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ON FINITE GROUPS ISOSPECTRAL TO SIMPLE LINEAR AND UNITARY GROUPS

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Abstract: Let L be a simple linear or unitary group of dimension larger than 3 over a finite field of characteristic p. We deal with the class of finite groups isospectral to L. It is known that a group of this class has a unique nonabelian composition factor. We prove that if $L \neq U_4(2), U_5(2)$ then this factor is isomorphic either to L or a group of Lie type over a field of characteristic different from p.

Keywords: finite group, spectrum of a group, simple group, linear group, unitary group, composition factor

The *spectrum* $\omega(G)$ of a finite group G is the set of its element orders. Two groups are said to be *isospectral* if they have the same spectra. A finite group L is *recognizable by spectrum* if every finite group G with $\omega(G) = \omega(L)$ is isomorphic to L. We say that the *problem of recognition by spectrum* is solved for L if the number $h(L)$ of pairwise nonisomorphic finite groups isospectral to L is known. Thus the recognizability of L is equivalent to the equality $h(L) = 1$. The recent results concerning the problem of recognition by spectrum can be found in the surveys [1, 2].

When solving the problem of recognition by spectrum for a finite group L, one naturally faces the question of what composition structure finite groups isospectral to L have. The present paper investigates this aspect of the recognition problem for finite simple linear and unitary groups. To denote these groups we use the matrix notation from [3] and the abbreviation $L_n^\varepsilon(q)$, where $\varepsilon \in \{+,-\}$, $L_n^+(q) = L_n(q)$, and $L_n^-(q) = U_n(q)$. Since the recognition problem is completely solved for $L_2(q)$ [4, 5], $L_3(q)$ [6–8], and $U_3(q)$ [6, 9], these groups are excluded from the article.

Theorem 1. *Let $L = L_n^\varepsilon(q)$, where $n \geq 4$, $\varepsilon \in \{+,-\}$, and $(n,q) \neq (4,2)$. Then there are no alternating groups among nonabelian composition factors of finite groups isospectral to L.*

Theorem 2. *Let $L = L_n^\varepsilon(q)$, where $n \geq 4$, $\varepsilon \in \{+,-\}$, and $(\varepsilon,n,q) \neq (-,5,2)$. Then there are no sporadic groups nor the Tits group ${}^2F_4(2)'$ among nonabelian composition factors of finite groups isospectral to L.*

Theorem 3. *Let $L = L_n^\varepsilon(q)$, where $n \geq 4$, $\varepsilon \in \{+,-\}$, $(\varepsilon,n,q) \neq (-,4,2)$, and q is a power of a prime p. Then among nonabelian composition factors of finite groups isospectral to L, there are no groups of Lie type over a field of characteristic p other than L.*

Note that some similar results for symplectic and orthogonal groups were established in [10].

§ 1. Preliminaries

Our notation of sporadic groups, simple classical groups and simple exceptional groups of Lie type follows [3]. The alternating group of degree n is denoted by Alt_n.

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Given a nonzero integer n, let $\pi(n)$ denote the set of prime divisors of n, and n_r, with r prime, denote the r-part of n, i.e., the largest power of r that divides n, while $n_{r'}$ denotes the r'-part of n, i.e., the absolute value of n/n_r. If n is a nonzero integer, m is an odd prime, and $(n,m)=1$ then we write $e(m,n)$ to denote the multiplicative order of n modulo m. Given an odd n, we put $e(2,n)=1$ if $n\equiv 1 \pmod 4$ and put $e(2,n)=2$ if $n\equiv 3 \pmod 4$.

Let n be an integer and $|n|>1$. A prime r is said to be a *primitive prime divisor* of the difference n^i-1 if $e(r,n)=i$. The existence of primitive divisors for almost all pairs of n and i was established by Zsigmondy.

Lemma 1.1 (Zsigmondy [11]). *Let n be an integer and $|n|>1$. Then for every natural number i there is a prime r with $e(r,n)=i$, except when $(n,i)\in\{(2,1),(2,6),(-2,2),(-2,3),(3,1),(-3,2)\}$.*

In what follows the notation $r_i(n)$ means a primitive prime divisor of n^i-1 if such exist. The product of all primitive divisors of n^i-1 taken with multiplicities is said to be the *greatest primitive divisor* and denoted by $k_i(n)$. Note that for a divisor, the property of being primitive depends on the pair (n,i) and is not determined by the number n^i-1. For example, $k_6(2)=1$ while $k_3(4)=7$ and $k_6(-2)=7$, and $k_2(2)=3$ while $k_2(-2)=1$.

It is easy to check that $k_1(n)=|n-1|/2$ if $n\equiv 3 \pmod 4$ and $k_1(n)=|n-1|$ otherwise, and also that $k_2(n)=|n+1|/2$ if $n\equiv 1 \pmod 4$ and $k_2(n)=|n+1|$ otherwise. It follows from [12] that for $i>2$

$$k_i(n)=\frac{|\Phi_i(n)|}{(r,\Phi_{i_{r'}}(n))}, \tag{1}$$

where $\Phi_i(x)$ is the ith cyclotomic polynomial and r is the largest prime number dividing i, and if $i_{r'}$ does not divide $r-1$, then $(r,\Phi_{i_{r'}}(n))=1$.

Lemma 1.2. *Let i be an odd prime, let q be a power of a prime p, and $\varepsilon\in\{+,-\}$. If $(i,\varepsilon q)\neq(3,-2)$ then $k_i(\varepsilon q)>q^{i-2}/p$. If, in addition, $(i,\varepsilon)\neq(q+1,-)$ then $k_i(\varepsilon q)>q^{i-2}$.*

The proof follows from [10, Lemma 3.1].

The *Gruenberg–Kegel graph* $GK(G)$, or the *prime graph*, of G is the graph with vertex set $\pi(G)$ in which two distinct vertices p and q are adjacent if and only if $pq\in\omega(G)$. The number of connected components of $GK(G)$ is denoted by $s(G)$, and the connected components are denoted by $\pi_i(G)$ with $1\leq i\leq s(G)$. If G has even order then by default $2\in\pi_1(G)$. According to this partition, $\omega_i(G)$ is the subset of $\pi_i(G)$-numbers in $\omega(G)$ for every $1\leq i\leq s(G)$. The structure of finite groups with disconnected prime graph is described by Gruenberg and Kegel.

Lemma 1.3 [13]. *If G is a finite group with $s(G)>1$ then one of the following holds:*

(1) *$s(G)=2$, G is a Frobenius group;*

(2) *$s(G)=2$, $G=ABC$, where A and AB are normal subgroups of G, B is a normal subgroup of BC, and AB and BC are Frobenius groups;*

(3) *there is a nonabelian simple group S such that $S\leq\overline{G}=G/K\leq\operatorname{Aut}S$ for some nilpotent normal subgroup K of G; moreover, K and $\overline{G}/S$ are $\pi_1(G)$-groups, $s(S)\geq s(G)$, and for every $1<i\leq s(G)$ there is $1<j\leq s(S)$ such that $\omega_i(G)=\omega_j(S)$.*

Finite simple groups with disconnected prime graph were described by Williams [13] and Kondrat′ev [24]. The complete list of these groups, with corrected inaccuracies, can be found in [15, Tables 1a–1c]. It follows from the results of Williams and Kondrat′ev that if S is a simple group and $s(S)>1$, then for every $1<i\leq s(S)$, the set $\omega_i(S)$ has the unique element maximal under divisibility [16, Lemma 4]. In the tables mentioned above and the present paper this maximal element is denoted by $n_i(S)$.

