normal closed subset of $H/\!/D$ follows from Lemma 5.3(i). It remains to prove that $H/\!/D$ and $(H/\!/D)/\!/(E/\!/D) \cong H/\!/E$.

Let a and b be elements in H with $a^D = b^D$. Then $DaD = DbD$. Thus, as $D \subseteq E$, $EaE = EbE$. It follows that $a^E = b^E$. This shows that

$$\phi\colon\ H/\!/D \ \to\ H/\!/E, \ \ h^D \ \mapsto\ h^E$$

is a surjective map.

Note that

$$\phi(a^D b^D) = \phi(\{h^D \mid h \in aDb\}) = \{h^E \mid h \in aDb\} = aDb/\!/E$$

and that

$$\phi(a^D)\phi(b^D) = a^E b^E = \{h^E \mid h \in aEb\} = aEb/\!/E$$

for any two elements a and b in H. On the other hand, since E is assumed to be normal in H, we obtain from Lemma 4.1(i) that

$$aDb//E = EaDbE//E = aEb//E$$

for any two elements a and b in H. Thus, we have

$$\phi(a^D b^D) = \phi(a^D)\phi(b^D)$$

for any two elements a and b in H. Since we also have $\phi(1^D) = 1^E$, ϕ is a homomorphism from $H//D$ to $H//E$.

Note finally that, for each element h in H, $\phi(h^D) = 1^E$ if and only if $h \in E$. Thus, $\ker(\phi) = E//D$. It follows that

$$(H//D)//(E//D) = (H//D)//\ker(\phi) \cong \mathrm{im}(\phi) = H//E;$$

cf. Theorem 6.3. □

Theorem 6.5

Let H be a hypergroup, let D and E be closed subsets of H, and assume that D normalizes E. Then ED is a closed subset of H, E is a normal closed subset of ED, $E \cap D$ is a normal closed subset of D, and $ED//E \cong D//(E \cap D)$.

Proof. That ED is a closed subset of H was shown in Lemma 4.1(ii), that E is a normal closed subset of ED was shown in Lemma 4.1(iii), and that $E \cap D$ is a normal closed subset of D was shown in Lemma 4.1(iv). It remains to show that $ED//E \cong D//(E \cap D)$.

For each element d in D, define $\psi(d) := d^E$. Then

$$\psi(ab) = \{\psi(d) \mid d \in ab\} = \{d^E \mid d \in ab\} = ab//E$$

and

$$\psi(a)\psi(b) = a^E b^E = \{d^E \mid d \in aEb\} = aEb//E$$

for any two elements a and b in D.

On the other hand, since we are assuming that D normalizes E, we obtain from Lemma 4.1(i) that

$$ab//E = EabE//E = aEb//E$$

for any two elements a and b in D. Thus, we have $\psi(ab) = \psi(a)\psi(b)$ for any two elements a and b in D. Since we also have $\psi(1) = 1^E$, ψ is a homomorphism from D to $ED//E$.

Note finally that

$$\ker(\psi) = \{d \in D \mid d \in E\} = E \cap D.$$

Thus, by Theorem 6.3,

$$D//(E \cap D) = D//\ker(\psi) \cong \mathrm{im}(\psi) = ED//E,$$

and we are done. □

7. Solvable hypergroups

A hypergroup H is called *solvable* if it contains closed subsets $F_0, \ldots, F_n$ such that $F_0 = \{1\}$, $F_n = H$, and, for each element i in $\{0, \ldots, n\}$ with $1 \leq i$, $F_{i-1} \subseteq F_i$, $F_i//F_{i-1}$ is thin, and $|F_i//F_{i-1}|$ is a prime number.

Lemma 7.1

Closed subsets of solvable hypergroups are solvable.

Proof. Let H be a solvable hypergroup. Then H contains closed subsets $F_0, \ldots, F_n$ such that $F_0 = \{1\}$, $F_n = H$, and, for each element i in $\{1, \ldots, n\}$, $F_{i-1} \subseteq F_i$, $F_i//F_{i-1}$ is thin, and $|F_i//F_{i-1}|$ is a prime number.

Let E be a closed subset of H. For each element i in $\{0, \ldots, n\}$, we set $E_i := F_i \cap E$.

Let i be an element in $\{1, \ldots, n\}$. Then $E_i \subseteq F_i$. Thus, as F_i normalizes F_{i-1}, E_i normalizes F_{i-1}. It follows that $F_{i-1}E_i$ is a closed subset of F_i; cf. Lemma 4.1(ii). Thus, by Theorem 5.2, $F_{i-1}E_i//F_{i-1}$ is a closed subset of $F_i//F_{i-1}$. Since $F_i//F_{i-1}$ is thin and $|F_i//F_{i-1}|$ is a prime number, this implies that

$$F_{i-1}//F_{i-1} = F_{i-1}E_i//F_{i-1} \qquad \text{or} \qquad F_{i-1}E_i//F_{i-1} = F_i//F_{i-1}.$$

In the first case, we obtain that $F_{i-1} = F_{i-1}E_i$, and that implies that $E_i \subseteq F_{i-1}$. Now, as $E_{i-1} = F_{i-1} \cap E$, we conclude that $E_{i-1} = E_i$.

In the second case, we recall that, since E_i normalizes F_{i-1},

$$F_{i-1}E_i//F_{i-1} \cong E_i//E_{i-1};$$

cf. Theorem 6.5. Thus, we obtain in this case that $E_i//E_{i-1} \cong F_i//F_{i-1}$. Now, as $F_i//F_{i-1}$ is thin and $|F_i//F_{i-1}|$ is a prime number, $E_i//E_{i-1}$ is thin, and $|E_i//E_{i-1}|$ is a prime number. Since $E_0 = \{1\}$ and $E_n = E$, this shows that E is solvable. □

Lemma 7.2

Let H be a solvable hypergroup, and let E be a normal closed subset of H. Then $H//E$ is solvable.

Proof. We are assuming that H is solvable. Thus, H contains closed subsets $F_0, \ldots, F_n$ such that $F_0 = \{1\}$, $F_n = H$, and, for each element i in $\{1, \ldots, n\}$, $F_{i-1} \subseteq F_i$, $F_i//F_{i-1}$ is thin, and $|F_i//F_{i-1}|$ is a prime number.

Let i be an element in $\{1, \ldots, n\}$. Then F_{i-1} normalizes E. Thus, by Lemma 4.1(ii), EF_{i-1} is a closed subset of H. It follows that $EF_{i-1} \cap F_i$ is a closed subset of F_i and $F_{i-1} \subseteq EF_{i-1} \cap F_i$. Since $F_i//F_{i-1}$ is thin and $|F_i//F_{i-1}|$ is a prime number, this implies that

$$F_{i-1} = EF_{i-1} \cap F_i \qquad \text{or} \qquad EF_{i-1} \cap F_i = F_i$$

On the other hand, since F_i normalizes E and F_{i-1}, we have $EF_{i-1}h \subseteq EhF_{i-1} \subseteq hEF_{i-1}$ for each element h in F_i, so F_i normalizes EF_{i-1}. Thus, by Theorem 6.5,

$$EF_i//EF_{i-1} \cong F_i//(EF_{i-1} \cap F_i).$$

It follows that $EF_i//EF_{i-1} = F_i//F_{i-1}$ or $EF_i//EF_{i-1} = F_i//F_i$.

In the first case, we obtain that $EF_i//EF_{i-1}$ is thin and $|EF_i//EF_{i-1}|$ is a prime number. From the fact that $EF_i//EF_{i-1}$ is thin we obtain that EF_{i-1} is strongly normal in EF_i; cf. Lemma 5.4. In particular, EF_{i-1} is normal in EF_i. Thus, by Theorem 6.4,

$$(EF_i//E)//(EF_{i-1}//E) \cong EF_i//EF_{i-1}.$$

It follows that $(EF_i//E)//(EF_{i-1}//E)$ is thin and $|(EF_i//E)//(EF_{i-1}//E)|$ is a prime number.

In the second case, $EF_{i-1} = EF_i$.

Since $EF_n//E = H//E$ and $EF_0//E = E//E$, this shows that $H//E$ is solvable. □

Lemma 7.3

Let H be a hypergroup, and let E be a closed subset of H. Assume that E and $H//E$ are solvable. Then H is solvable.

Proof. We are assuming that E is solvable. Thus, E contains closed subsets F_0, …, F_m such that $F_0 = \{1\}$, $F_m = E$, and, for each element i in $\{1, \dots, m\}$, $F_{i-1} \subseteq F_i$, $F_i//F_{i-1}$ is thin, and $|F_i//F_{i-1}|$ is a prime number.