Recall that an *independent set of vertices*, or *coclique*, in a graph Γ is a set of vertices pairwise nonadjacent to each other in Γ. We write $t(\Gamma)$ to denote the independence number of Γ, i.e., the maximal number of vertices in its cocliques. Given a group G, put $t(G)=t(GK(G))$. By analogy, for each prime r, define the r-independence number $t(r,G)$ to be the maximal number of vertices in cocliques of $GK(G)$ containing the vertex r.

Lemma 1.4 [17, 18]. *Let L be a finite nonabelian simple group such that $t(L) \geq 3$ and $t(2, L) \geq 2$, and let G be a finite group with $\omega(G) = \omega(L)$. Then the following hold:*

(1) *There exists a nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \operatorname{Aut} S$, where K is the maximal normal soluble subgroup of G.*

(2) *For every coclique ρ of $GK(G)$ of size at least 2, at most one prime of ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*

(3) *Any prime $r \in \pi(G)$ not adjacent to 2 in $GK(G)$ does not divide the product $|K| \cdot |\overline{G}/S|$. In particular, $t(2, S) \geq t(2, G)$.*

Lemma 1.5. *Let G be a finite group, let K be a normal subgroup of G, and $r \in \pi(K)$. Suppose that the factor group G/K contains a section isomorphic to a noncyclic abelian p-group for some prime p other than r. Then $rp \in \omega(G)$.*

Proof. Let R be a Sylow r-subgroup of K and let N be its normalizer in G. By the Frattini argument $G/K \simeq N/(K \cap N)$ and so N and its normal subgroup R satisfy the hypothesis of the lemma. Therefore we can assume that $G = N$ and $K = R$. Since $p \neq 2$, the group G/K which contains a section isomorphic to a noncyclic abelian p-group must also contain a noncyclic abelian p-subgroup. The rest of the proof follows from [19, Chapter 5, Theorem 3.16].

§ 2. Properties of Simple Linear and Unitary Groups

The formulas of the orders of simple linear and unitary groups imply that the set of primes dividing the order of $L_n^\varepsilon(q)$ looks as follows:

$$r \in \pi(L_n^\varepsilon(q)) \text{ if and only if } e(r, \varepsilon q) \leq n \text{ or } r \text{ divides } q. \tag{2}$$

We will use the adjacency criterion for the prime graphs of linear and unitary groups from [20]. In that paper, the criterion for unitary groups is formulated in terms of a function $\nu(x)$ acting from $\mathbb{N}$ to $\mathbb{N}$ by the following rule:

$$\nu(x) = \begin{cases} 2x, & \text{if } x \text{ is odd}, \\ x/2, & \text{if } x \equiv 2 \pmod 4, \\ x, & \text{if } x \equiv 0 \pmod 4. \end{cases} \tag{3}$$

It is easy to check that

$$\nu(\nu(x)) = x, \tag{4}$$

$$e(m, -x) = \nu(e(m, x)) \tag{5}$$

for every nonzero integer x and prime m. Using this observation we unite the criteria for linear and unitary groups in the following three lemmas.

Lemma 2.1 [20, Propositions 2.1 and 2.2]. *Let $L = L_n^\varepsilon(q)$ be a simple group over a field of characteristic p. Let r and s be odd primes in $\pi(L)$ other than p. Let $k = e(r, \varepsilon q)$, $l = e(s, \varepsilon q)$ and suppose that $2 \leq k \leq l$. Then r and s are not adjacent in $GK(L)$ if and only if $k + l > n$ and k does not divide l.*

Lemma 2.2 [20, Proposition 3.1]. *Let $L = L_n^\varepsilon(q)$ be a simple group over a field of characteristic p. Let $r \in \pi(L)$ and $r \neq p$. Then r and p are not adjacent in $GK(L)$ if and only if one of the following holds*:

1) *r is odd and $e(r, \varepsilon q) > n - 2$;*

2) *$L = L_2(q)$ and $r = 2$;*

3) *$L = L_3^\varepsilon(q)$, $r = 3$, and $(\varepsilon q - 1)_3 = 3$.*

Lemma 2.3 [20, Propositions 4.1 and 4.2]. *Let $L = L_n^\varepsilon(q)$ be a simple group over a field of characteristic p. Let r be a prime divisor of $\varepsilon q - 1$ and let s be an odd prime other than p. Let $k = e(s, \varepsilon q)$. Then s and r are not adjacent in $GK(L)$ if and only if one of the following holds*:

1) *$k = n$, $n_r \leq (\varepsilon q - 1)_r$, and if $n_r = (\varepsilon q - 1)_r$ then $2 < (\varepsilon q - 1)_r$;*

2) *$k = n - 1$ and $(\varepsilon q - 1)_r \leq n_r$.*

It was noted in the introduction that the recognition problem is already solved for linear and unitary groups of dimension at most 3. There is also a number of results concerning groups of larger dimensions.

Lemma 2.4. 1. *If L is one of the simple groups $L_n(2^m)$, with $m \geq 1$ and $n \geq 4$, or $U_4(2^m)$, with $m > 1$, while G is a finite group with $\omega(G) = \omega(L)$ then $L \leq G \leq \operatorname{Aut} L$.*

2. *If L is one of the groups $L_4(3)$, $L_5(3)$, $L_6(3)$, $U_6(2)$, $U_4(3)$, and $U_4(5)$, while G is a finite group with $\omega(G) = \omega(L)$ then $L \leq G \leq \operatorname{Aut} L$.*

3. *The group $L = U_4(2)$ is isospectral to a group that has a nonabelian composition factor isomorphic to $\mathrm{Alt}_5 \simeq L_2(4)$.*

4. *The group $L = U_5(2)$ is isospectral to a group that has a nonabelian composition factor isomorphic to the sporadic group Mathieu M_{11}.*

Proof. The complete references to proofs of the assertions 1 and 2 can be found in [21, 22] and [1] respectively. The assertions 3 and 4 were proved in [23].

If L is one of the groups listed in (1) and (2) of Lemma 2.4 then the unique nonabelian composition factor of a finite group isospectral to L is isomorphic to L. Among the listed groups, only $L_4(2)$ is isomorphic to an alternating group, namely to Alt_8, and there is no ones isomorphic to a sporadic group. Hence if $L \neq L_4(2)$ then the conclusions of Theorems 1–3 are true for L. On the other hand, (3) and (4) of Lemma 2.4 show that $U_4(2)$ does not satisfy the conclusions of Theorems 1 and 3, while $U_5(2)$ does not satisfy the conclusion of Theorem 2. Furthermore, $\pi(U_4(2)) = \{2, 3, 5\}$ and using [3] it is not hard to check that the conclusion of Theorem 2 is true for $L = U_4(2)$. Thus proving Theorems 1–3 we may assume that L is not contained in the set

$$\mathscr{L}=\{L_n(2^m) \mid m \geq 1\} \cup \{U_4(2^m) \mid m \geq 1\} \cup \{L_4(3), L_5(3), L_6(3), U_6(2), U_4(3), U_4(5)\}.$$

Then [20, 24] guarantee that $t(L) \geq 3$ and $t(2, L) \geq 2$, and so the conclusion of Lemma 1.4 holds for groups isospectral to L.