We are assuming that $H//E$ is solvable. Thus, by Theorem 5.2, H contains closed subsets F_{m+1}, …, F_n such that $F_n//E = H//E$ and, for each element i in $\{m+1, \dots, n\}$, $F_{i-1} \subseteq F_i$, $(F_i//E)//(F_{i-1}//E)$ is thin, and $|(F_i//E)//(F_{i-1}//E)|$ is a prime number.

Let i be an element in $\{m+1, \dots, n\}$. Then $(F_i//E)//(F_{i-1}//E)$ is thin. Thus, by Lemma 5.4, $F_{i-1}//E$ is strongly normal in $F_i//E$. Thus, by Lemma 5.3(ii), F_{i-1} is a strongly normal in F_i. In particular, F_{i-1} is a normal in F_i. Thus, by Theorem 6.4,

$$(F_i//E)//(F_{i-1}//E) \cong F_i//F_{i-1}.$$

Now, as $(F_i//E)//(F_{i-1}//E)$ is thin and $|(F_i//E)//(F_{i-1}//E)|$ is a prime number, $F_i//F_{i-1}$ is thin and $|F_i//F_{i-1}|$ is a prime number.

What we have seen so far is that, for each element i in $\{1, \dots, n\}$, $F_{i-1} \subseteq F_i$, $F_i//F_{i-1}$ is thin, and $|F_i//F_{i-1}|$ is a prime number. From $F_n//E = H//E$ we also obtain that $F_n = H$. Thus, as $F_0 = \{1\}$, H is solvable. □

Let H be a hypergroup. Recall from Section 4 that a closed subset E of H is said to be subnormal in H if H contains closed subsets F_0, …, F_n such that $F_0 = E$, $F_n = H$, and, for each element i in $\{1, \dots, n\}$, F_{i-1} is normal in F_i.

We are now in the position to weaken the hypothesis of Lemma 7.2.

Theorem 7.4

Let H be a solvable hypergroup, and let E be a subnormal closed subset of H. Then $H//E$ is solvable.

Proof. Clearly, if $E = H$, $H//E$ is solvable. Therefore, we assume that $E \neq H$.

From $E \neq H$ together with the fact that E is a subnormal closed subset of H we obtain a proper normal closed subset F of H such that E is subnormal in F. By induction, $F//E$ is solvable.

Since H is solvable and F is a normal closed subset of H, $H//F$ is solvable; cf. Lemma 7.2. On the other hand, as F is a normal closed subset of H, we obtain from Theorem 6.4 also that

$$(H//E)//(F//E) \cong H//F.$$

Thus, $(H//E)//(F//E)$ is solvable.

Now, as both $F//E$ and $(H//E)//(F//E)$ are solvable, so is $H//E$; cf. Lemma 7.3. □

Corollary 7.5

Let H be a solvable hypergroup, and let D and E be closed subsets of H with $D \subseteq E$. Assume that D is subnormal in H, and that $E//D$ is subnormal in $H//D$. Then E is subnormal in H.

Proof. We are assuming that H is solvable and that D is subnormal in H. Thus, by Theorem 7.4, $H//D$ is solvable. Since $H//D$ is solvable and $E//D$ is assumed to be subnormal in $H//D$, $(H//D)//(E//D)$ is solvable; cf. Theorem 7.4. Thus, by Theorem 5.2, H contains closed subsets F_0, ..., F_n such that $F_0//D = E//D$, $F_n//D = H//D$, and, for each element i in $\{1, \ldots, n\}$, $F_{i-1}//D \subseteq F_i//D$ and $(F_i//D)//(F_{i-1}//D)$ is thin.

Let i be an element in $\{1, \ldots, n\}$. Since $(F_i//D)//(F_{i-1}//D)$ is thin, $F_{i-1}//D$ is strongly normal in $F_i//D$; cf. Lemma 5.4. Thus, by Lemma 5.3(ii), F_{i-1} is strongly normal in F_i. In particular, F_{i-1} is normal in F_i; cf. Lemma 4.3.

From $E//D = F_0//D$ we obtain that $E = F_0$, from $F_n//D = H//D$ that $F_n = H$. Also, for each element i in $\{1, \ldots, n\}$, F_{i-1} is normal in F_i. Thus, E is subnormal in H. □

8. Hall subsets of solvable association schemes

In this section, we turn to association schemes. We will refer freely to some notation introduced in [8].

Our first lemma is a slight generalization of [4; Lemma 5.2].

Lemma 8.1

Let S be an association scheme, let T be a closed subset of S, let s be an element of S, and let π be a set of prime numbers. Assume that s is π-valenced and that n_T is a π-number. Then s^T is π-valenced.

Proof. By [8; Theorem 4.1.3(ii)], $n_{s^T} n_T = n_{TsT}$. Moreover, by [8; Lemma 2.3.3], n_{TsT} divides $n_T n_s n_T$. Thus, n_{s^T} divides $n_T n_s$. Thus, as n_T and n_s are assumed to be π-numbers, n_{s^T} is a π-number, and that means that s^T is π-valenced. □

Lemma 8.2

Let π be a set of prime numbers, and let S be a solvable and π-valenced association scheme. Assume that S has no thin subnormal closed subset different from $\{1\}$ the valency of which is a π-number. Then S is thin.

Proof. Since S is assumed to be solvable, S contains closed subsets T_0, ..., T_n such that $T_0 = \{1\}$, $T_n = S$, and, for each element i in $\{1, \ldots, n\}$, T_{i-1} is strongly normal in T_i.

Assume, by way of contradiction, that S is not thin. Then $\{1, \ldots, n\}$ contains an element i such that T_{i-1} is thin and T_i is not thin. Since T_{i-1} is strongly normal in T_i, $O^\vartheta(T_i) \subseteq T_{i-1}$. Thus, as T_{i-1} is thin, $O^\vartheta(T_i)$ is thin.

Let s be an element in T_i. Then, as $O^\vartheta(T_i)$ is thin, s^*s is a thin closed subset of S; cf. [8; Lemma 6.7.1(i), (iv)]. Furthermore, by [8; Lemma 6.7.1(v)], s^*s is strongly normal in $O^\vartheta(T_i)$, and, by [8; Lemma 6.7.1(iii)], $\{s\} = ss^*s$. This latter equation yields $n_{s^*s} = n_s$; cf. [8; Lemma 1.4.4(ii)]. Thus, as S is assumed to be π-valenced, n_{s^*s} is a π-number.

What we have seen is that s^*s is a thin subnormal closed subset of S the valency of which is a π-number. Since S is assumed to have no thin subnormal closed subset different from $\{1\}$ the valency of which is a π-number, $s^*s = \{1\}$. This implies that s is thin.

Since s has been chosen arbitrarily from T_i, we have shown that T_i is thin, contradiction. □

Let S be an association scheme, and recall from Section 1 that S is a hypergroup with respect to the hypermultiplication defined by the complex multiplication in S. Note that the closed subsets of S are exactly the closed subsets of this hypergroup. Furthermore, the strongly normal closed subsets of S are exactly the strongly normal closed subsets of this hypergroup, so that, by [9; Lemma 4.2.5(ii)], S is solvable if and only if this hypergroup is solvable. We shall now see that the transition from scheme theory to hypergroup theory is also compatible with quotients.

Lemma 8.3

Let S be an association scheme, and let T be a closed subset of S. Then the hypergroup defined by the complex multiplication in the quotient scheme $S/\!/T$ is equal to the quotient of the hypergroup defined by the complex multiplication in the scheme S over T.

Proof. By [8; Lemma 4.1.4], the map which associates to each element s^T in $S/\!/T$ the double coset TsT is a bijective map from the quotient scheme $S/\!/T$ to the set of all double cosets of T in S. The elements of the quotient of the hypergroup defined by the complex multiplication in the scheme S over T are the double cosets of T in the hypergroup S.

Let p and q be elements in S. Then, by [8; Lemma 4.1.4],

$$p^T q^T := \{s^T \mid s \in TpTqT\}.$$

The product p^T and q^T in the quotient of the hypergroup defined by the complex multiplication in the scheme S over T is defined to be the set $\{s^T \mid s \in pTq\}$. On the other hand, by Lemma 5.1, $\{s^T \mid s \in pTq\} = \{s^T \mid s \in TpTqT\}$. □

A closed subset T of an association scheme S is said to be *subnormal in* S if T is subnormal in the hypergroup defined by the complex multiplication in S.

Lemma 8.4

Let S be a solvable association scheme, and let T be a subnormal closed subset of S. Then we have the following.

(i) *The quotient scheme $S//T$ is solvable.*

(ii) *Let U be a closed subset of S with $T \subseteq U$. Assume that $U//T$ is subnormal in $S//T$. Then U is subnormal in S.*

Proof. (i) We are assuming that T is a subnormal closed subset of S. Thus, T is a subnormal closed subset of the hypergroup S. Thus, by Theorem 7.4, the quotient $S//T$ is solvable. Thus, by Lemma 8.3, the quotient scheme $S//T$ is solvable.