§ 3. Proof of Theorem 1

Let $L = L_n^{\varepsilon}(q)$, where $n \geq 4$, $\varepsilon \in \{+, -\}$, and q is a power of a prime p. Let G be a finite group isospectral to L and let K be the soluble radical of G. Assume that the conclusion of Theorem 1 does not hold. Then $L \notin \mathscr{L}$ and applying Lemma 1.4 we obtain $S \leq \overline{G} = G/K \leq \operatorname{Aut}(S)$, where $S \simeq \mathrm{Alt}_m$ for some $m \geq 5$.

Suppose that for L there is a set M of three natural numbers which satisfies the following:

$(*)$ for every $i \in M$, the number $k_i(q)$ is not equal to 1;

$(**)$ primitive prime divisors $r_i(\varepsilon q)$ and $r_j(\varepsilon q)$, where $i, j \in M$, are not adjacent in $GK(L)$ if $i \neq j$.

Consider the numbers $k_i(\varepsilon q)$, where i runs over M. It follows from (2) of Lemma 1.4 that at least two of these three numbers are coprime to $|K| \cdot |\overline{G}/S|$ and lie in $\omega(S)$. Denote them by a and b. Suppose that there is a prime devisor r of a such that $r \leq m/2$. Since all prime divisors of b are not adjacent to r in $GK(G)$, they all are larger than $m/2$. Therefore, either all prime divisors of a or all prime divisors of b are greater than $m/2$. Denote by k that number for which this is true.

Let r' and r'' be two different prime divisors of k. Then $r' + r'' > m$. Thus $r'r'' \notin \omega(S)$ and so $r'r'' \in \omega(L) \setminus \omega(G)$, which is impossible. Let k be a power of a prime r larger than r. Then $r^2 > (m/2)^2 > m$ and hence $r^2 \in \omega(L) \setminus \omega(G)$; a contradiction. Therefore, k is a prime and the fact that $k \in \omega(S)$ yields $m \geq k$. Thus $m \geq k_i(\varepsilon q)$ for some $i \in M$.

The idea of the further proof is as follows: Choosing M in a special way, we bound m from below in terms of n and q. Then we arrive at a contradiction by showing that the maximal power of p in $\omega(S)$ is strictly larger than the maximal power of p in $\omega(L)$ with a few exceptions which are analyzed separately. We will refer to the maximal power of p in the spectrum of a finite group as the *p-period* of this group.

Denote the p-period of L by p^l. It follows from [25, Proposition 0.5] that l satisfies the inequalities

$$p^{l-1} + 1 \leq n \leq p^l. \tag{6}$$

Suppose that $n \geq 17$. Then there are at least three different primes in the interval $(n/2, n]$ and obviously all of them are at least $\max\{(n + 1)/2, 11\}$. By Lemma 2.1 the set M constituted by these three numbers satisfies $(*)$ and $(**)$. Thus for at least one prime $i \in M$, the number $k_i(\varepsilon q)$ is a prime not exceeding m.

Since $i \geq \max\{(n+1)/2, 11\}$, Lemma 1.2 implies that

$$m \geq k_i(\varepsilon q) > \max\{q^{\frac{n-3}{2}}/p, q^9/p\}.$$

Since $m > q^9/p > p^7+1$, there is an element of order p^7 in S. Hence $p^7 \in \omega(L)$ and $l \geq 7$. For $l \geq 7$, we have $l+2 < (2^{l-1}-2)/2$, and so $l+2 < (p^{l-1}-2)/2$. It follows from (6) that $(p^{l-1}-2)/2 \leq (n-3)/2$. Thus $l+2 < (n-3)/2$ and so $m > q^{(n-3)/2}/p > p^{l+2}/p = p^{l+1}$, which results in $p^{l+1} \in \omega(G) \setminus \omega(L)$; a contradiction.

Suppose that $n \in \{13, 14, 15, 16\}$. The set $M = \{7, 11, 13\}$ satisfies $(*)$ and $(**)$ and therefore $m \geq \min\{k_7(\varepsilon q), k_{11}(\varepsilon q), k_{13}(\varepsilon q)\}$. Since $q+1$ cannot be equal to 7, it follows from Lemma 1.2 that $m > q^5$. If $q > 2$ then S contains an element of order p^5 and so $l \geq 5$. But then $n \geq p^{l-1}+1 \geq 2^4+1 = 17$; a contradiction. If $q = 2$ then $m \geq \min\{k_7(\varepsilon 2), k_{11}(\varepsilon 2), k_{13}(\varepsilon 2)\} = 43$ and thus $37 \in \omega(S)$. But $e(37, 2) = 36$ and so $37 \notin \omega(L)$; a contradiction.

Let $n = 11, 12$. The set $M = \{7, 9, 11\}$ satisfies $(*)$ and $(**)$. It is not hard to derive from the equality $k_9(\varepsilon q) = (q^6 + \varepsilon q^3 + 1)/(3, \varepsilon q - 1)$ and Lemma 1.2 that $m > q^4 + 2$. Therefore S contains an element of order q^4 and so $l \geq 4$. If $p \neq 2$ then $n \geq p^{l-1}+1 \geq 3^3+1 = 28$, which is impossible. If $p = 2$ and $q > 2$ then $m > 4^4 + 2$ and so $2^8 \in \omega(G) \setminus \omega(L)$. If $q = 2$ then $m \geq 19$ and $2 \cdot 13 \in \omega(S)$. But $2 \cdot 13 \notin \omega(L)$; a contradiction.

Suppose that $n = 9, 10$. The set $M = \{7, 8, 9\}$ satisfies $(*)$ and $(**)$. Thus $m \geq k_8(\varepsilon q) = (q^4+1)/(2, \varepsilon q - 1)$. If $q > 3$ then we immediately get a contradiction since L has no elements of order p^3 for $p > 3$ and of order p^6 for $p \in \{2, 3\}$. If $q = 3$ then $m \geq 42$, and so $31 \in \omega(S)$. Since $e(31, 3) = 30$, there are no elements of order 31 in L. If $q = 2$ then $m \geq 17$ and similarly to the previous paragraph we infer that $2 \cdot 13 \in \omega(S) \setminus \omega(L)$, which is impossible.

Let $n = 8$. The set $M = \{5, 7, 8\}$ satisfies $(*)$ and $(**)$. Let first $p \neq 2$. Then $m > q^3$ and hence $p^3 \in \omega(S) \setminus \omega(L)$; a contradiction. If $p = 2$ then the 2-period of group L is equal to 8. To arrive at a contradiction, it suffices to show that $m > 17$. For $q > 2$ we have $m \geq \min\{k_5(\varepsilon q), k_7(\varepsilon q), k_8(\varepsilon q) \mid q = 2^m > 2\} > 4^3/2 > 17$ as required. It remains to consider the case $q = 2$. Observe that by Lemma 2.2 the primitive divisors $r_7(\varepsilon 2)$ and $r_8(\varepsilon 2)$ are not adjacent to 2 in $GK(L)$. Thus the numbers $k_7(\varepsilon 2)$ and $k_8(\varepsilon 2)$ are in $\omega(S)$ and both of them are primes between $m-3$ and m. In particular, $m \geq 43$; a contradiction.