(ii) We are assuming that T is a subnormal closed subset of the scheme S. Thus, T is a subnormal closed subset of the hypergroup S defined by the complex multiplication in S.

We are assuming that the quotient scheme $U//T$ is subnormal in the quotient scheme $S//T$. Thus, by Lemma 8.3, the quotient $U//T$ is subnormal in the quotient $S//T$. Thus, by Corollary 7.5, U is subnormal in the hypergroup S defined by its complex multiplication. Thus, U is subnormal in the scheme S. □

Theorem 8.5

Let π be a set of prime numbers, and let S be a solvable and π-valenced association scheme. Then S contains a strongly normal closed π-subset which contains all subnormal closed π-subsets of S.

Proof. Let U be maximal among the subnormal closed π-subsets of S. By hypothesis, S is solvable. Thus, as U is a subnormal closed subset of S, $S//U$ is solvable; cf. Lemma 8.4(i). Furthermore, since S is π-valenced and n_U is a π-number, we obtain from Lemma 8.1 that $S//U$ is π-valenced. Thus, we may apply Lemma 8.2 to the quotient scheme $S//U$ in place of S.

Let V be a closed subset of S with $U \subseteq V$. Assume that $V//U$ is thin, that $V//U$ is subnormal in $S//U$, and that $n_{V//U}$ is a π-number.

Since U is subnormal in S and $V//U$ is subnormal in $S//U$, we obtain from Lemma 8.4(ii) that V is subnormal in S. On the other hand, n_U as well as $n_{V//U}$ are π-numbers. Thus, by [8; Lemma 4.3.3(i)], n_V is a π-number, so that the choice of U forces $U = V$.

This shows that $S//U$ does not contain a thin subnormal closed subset different from $U//U$ the valency of which is a π-number. Thus, by Lemma 8.2, $S//U$ is thin. It follows that U is strongly normal in S; cf. [8; Lemma 4.2.5(ii)].

Let T be a subnormal closed π-subset of S. Since U is strongly normal in S, U is normal in S; cf. [8; Lemma 2.5.5]. Thus, by [8; Lemma 2.5.2(iii)], UT is a closed subset of S and is subnormal in S.

On the other hand, by [8; Lemma 2.3.6(i)], $n_U n_T = n_{UT} n_{U \cap T}$. Thus, UT is a closed π-subset of S. Thus, as $U \subseteq UT$, the choice of U forces $T \subseteq U$. □

Let π be a set of prime numbers, and let S be a solvable and π-valenced association scheme. From Theorem 8.5 we obtain that S contains a strongly normal closed π-subset which contains all subnormal closed π-subsets of S. In the following, this closed subset of S will be denoted by $O_\pi(S)$.

Proposition 8.6

Let π be a set of prime numbers, and let S be a solvable and π-valenced association scheme. Set $O := O_\pi(S)$. Then we have the following.

(i) *The scheme $S//O$ possesses at least one Hall π-subset.*

(ii) *Any two Hall π-subsets of $S//O$ are conjugate in S.*

(iii) *Any closed π-subset of $S//O$ is contained in a Hall π-subset of $S//O$.*

Proof. Since O is a strongly normal closed subset of S, $S//O$ is thin; cf. [8; Lemma 4.2.5(ii)]. Moreover, by Lemma 8.4(i), $S//O$ is solvable. Thus, with a reference to the group correspondence, the claim follows from [2; Theorem]. □

Now we are in the position to prove the main result of this article. We split the proof into three parts.

Theorem 8.7

Let π be a set of prime numbers, and let S be a solvable and π-valenced association scheme. Then S possesses at least one Hall π-subset.

Proof. Set $O := O_\pi(S)$. By Proposition 8.6(i), the scheme $S//O$ possesses at least one Hall π-subset. Let T be a closed subset of S such that $T//O$ is a Hall π-subset of $S//O$; cf. [8; Lemma 4.1.7(ii)]. We will see that T is a Hall π-subset of S.

Since $T//O$ is a Hall π-subset of $S//O$, $n_{T//O}$ is a π-number. Since n_O is a π-number, too, we obtain from [8; Lemma 4.3.3(i)] that n_T is a π-number. But T is also π-valenced, since S is π-valenced. Thus, T is a closed π-subset of S, and it remains to be shown that $n_{S//T}$ is a π'-number.