Let $n = 5, 7$. Then by [15, Table 1a] the prime graph of L has two connected components and $n_2(L) = k_n(\varepsilon q)$. By Lemma 1.3 the graph $GK(S)$ is also disconnected and $n_2(L) = n_j(S)$ for some $j > 1$. Thus $n_2(L)$ is a prime and $m - 2 \leq n_2(L) \leq m$. Hence $m \geq k_5(\varepsilon q)$. Repeating the argument of the preceding paragraph we arrive at a contradiction, provided that $q > 2$. The group $L_5(2)$ is eliminated as in $\mathscr{L}$. Let $L = U_5(2)$. Since $k_5(-2) = 11$ lies in $\omega(S)$, it follows that $m \geq 11$. Therefore $7 \in \omega(S) \setminus \omega(L)$.

Let $n = 6$. Since $L_6^{\varepsilon}(2) \in \mathscr{L}$, we can assume that $q > 2$. If $p = 2$ or $\varepsilon q \equiv 1 \pmod 4$, it follows from Lemmas 2.2 and 2.3 that every primitive divisor $r_5(\varepsilon q)$ is not adjacent to 2 in $GK(L)$. Then by (3) of the Lemma 1.4 the number $k_5(\varepsilon q)$ is coprime to the product $|K| \cdot |\overline{G}/S|$, and hence it is a prime satisfying $m - 3 \leq k_5(\varepsilon q) \leq m$; a contradiction.

Thus $\varepsilon q \equiv 3 \pmod 4$. Moreover, $L_6^{\varepsilon}(3) \in \mathscr{L}$; therefore, $q \geq 5$. Hence $r_6(\varepsilon q)$ is not adjacent to 2 and $k_6(\varepsilon q)$ is a prime satisfying $m - 3 \leq k_6(\varepsilon q) \leq m$. Then $m \geq (q^2 - q + 1)/2 \geq 7$ and thus a Sylow 3-subgroup of S is not cyclic. Applying Lemma 1.5 we infer that 3 is adjacent to each prime divisor $r \neq 3$ of $|K|$ in $GK(G)$. If $(\varepsilon q - 1)_3 \leq 3$ then Lemmas 2.2 and 2.3 imply that the primitive prime divisor $r_5(\varepsilon q)$ is not adjacent to 3 in $GK(L)$ and so $k_5(\varepsilon q)$ must be a prime with $m - 2 \leq k_5(\varepsilon q) \leq m$; a contradiction.

Therefore $(\varepsilon q - 1)_3 \geq 9$. This, in particular, implies that $k_6(\varepsilon q) = q^2 - \varepsilon q + 1$. Hence $m \geq q^2 - \varepsilon q + 1$. If $q > p$ then $m > p^3$, contrary to the fact that $p^3 \notin \omega(L)$. Thus $q = p$ and the conditions $(\varepsilon q - 1)_3 \geq 9$ and $\varepsilon q \equiv 3 \pmod 4$ yield $p \geq 19$. So the p-period of L is equal to p. If $L = U_6(p)$ then $m \geq p^2 + p + 1$; a contradiction. Let $L = L_6(p)$. Then $m \geq p^2 - p + 1 > 2p$ and a Sylow p-subgroup of S is not cyclic. Applying Lemma 1.5 we infer that p is adjacent to each prime divisor $r \neq p$ of $|K|$ in $GK(G)$. Thus none of the primes dividing $k_5(p)$ can divide the order of K and so $k_5(p) \in \omega(S)$. Since $k_5(p) > p^3$, the number

$k_5(p)$ cannot be a prime. Let the product of primes r_1 and r_2, not necessarily different, divides $k_5(p)$. Every prime divisor of $k_5(p)$ is not adjacent to p, therefore, $r_1 > m-p > m/2$ and $r_2 > m-p > m/2$. Then $r_1r_2 > m$ and $r_1 + r_2 > m$, and there is no element of order r_1r_2 in S; a contradiction.

Finally, let $n = 4$. Since $L \notin \mathscr{L}$, we have $p > 2$, $q > 3$ and $(\varepsilon, p) \neq (-, 5)$. By Lemmas 2.2 and 2.3 for at least one i of the pair $3, 4$, every primitive divisor $r_i(\varepsilon q)$ is not adjacent to 2 in $GK(L)$. But then $k_i(\varepsilon q)$ is a prime and $m \geq k_i(\varepsilon q) \geq m-3$. If $q > p$ then $m \geq \min\{k_3(\varepsilon q), k_4(\varepsilon q)\} > p^3$, which is impossible for the p-period of L is at most p^2.

Thus $q = p > 3$ and the p-period of L is equal to p. Note that $k_4(\varepsilon p) = (p^2+1)/2 > 2p$, and since $(\varepsilon, p) \neq (-, 5)$, it follows that $k_3(\varepsilon p) = (p^2 + \varepsilon p + 1)/(3, \varepsilon p - 1) > 2p$. Hence $m > 2p$ and a Sylow p-subgroup of S is not cyclic. Therefore both $k_3(\varepsilon p)$ and $k_4(\varepsilon p)$ are coprime to the order of K and lie in the spectrum of S. Repeating the argument of the preceding paragraph we deduce that both of these numbers must be primes greater than $m-p$ and less than m. If $\varepsilon = +$ and $p \equiv -1 \pmod 3$ then $m \geq p^2+p+1$ and $p^2 \in \omega(S)$; a contradiction. If $\varepsilon = -$ and $p \equiv 1 \pmod 3$ then $m \geq p^2 - p + 1 > p + (p^2+1)/2$, contrary to the inequality $k_4(\varepsilon p) > m - p$. In the remaining cases, unless $L = L_4(7)$, we have $k_3(\varepsilon p) = (p^2+\varepsilon p+1)/3 < (p^2+1)/2 - p \leq m - p$, which is impossible. It remains to observe, for the case $L = L_4(7)$, that $k_4(\varepsilon p) = 25$ is not a prime.

Theorem 1 is proved.

§ 4. Proof of Theorem 2

Lemma 4.1. *If $3 \leq i \leq 20$, q is a power of a prime, and $k_i(q)$ lies in the spectrum of a sporadic group or the Tits group ${}^2F_4(2)'$ then the triple $(i, q, k_i(q))$ is in Table 1.*

Table 1

$q \setminus i$	3	4	5	6	8	10	12	14	18	20
2	7	5	31	1	17	11	13	43	19	41
3	13	5		7	41					
4	7	17		13		41				
5	31	13		7						
7	19	25		43						
8				19						
9		41								
11				37						

Proof. Using (1) it is easy to show that $k_i(q) \geq (q^2-q+1)/3$ for $i \geq 3$. According to [3], the orders of elements in the sporadic groups do not exceed 119. Thus $(q^2-q+1)/3 \leq 119$ and so $q \leq 19$. Now direct calculations show that the lemma holds.

Let $L = L_n^\varepsilon(q)$, where $n \geq 4$, $\varepsilon \in \{+, -\}$, q is a power of a prime p, and $L \notin \mathscr{L}$. Let G be a finite group isospectral to L and let K be the soluble radical of G. By Lemma 1.4 we have $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$, where S is a nonabelian simple group. Theorem 2 ensues from the following two propositions.

Proposition 4.1. *Let $L = L_4^\varepsilon(q)$. Then S is neither a sporadic group nor the Tits group.*

Proof. Assume the opposite. The groups $L_4^\varepsilon(2^m)$, $L_4(3)$, and $U_4(5)$ are in $\mathscr{L}$, and so q is odd, greater than 3, and also not equal to 5 in unitary case. By Lemma 2.3 all prime divisors of at least one of the numbers $k_4(\varepsilon q)$ and $k_3(\varepsilon q)$ are not adjacent to 2 in $GK(L)$, and by Lemma 1.4 at least one of these numbers is in $\omega(S)$. Hence $q \leq 11$ by Lemma 4.1.