Since $T//O$ is a Hall π-subset of $S//O$, $n_{(S//O)//(T//O)}$ is a π'-number. From [8; Lemma 4.3.3] we also obtain that $n_{S//T} = n_{(S//O)//(T//O)}$. Thus, $n_{S//T}$ is a π'-number. □

Theorem 8.8

Let π be a set of prime numbers, and let S be a solvable and π-valenced association scheme. Then any two Hall π-subsets of S are conjugate in S.

Proof. Let T and U be Hall π-subsets of S, and set $O := O_\pi(S)$. Since S is assumed to be π-valenced, so is T. Thus, as n_O is a π-number, $T//O$ is π-valenced; cf. Lemma 8.1.

Since n_T is a π-number, so is $n_{T//O}$; cf. [8; Lemma 4.3.3(i)]. Thus, $T//O$ is a closed π-subset of $S//O$. From [8; Lemma 4.3.3] we also obtain that $n_{S//T} = n_{(S//O)//(T//O)}$. Thus, as $n_{S//T}$ is a π'-number, so is $n_{(S//O)//(T//O)}$. It follows that $T//O$ is a Hall π-subset of $S//O$.

Similarly, $U//O$ is a Hall π-subset of $S//O$. Thus, by Proposition 8.6(ii), S contains an element s such that $(s^O)^*(T//O)s^O = U//O$. Thus, by [8; Lemma 4.1.4], $s^*Ts = U$. □

Theorem 8.9

Let π be a set of prime numbers, and let S be a solvable and π-valenced association scheme. Then any closed π-subset of S is contained in a Hall π-subset of S.

Proof. Let T be a closed π-subset of S, and set $O := O_\pi(S)$. Then, OT is a closed subset of S. Thus, by [8; Lemma 4.1.7(i)], $OT//O$ is a closed subset of $S//O$.

Since S is assumed to be π-valenced, so is OT. Since T is a closed π-subset of S, n_T is a π-number. Also n_O is a π-number. Thus, by [8; Lemma 4.3.4], $n_{OT//O}$ is a π-number. Thus, $OT//O$ is a closed π-subset of $S//O$. Thus, by Proposition 8.6(iii), $S//O$ contains a Hall π-subset which contains $OT//O$. Let U be a closed subset of S such that $O \subseteq U$, $U//O$ is a Hall π-subset of $S//O$, and $OT//O \subseteq U//O$; cf. [8; Lemma 4.1.7(ii)].

We will see that U is a Hall π-subset of S which contains T.

Since $U//O$ is a Hall π-subset of $S//O$, $n_{U//O}$ is a π-number. Thus, as n_O is a π-number, n_U is a π-number; cf. [8; Lemma 4.3.3(i)]. Since $U//O$ is a Hall π-subset of $S//O$, $n_{(S//O)//(U//O)}$ is a π'-number. Furthermore, by [8; Lemma 4.3.3], $n_{S//U} = n_{(S//O)//(U//O)}$. Thus, $n_{S//U}$ is a π'-number. It follows that U is a Hall π-subset of S.

From $OT//O \subseteq U//O$ we obtain that $OT \subseteq U$; cf. [8; Lemma 4.1.4]. Thus, U is a Hall π-subset of S, and U contains T. $\square$

References

1. French, C., Zieschang, P.-H.: A Schur-Zassenhaus Theorem for association schemes, *J. Algebra* **435**, 88-123 (2015)

2. Hall, P.: A note on soluble groups, *J. London Math. Soc.* **3**, 98-105 (1928)

3. Hanaki, A., Miyamoto, I.: Classification of association schemes of small order, Online catalogue. http://kissme.shinshu-u.ac.jp/as

4. Hirasaka M., Muzychuk, M., Zieschang, P.-H.: A generalization of Sylow's theorems on finite groups to association schemes, *Math. Z.* **241**, 665-672 (2002)

5. Marty, F.: Sur une généralisation de la notion de groupe, in: Huitième Congres des Mathématiciens, Stockholm 1934, 45-59

6. Rassy, M., Zieschang, P.-H.: Basic structure theory of association schemes, *Math. Z.* **227**, 391-402 (1998)

7. Tanaka, R., Zieschang, P.-H.: On a class of wreath products of hypergroups and association schemes, *J. Algebraic Combin.* **37**, 601-619 (2013)

8. Zieschang, P.-H.: *Theory of Association Schemes*, Springer Monographs in Mathematics, Berlin Heidelberg New York, 2005

9. Zieschang, P.-H.: Hypergroups all non-identity elements of which are involutions, in: *Advances in algebra* 305-322, Springer Proc. Math. Stat. **277**, Springer, Cham, 2019