Let $q = 11$. Then $k_3(\varepsilon 11) \in \omega(S)$. Lemma 4.1 implies that $L = U_4(11)$ and $37 \in \omega(S)$. Therefore S is isomorphic to one of the sporadic groups Ly and J_4. In the former case $67 \in \omega(S) \setminus \omega(L)$ and in the latter $43 \in \omega(S) \setminus \omega(L)$; a contradiction.

Let $q = 9$. If $L \neq L_4(9)$ then $k_3(\varepsilon 9) \in \omega(S)$, contrary to Lemma 4.1. And if $L = L_4(9)$ then $41 = r_4(9) \in \omega(S)$. Hence $S = F_1$ and $47 \in \omega(S) \setminus \omega(L)$; a contradiction.

Let $q = 7$. If $\varepsilon = +$ then $\pi(S) \subseteq \pi(L_4(7)) = \{2, 3, 5, 7, 19\}$ and $19 = r_3(7) \in \omega(S)$. Using [3] it is easy to verify that this is impossible. For unitary groups, we have $\pi(U_4(7)) = \{2, 3, 5, 7, 43\}$ and therefore $S = J_2$. On the other hand, $5 = r_4(7)$ is not adjacent to 2 in $GK(L)$. But $10 \in \omega(J_2)$; a contradiction.

Let $q = 5$. Then $\pi(S) \subseteq \pi(L_4(5)) = \{2, 3, 5, 13, 31\}$ and hence $S = {}^2F_4(2)'$. On the other hand, $31 = r_3(5) \in \omega(S)$. But $31 \notin \omega({}^2F_4(2)')$; a contradiction. The proposition is proved.

Proposition 4.2. *Let $L = L_n^{\varepsilon}(q)$, with $n \geq 5$, and S is a sporadic group or the Tits group. Then $L = U_5(2)$ and $S \simeq M_{11}$.*

Proof. Note that for $n \neq 6, 10$, there are two different primes i and j in the interval $(n/2, n]$. According to the parity of n either $\{r_i(\varepsilon q), r_j(\varepsilon q), r_n(\varepsilon q)\}$ or $\{r_i(\varepsilon q), r_j(\varepsilon q), r_{n-1}(\varepsilon q)\}$ is a coclique of size 3 in $GK(L)$. By Lemma 1.4 at least one of the numbers $k_i(\varepsilon q)$ and $k_j(\varepsilon q)$ is in $\omega(S)$. Hence there is a prime i such that $i > n/2$ and $k_i(\varepsilon q) \in \omega(S)$.

If $n = 6$ or $n = 10$ then by Lemma 2.1 there is a coclique of three numbers in $GK(L)$ containing $r_3(\varepsilon q)$, $r_5(\varepsilon q)$ for $n = 6$ and $r_5(\varepsilon q)$, $r_7(\varepsilon q)$ for $n = 10$. Therefore in these cases there is a prime i such that $i \geq n/2$ and $k_i(\varepsilon q) \in \omega(S)$.

Suppose that $q \neq 2$.

Let $i \geq 7$. Exploiting Lemma 1.2 we infer that $k_i(\varepsilon q) > q^{i-2} \geq q^5 \geq 3^5 = 243 > 119$; a contradiction.

Let $i = 5$. Then $k_i(\varepsilon q) > q^{i-2} \geq q^5 \geq 5^3 = 125 > 119$ for $q \geq 5$, and so $q = 3$ or $q = 4$. Moreover, $n \leq 2i = 10$. By Lemma 4.1 we find that $q = 4$, $\varepsilon = -$ and $41 = k_5(-4) \in \omega(S)$, and thus $S \simeq F_1$. If $n \leq 10$ then it is easily seen that $31 \in \omega(S) \setminus \omega(L)$. If $n = 10$ then $109 = r_9(-4)$ is not adjacent to 2 in $GK(L)$ but it does not divide the order of S.

Let $i = 3$ and $L = U_n(q)$. Then $q \leq 11$ and $q \neq 9$ by Lemma 4.1. Moreover, $n \leq 2i = 6$. If $n = 5$ then Lemmas 2.3 and 1.4 imply that $k_5(-q) \in \omega(S)$. Then $q = 4$, $k_5(-4) = 41$, and therefore $S \simeq F_1$ and $31 \in \omega(S) \setminus \omega(L)$; a contradiction. Hence $n = 6$. For $q = 11$ it follows from Lemmas 2.3 and 1.4 that $k_6(-11) \in \omega(S)$, contrary to Lemma 4.1. Thus $q \leq 7$.

If $q = 7$ then $k_i(q) = k_3(-7) = 43$ and so $S \simeq J_4$. In this case $37 \in \omega(S) \setminus \omega(L)$; a contradiction.

Let $q = 5$. Lemmas 2.3 and 1.4 imply that $521 = r_5(-5) \in \omega(S)$; a contradiction.

Suppose that $q = 4$. Lemmas 2.2 and 1.4 imply that $41 = r_5(-4) \in \omega(S)$ and therefore $S \simeq F_1$. But then $47 \in \omega(S) \setminus \omega(L)$; a contradiction.

Thus $q = 3$. Observing that $\pi(U_6(3)) = \{2, 3, 5, 7, 13, 61\}$, we deduce that $S \simeq {}^2F_4(2)'$ or $S \simeq J_2$. Let $S \simeq {}^2F_4(2)'$. Since $61, 91 \in \omega(G) \setminus \omega(S)$, it follows that 61 and at least one of the numbers 7 and 13, denote it by r, must lie in $\omega(K)$. Let T be a preimage of a Sylow 5-subgroup of G/K in G. Then T is soluble and by the Hall theorem it has a Hall $\{5, r, 61\}$-subgroup H. The numbers 5, r, and 61 form a coclique in $GK(G)$ and so in $GK(H)$ as well. This contradicts the solubility of H in view of [17, Lemma 1.1]. The case $S \simeq J_2$ is handled in a similar manner with the only difference that r is exactly 13.

Let now $i = 3$ and $L = L_n(q)$. Then $5 \leq n \leq 6$ and $q \leq 7$ by Lemma 4.1.

If $q = 7$ then Lemmas 2.3 and 1.4 imply that $k_5(7) \in \omega(S)$; a contradiction. If $q = 5$ and $L = L_5(5)$ then $k_5(5) \in \omega(S)$; a contradiction. If $L = L_6(5)$ then $31 = k_3(5) \in \omega(S)$. It is not hard to check by [3] that whenever 31 divides the order of a sporadic group, so does at least one of the numbers 19 or 37. But 19 and 37 are not in $\omega(L_6(5))$; a contradiction. Thus $q = 3$ and Lemmas 2.3 and 1.4 imply that $121 = k_5(3) \in \omega(S)$; a contradiction.

It remains to consider the case of $q = 2$. Since $L \notin \mathscr{L}$, we can assume that $L = U_n(2)$ and $n \neq 6$. If $n \geq 13$ then $i \geq 11$ and hence $k_i(-2) > 2^9/2 > 119$; a contradiction. For $n = 11, 12$ it follows from Lemmas 2.2 and 1.4 that $683 = r_{11}(-2) \in \omega(S)$; a contradiction. Similarly, for $n = 7, 8$ we infer that $43 = r_7(-2) \in \omega(G)$. Then $S \simeq J_4$ and so $37 \in \omega(G)$; a contradiction. Let $n = 9, 10$. The previous argument implies that $r_7(-2) \notin \omega(S)$, and it follows from (2) of Lemma 1.4 that $r_5(-2) = 11$, $r_8(-2) = 17$, and $r_9(-2) = 19$ are in $\omega(S)$. Moreover, all prime divisors of the order of S are at most 31. It is easily seen from [3] that there are no sporadic groups with such properties. Hence

$n = 5$. Since $\pi(U_5(2)) = \{2, 3, 5, 11\}$, the group S can be isomorphic only to M_{11} or M_{12}. Note that $10 \in \omega(M_{12}) \setminus \omega(U_5(2))$. Thus $S \simeq M_{11}$.

The proposition and Theorem 2 are proved.

§ 5. Proof of Theorem 3

Let $L = L_n^\varepsilon(q)$, where $n \geq 4$, $\varepsilon \in \{+, -\}$ and $q = p^\alpha$. Let G be a finite group isospectral to L, and K be the soluble radical of G. Applying Lemma 1.4 under conditions of Theorem 3 we infer that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$, where S is a simple group of Lie type over a field of characteristic p.

Lemma 5.1. *If L is one of the groups $U_6(4)$, $U_7(2)$, and $U_7(4)$, the conclusion of Theorem 3 holds for L.*

Proof. Let $L = U_6(4)$. By Lemma 2.2 the numbers $r_6(-4) = 7$ and $r_5(-4) = 41$ are not adjacent to 2 in $GK(L)$ and hence they are in $\pi(S)$. Thus $7, 41 \in \pi(S) \subseteq \pi(L) = \{2, 3, 5, 7, 13, 17, 41\}$. In [26, Table 1] one can find the set of simple groups for which the largest divisor of the order is equal to 41. This set contains five groups of Lie type over fields of characteristic 2 but of them only $U_6(4)$ fails to have 7 in its spectrum.

If $L = U_7(2)$ then $r_7(-2) = 43$ is not adjacent to 2 in $GK(L)$. Examining the set of simple groups satisfying the condition $43 \in \pi(S) \subseteq [2; 43]$ from [26, Table 1], we see that it contains only five groups of Lie type over fields of characteristic 2 but of them only $U_7(2)$ fails to have 17 in its spectrum.

If $L = U_7(4)$ then $r_7(-4) = 113$ is not adjacent to 2 in $GK(L)$. It follows from [26, Table 1] that the only simple group of Lie type over a field of characteristic 2 with $113 \in \pi(S) \subseteq [2; 113]$ is the group $U_7(4)$.

In all considered cases $S \simeq L$. The lemma is proved.

Thus we may assume that L is different not only from groups of $\mathscr{L}$ but also from $U_6(4)$, $U_7(2)$, and $U_7(4)$. In particular, p is odd if $\varepsilon = +$ or $n = 4$. We choose two prime numbers in $\pi(L)$ as follows. If $\varepsilon = +$ then put $r_n = r_{n\alpha}(p)$ and $r_{n-1} = r_{(n-1)\alpha}(p)$. If $\varepsilon = -$ then put $r_n = r_{\nu(n)\alpha}(p)$ and $r_{n-1} = r_{\nu(n-1)\alpha}(p)$, where the function $\nu(x)$ is defined in (3). Note that these primitive divisors exist in view of conditions on L and they are greater than 3 since $n \geq 4$. Using (4) and (5) we calculate $e(r_n, \varepsilon q) = n$ and $e(r_{n-1}, \varepsilon q) = n - 1$. Hence r_n and r_{n-1} are not adjacent to p in $GK(L)$ by Lemma 2.2.

Lemma 5.2. *S is not isomorphic to $L_2(p)$.*

Proof. Suppose that $S \simeq L_2(p)$. By Lemma 2.3 one of r_n and r_{n-1} is not adjacent to 2 in $GK(L)$ and hence lies in $\pi(S)$ by Lemma 1.4. On the other hand, each of $n\alpha$, $(n-1)\alpha$, $\nu(n)\alpha$, and $\nu(n-1)\alpha$ is greater than 3 while $\pi(S)$ consists of p and divisors of $p^2 - 1$. This contradicts the definition of primitive divisor. The lemma is proved.

Lemma 5.3. *If $r \in \pi(L)$ is not adjacent to p in $GK(L)$ then it does not divide $|K| \cdot |\overline{G}/S|$; in particular, $t(p, S) \geq t(p, L)$.*

Proof. Let $r \in \pi(L)$, $rp \notin \pi(L)$, and $r \notin \pi(S)$. By Lemma 1.4 we can assume that p is odd. Lemma 2.2 and the condition $n \geq 4$ imply that $r > 3$.

Suppose that $r \in \pi(\overline{G}/S)$. Since $r \notin \pi(S)$ and $r > 3$, there is a field automorphism of S of order r in G. In all groups of Lie type over a field of characteristic p the centralizer of a field automorphism contains an element of order p, so we have $rp \in \omega(G)$; a contradiction.

Suppose that $r \in \pi(K)$. Since S differs from $L_2(p)$ by Lemma 5.2, its Sylow p-subgroup includes an elementary abelian subgroup of order p^2. By Lemma 1.5 there is an element of order pr in G; a contradiction. The lemma is proved.

Let S be a group over a field of order p^β (in the notation of [3]). Denote by $e(p, S)$ the set $\{e(r, p^\beta) \mid r \in \pi(S), r \neq p, r > 3, pr \notin \omega(S)\}$. By Lemma 5.3 the numbers r_n and r_{n-1} lie in $\pi(S)$ and are not adjacent to p in $GK(S)$. Therefore $e(r_n, p^\beta)$ and $e(r_{n-1}, p^\beta)$ are in $e(p, S)$.

Let $e_n = e(r_n, p^\beta)$ and $e_{n-1} = e(r_{n-1}, p^\beta)$. Since $r_n \in \pi(S)$, the number $k_{e_n}(p^\beta)$ divides the order of S. If a primitive divisor $r_{e_n\beta}(p)$ exists, then it divides $k_{e_n}(p^\beta)$ and hence lies in $\pi(S) \subseteq \pi(L)$. By the same argument if $r_{e_{n-1}\beta}(p)$ exists then it also lies in $\pi(L)$.

Let $\varepsilon = +$. By the definition of primitive divisor, $e_{n-1}\beta = a(n-1)\alpha$ for some positive integer a. Since $e_{n-1}\beta \geq 3$ and p is odd, a primitive divisor $r_{e_{n-1}\beta}(p)$ exists and thus lies in $\pi(L)$. Then $e(r_{e_{n-1}\beta}(p), q) \leq n$ by (2). On the other hand, $e(r_{e_{n-1}\beta}(p), q) = e(r_{a(n-1)\alpha}(p), p^\alpha) = a(n-1)$. Thus $a(n-1) \leq n$ and so $a = 1$ and $e_{n-1}\beta = (n-1)\alpha$. By the same argument $e_n\beta = n\alpha$. In particular, $e_n/e_{n-1} = n/(n-1)$.

Let now $\varepsilon = -$. Then $e_n\beta = a\nu(n)\alpha$ and $e_{n-1}\beta = b\nu(n-1)\alpha$ for some positive integers a and b. In view of conditions on L primitive divisors $r_{e_n\beta}(p)$ and $r_{e_{n-1}\beta}(p)$ exist and lie in $\pi(S) \subseteq \pi(L)$. By (2) this implies that $e(r_{e_n\beta}(p), -q)$ and $e(r_{e_{n-1}\beta}(p), -q)$ do not exceed n. On the other hand, $e(r_{e_n\beta}(p), -q) = \nu(e(r_{e_n\beta}(p), p^\alpha)) = \nu(a\nu(n))$ and $e(r_{e_{n-1}\beta}(p), -q) = \nu(e(r_{e_{n-1}\beta}(p), p^\alpha)) = \nu(b\nu(n-1))$ by (5). Hence $\nu(a\nu(n)) \leq n$ and $\nu(b\nu(n-1)) \leq n$. Examining these inequalities according to the remainder of n modulo 4, we infer that $a \leq 2$ for $n \equiv 2 \pmod 4$ and $a = 1$ otherwise, and $b \leq 2$ for $n \equiv 3 \pmod 4$ and $b = 1$ otherwise. Thus $e_n/e_{n-1} = a\nu(n)/b\nu(n-1)$, where $a, b \in \{1, 2\}$, $n/4(n-1) \leq e_n/e_{n-1} \leq 4n/(n-1)$ and $e_n/e_{n-1} \neq n/(n-1)$.

Therefore for each value of ε there must be two numbers e_n and e_{n-1} in $e(p, S)$ such that the ratio e_n/e_{n-1} belongs to the set

$$R_n = \{2^\gamma n/(n-1) \mid \gamma = -2, -1, 0, 1, 2\},$$

and $\gamma = 0$ if and only if $\varepsilon = +$.

Lemma 5.4. *Let $n \geq 4$ and $m \geq 2$ be natural numbers and let δ be an integer. If $2^\delta n/(n-1) = m/(m-1)$ then $m = n$ and $\delta = 0$. If $2^\delta n/(n-1) = (m-1)/m$ then $n = 4$, $m = 3$, and $\delta = -1$.*

Proof. If $\delta < 0$ then $2^\delta n/(n-1) \leq n/(2n-2) < 1 < m/(m-1)$. If $\delta > 0$ then $2^\delta n/(n-1) \geq 2n/(n-1) > 2 \geq m/(m-1)$. Thus $2^\delta n/(n-1) = m/(m-1)$ yields $\delta = 0$ and so $n = m$.

If $\delta < -1$ then $2^\delta n/(n-1) \leq n/(4n-4) < 1/2 \leq (m-1)/m$. If $\delta > -1$ then $2^\delta n/(n-1) \geq n/(n-1) > 1 > m-1/m$. Hence $2^\delta n/(n-1) = (m-1)/m$ yields $\delta = -1$ and $n/(2n-2) = (m-1)/m$. If n is odd then both sides of the last equality are irreducible fractions. Therefore $2n-2-n = 1$ and $n = 3$, which is not the case. If n is even then $(n/2)/(n-1)$ is an irreducible fraction and hence $n-1-n/2 = 1$. Then $n = 4$ and $m = 3$. The lemma is proved.

Deriving a corollary of Lemma 5.4 we show that R_n does not contain numbers of the form 2^δ and $2^\delta(m-1)/m$, where δ is an integer and $m \geq 4$.

Below we will consider all groups of Lie type one at a time. We use results of [20, § 3] to find the set $e(p, S)$.

Suppose that $S \simeq L_m(p^\beta)$, where $m \geq 3$ or $\beta > 1$. Then $e(p, S) = \{m, m-1\}$. Hence $e_n/e_{n-1} = m/(m-1)$ or $e_n/e_{n-1} = (m-1)/m$. Let $2^\gamma n/(n-1) = m/(m-1)$. It follows from Lemma 5.4 that $n = m$ and $\gamma = 0$. Thus $\varepsilon = +$ and $S \simeq L$. Let $2^\gamma n/(n-1) = (m-1)/m$. Then $n = 4$, $m = 3$ and $\gamma = -1$, thus $\varepsilon = -$ and $a = 1$. Now we conclude from $(m-1)\beta = e_n\beta = \nu(n)\alpha = 4\alpha$ that $\beta = 2\alpha$. Hence $L = U_4(q)$, $S \simeq L_3(q^2)$, and $r_3(q) \in \pi(S) \setminus \pi(L)$; a contradiction.

Suppose that $S \simeq U_m(p^\beta)$, where $m \geq 3$. Then $e(p, S) = \{\nu(m), \nu(m-1)\}$. Let $2^\gamma n/(n-1) = \nu(m)/\nu(m-1)$. By Lemma 5.4 we have $n = m$ and $\gamma \neq 0$. Thus $\varepsilon = -$ and $S \simeq L$. Let $2^\gamma n/(n-1) = \nu(m-1)/\nu(m)$. Then $m = 3$ and $\nu(m-1)/\nu(m) = (m-1)/4m$ and so $\gamma + 2 = -1$, contrary to the inequality $\gamma \geq -2$.

Suppose that $S \simeq O_{2m+1}(p^\beta)$ or $S \simeq S_{2m}(p^\beta)$. Then $e(p, S) \subseteq \{m, 2m\}$. Therefore a ratio of any two elements of $e(p, S)$ is a power of 2 and cannot be in R_n; a contradiction. By the same argument S differs from the groups of types G_2, ${}^3D_4$, ${}^2F_4$, and ${}^2B_2$, for otherwise $e(p, S)$ is one of the sets $\{3, 6\}$, $\{12\}$, $\{6, 12\}$, and $\{1, 4\}$.

Suppose that $S \simeq O^+_{2m}(p^\beta)$, where $m \geq 4$. Then $e(p, S) = \{2m-2, m-1\}$ for even m and $e(p, S) = \{2m-2, m\}$ for odd m. The ratio e_n/e_{n-1} can be neither $2(m-1)/m$ nor 2, therefore, m is odd and $2^\gamma n/(n-1) = m/(2m-2)$. Then $n = m$ and $\gamma = -2$, which implies that n is odd and $\varepsilon = -$. Hence $e_n/e_{n-1} = \nu(n)/b\nu(n-1) = 2n/b\nu(n-1) \geq 2n/(n-1)$ and γ must be positive; a contradiction.

Suppose that $S \simeq O^-_{2m}(p^\beta)$, where $m \geq 4$. Then $e(p,S) = \{2m, 2m-2, m-1\}$ for even m and $e(p,S) = \{2m, 2m-2\}$ for odd m. The ratio e_n/e_{n-1} cannot be a power of 2 and cannot be equal to $(m-1)/2m$ or $(m-1)/m$, and so e_n/e_{n-1} is one of the numbers $2m/(m-1)$ and $m/(m-1)$. Let m be even and $2^\gamma n/(n-1) = 2m/(m-1)$. Then $n = m$ and $\gamma = 1$, whence $\varepsilon = -$. Hence n is even and $e_n/e_{n-1} = a\nu(n)/\nu(n-1) = a\nu(n)/2(n-1) \leq n/2(n-1)$; therefore, γ must be negative; a contradiction.

Let $2^\gamma n/(n-1) = m/(m-1)$. Then $n = m$ and $\gamma = 0$, whence $\varepsilon = +$. Now from $2m\beta = e_n\beta = n\alpha$ we conclude that $\alpha = 2\beta$. Hence $L = L_n(q_0^2)$ and $S \simeq O^-_{2n}(q_0)$, where $q_0^2 = q$. If n is odd then $e\big(r_n(q_0), q_0^2\big) = n$ and Lemma 2.2 implies that $r_n(q_0)$ is not adjacent to p in $GK(L)$ and so by Lemma 5.3 it must divide the order of S but this is not the case. If n is even then both $r_{2(n-1)}(q_0)$ and $r_{n-1}(q_0)$ are not adjacent to p in $GK(L)$, thus by Lemma 5.3 they are coprime to $|K| \cdot |\overline{G}/S|$. This contradicts the fact that L contains an element of order $r_{2(n-1)}(q_0)r_{n-1}(q_0)$ and S does not.

Suppose that $S \simeq E_8(p^\beta)$. Then $e(p,S) = \{30, 24, 20, 15\}$. The ratio e_n/e_{n-1} is neither a power of 2 nor a number of the form $2^\delta(m-1)/m$ for $m \geq 4$; therefore,

$$e_n/e_{n-1} \in \{2/3, 4/3, 5/4, 6/5, 5/8\}.$$

If $2^\gamma n/(n-1) = 5/8$ then $n = 5$, $\gamma = -1$ and $\varepsilon = -$. Hence $e_n/e_{n-1} = \nu(n)/\nu(n-1) = 2n/(n-1)$ and $\gamma = 1$; a contradiction.

Let $2^\gamma n/(n-1) = 2/3$ or $2^\gamma n/(n-1) = 4/3 = 2 \cdot 2/3$. Then $n = 4$. If $\varepsilon = +$ then $e_n/e_{n-1} = 4/3$ and $20\beta = e_n\beta = 4\alpha$ yields $\alpha = 5\beta$, thus $L = L_4\big(q_0^5\big)$ and $S \simeq E_8(q_0)$, where $q_0^5 = q$. If $\varepsilon = -$ then $e_n/e_{n-1} = \nu(n)/\nu(n-1) = 2/3$ and from $20\beta = e_n\beta = 4\alpha$ we calculate $\alpha = 5\beta$. Therefore $L = U_4\big(q_0^5\big)$ and $S \simeq E_8(q_0)$, where $q_0^5 = q$. Similarly, if $e_n/e_{n-1} = 5/4$ then $L = L_5\big(q_0^6\big)$ and $S \simeq E_8(q_0)$, where $q_0^6 = q$, and if $2^\gamma n/(n-1) = 6/5$ then $L = L_6\big(q_0^4\big)$ and $S \simeq E_8(q_0)$, where $q_0^4 = q$. In any case $r_{14}(q_0) \in \pi(S) \setminus \pi(L)$.

Suppose that $S \simeq E_7(p^\beta)$. Then $e(p,S) = \{18, 14, 9, 7\}$. The ratio e_n/e_{n-1} is not a power of 2 and therefore

$$e_n/e_{n-1} \in \{2^\delta \cdot 9/7, 2^\delta \cdot 7/9 \mid \delta = -1, 0, 1\}.$$

Let $2^\gamma n/(n-1) = 2^\delta \cdot 9/7$. Then $2^\gamma \cdot 7n = 2^\delta \cdot 9(n-1)$. If $\gamma \geq \delta$ then $n = 9$ and $2^\gamma \cdot 7 = 2^\delta \cdot 8$, which is impossible. If $\gamma \leq \delta$ then $n - 1 = 7$ and $2^\gamma \cdot 8 = 2^\delta \cdot 9$, which is impossible. Let $2^\gamma n/(n-1) = 2^\delta \cdot 7/9$. Then $\gamma \leq \delta$, whence $n - 1 = 9$ and $2^\gamma \cdot 10 = 2^\delta \cdot 7$, and we arrive at a contradiction again.

Suppose that $S \simeq E_6(p^\beta)$ or $S \simeq F_4(p^\beta)$. Then $\{12, 8\} \subseteq e(p,S) \subseteq \{12, 9, 8\}$. Hence $e_n/e_{n-1} \in \{9/8, 2/3, 4/3\}$. Let $2^\gamma n/(n-1) = 9/8$. Then $n = 9$ and $\gamma = 0$, whence $\varepsilon = +$. From $9\beta = e_n\beta = 9\alpha$ we calculate $\beta = \alpha$, and so $L = L_9(q)$, $S \simeq E_6(q)$, and $r_{12}(q) \in \pi(S) \setminus \pi(L)$; a contradiction.

Let $2^\gamma n/(n-1) = 2/3$ or $2^\gamma n/(n-1) = 4/3$. Then $n = 4$. If $\varepsilon = +$ then $e_n/e_{n-1} = 4/3$ and so S is of type E_6. It follows from $12\beta = e_n\beta = 4\alpha$ that $\alpha = 3\beta$, therefore, $L = L_4\big(q_0^3\big)$ and $S \simeq E_6(q_0)$, where $q_0^3 = q$. In this case $r_8(q_0) \in \pi(S) \setminus \pi(L)$. If $\varepsilon = -$ then $e_n/e_{n-1} = \nu(n)/\nu(n-1) = 2/3$ and from $8\beta = e_n\beta = 4\alpha$ we conclude that $\alpha = 2\beta$. Hence $L = U_4\big(q_0^2\big)$ and S is isomorphic to $E_6(q_0)$ or $F_4(q_0)$, where $q_0^2 = q$. But then $r_6(q_0) \in \pi(S) \setminus \pi(L)$.

Suppose that $S \simeq {}^2E_6(p^\beta)$. Then $e(p,S) = \{18, 12, 8\}$ and $e_n/e_{n-1} \in \{2/3, 9/4\}$. Let $2^\gamma n/(n-1) = 2/3$. Then $n = 4$ and $\varepsilon = -$. If $e_n = 12$ then $\alpha = 3\beta$ and $L = U_4\big(q_0^3\big)$, $S \simeq {}^2E_6(q_0)$, where $q_0^3 = q$. If $e_n = 8$ then $\alpha = 2\beta$, therefore, $L = U_4\big(q_0^2\big)$ and $S \simeq {}^2E_6(q_0)$, where $q_0^2 = q$. In any case $r_{10}(q_0) \in \pi(S) \setminus \pi(L)$. Let $2^\gamma n/(n-1) = 9/4$. Then $n = 9$ and $\gamma = -1$, whence $\varepsilon = -$. From $18\beta = e_n\beta = \nu(n)\alpha = 18\alpha$ we calculate that $\alpha = \beta$. Thus $L = U_9(q)$ and $S \simeq {}^2E_6(q)$, in which case $r_{12}(q) \in \pi(S) \setminus \pi(L)$.

Suppose that $S \simeq {}^2G_2(3^\beta)$, where $\beta \geq 3$ is odd. Then $e(3,S) = \{6, 2, 1\}$. Since $6 = 8 \cdot 3/4$ and $3 = 4 \cdot 3/4$, the ratio e_n/e_{n-1} can be equal to $1/6$ or $1/3$. Thus $2^\gamma n/(n-1) = 1/6 = 1/8 \cdot 4/3 = 1/4 \cdot 2/3$ or $2^\gamma n/(n-1) = 1/3 = 1/4 \cdot 4/3 = 1/2 \cdot 2/3$, whence $n = 4$ and $\gamma < 0$. This implies that $\varepsilon = -$. Then $e_n/e_{n-1} = \nu(n)/\nu(n-1) = 2/3$; a contradiction.

Theorem 3 is proved.

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