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# ON FINITE GROUPS ISOSPECTRAL TO SIMPLE SYMPLECTIC AND ORTHOGONAL GROUPS

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**Abstract:** The spectrum of a finite group is the set of its element orders. Two groups are said to be *isospectral* if their spectra coincide. We deal with the class of finite groups isospectral to simple and orthogonal groups over a field of an arbitrary positive characteristic $p$. It is known that a group of this class has a unique nonabelian composition factor. We prove that this factor cannot be isomorphic to an alternating or sporadic group. We also consider the case where this factor is isomorphic to a group of Lie type over a field of the same characteristic $p$.

**Keywords:** finite group, spectrum of a group, simple group, symplectic group, orthogonal group, composition factor

The *spectrum* $\omega(G)$ of a finite group $G$ is the set of its element orders. Two groups are said to be *isospectral* if their spectra coincide. A finite group $L$ is *recognizable by spectrum* if every finite group $G$ with $\omega(G) = \omega(L)$ is isomorphic to $L$. More generally, a group $L$ is a *group for which the problem of recognition by spectrum is solved* if the number (up to isomorphism) of finite groups isospectral to $L$ is known. The recent results on the problem of recognition by spectrum can be found in the survey by V. D. Mazurov [1].

The present paper addresses the problem of recognition for finite simple symplectic and orthogonal groups. Within this class, the problem is solved for the groups $B_2(q)$ [2, 3], $B_3(2)$ [4, 5], $B_3(3)$ [5], $B_{2^m}(2)$ [6, 7], $C_3(3)$ [2], $D_4(2)$ [4, 5], $D_4(3)$ [5], $D_5(2)$ [8], ${}^2D_4(2)$ [5], ${}^2D_p(3)$ [9], ${}^2D_{2^m+1}(2)$ [7], ${}^2D_{2^m+1}(3)$ [10], where $m \geq 2$ and $p$ is an odd prime. Also some series of symplectic and orthogonal groups are proved to be quasirecognizable by spectrum (see [7, 11, 12]). A finite nonabelian simple group $L$ is *quasirecognizable by spectrum* if each finite group $G$ with $\omega(G) = \omega(L)$ has a unique nonabelian composition factor and this factor is isomorphic to $L$.

The results of the paper provide restrictions on the composition structure of a finite group isospectral to a simple symplectic or orthogonal group.

**Theorem 1.** *Let $L$ be one of the simple groups $B_n(q)$ with $n \geq 2$ and $(n, q) \neq (2, 3)$, $C_n(q)$ with $n \geq 3$, and $D_n(q)$, ${}^2D_n(q)$ with $n \geq 4$. Then there are no alternating groups among nonabelian composition factors of finite groups isospectral to $L$.*

**Theorem 2.** *Let $L$ be one of the simple groups $B_n(q)$ with $n \geq 2$ and $(n, q) \neq (2, 3)$, $C_n(q)$ with $n \geq 3$, and $D_n(q)$, ${}^2D_n(q)$ with $n \geq 4$. Then there are no sporadic groups nor the Tits group ${}^2F_4(2)'$ among nonabelian composition factors of finite groups isospectral to $L$.*

**Theorem 3.** *Let $q$ be a power of a prime $p$, let $L$ be one of the simple groups $B_n(q)$ with $n \geq 2$ and $(n, q) \neq (2, 3)$, $C_n(q)$ with $n \geq 3$, and $D_n(q)$, ${}^2D_n(q)$ with $n \geq 4$, and let $G$ be the finite group with $\omega(G) = \omega(L)$. Suppose that there is a factor $S$ among nonabelian composition factors of $G$ which is isomorphic to a group of Lie type over a field of characteristic $p$.*

*(1) If $L = B_2(q)$, where $q > 3$, then $S \in \{A_1(q^2), B_2(q)\}$.*

*(2) If $L \in \{B_3(q), C_3(q), D_4(q)\}$ then $S \in \{A_1(q^3), B_3(q), C_3(q), D_4(q), G_2(q)\}$.*

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(3) *If* $n \geq 4$ *and* $L \in \{B_n(q), C_n(q), {}^2D_n(q)\}$ *then* $S \in \{B_n(q), C_n(q), {}^2D_n(q)\}$.

(4) *If* $n \geq 6$ *is even and* $L = D_n(q)$ *then* $S \in \{B_{n-1}(q), C_{n-1}(q), D_n(q)\}$.

(5) *If* $n \geq 5$ *is odd and* $L = D_n(q)$ *then* $S \simeq L$.

Note that our choice of symplectic and orthogonal groups as the topic of study is partially caused by Question 12.39 of the Kourovka Notebook [13] which concerns the validity of Shi's conjecture first appeared in [14]. The conjecture says that every finite simple group is uniquely determined by its spectrum and order in the class of finite groups. At present the conjecture is verified for all simple groups except for symplectic and some orthogonal groups (see [15]).

## § 1. Preliminaries

Our notation for sporadic and simple groups of Lie type follows [16]. In this connection if a group of Lie type $L$ is denoted by ${}^tX_n(q)$ [16, pp. xiv–xv] then we say that $L$ is a *group of rank* $n$ *over a field of order* $q$. In particular, the rank of a twisted group is supposed equal to that of the associated untwisted group. The alternating group of degree $n$ is denoted by $\mathrm{Alt}_n$.

Given a natural number $n$, let $\pi(n)$ denote the set of prime divisors of $n$, and $n_r$, where $r$ is a prime, denotes the $r$-part of $n$; i.e., the largest power of $r$ that divides $n$, while $n_{r'}$ denotes the $r'$-part of $n$, i.e., the ratio $n/n_r$. If $n$ and $m > 2$ are coprime natural numbers, then we write $e(m, n)$ to denote the multiplicative order of $n$ modulo $m$. Given an odd $n$, we put $e(2, n) = 1$ if $n \equiv 1 \pmod 4$ and put $e(2, n) = 2$ if $n \equiv 3 \pmod 4$.

Let $n > 1$. A prime $r$ is said to be a *primitive prime divisor* of the difference $n^i - 1$ if $e(r, n) = i$. The existence of primitive divisors for almost all pairs of $n$ and $i$ was established by Zsigmondy.

**Lemma 1.1** [17]. *Let* $n > 1$ *be a natural number. Then for every natural number* $i$ *there is a prime* $r$ *with* $e(r, n) = i$, *except when* $n = 2$ *and* $i = 1$, $n = 3$ *and* $i = 1$, $n = 2$ *and* $i = 6$.

In what follows the notation $r_i(n)$ means a primitive prime divisor of $n^i - 1$ if such exist. The product of all primitive divisors of $n^i - 1$ taken with multiplicities is said to be the *greatest primitive divisor* and denoted by $k_i(n)$. Note that for a divisor, the property of being primitive depends on the pair $(n, i)$ and is not determined by the number $n^i - 1$. For example, $k_6(2) = 1$, $k_3(4) = 7$, $k_2(8) = 9$, and $k_1(64) = 63$.

It is not hard to check that $k_1(n) = n - 1$ if $n \not\equiv 3 \pmod 4$, and $k_1(n) = (n - 1)/2$ if $n \equiv 3 \pmod 4$, and also that $k_2(n) = (n + 1)/(2, n - 1)$ if $n \not\equiv 3 \pmod 4$, and $k_2(n) = n + 1$ if $n \equiv 3 \pmod 4$. It follows from [18] that for $i > 2$

$$k_i(n) = \Phi_i(n)/(r, \Phi_{i_{r'}}(n)),$$

where $\Phi_i(x)$ is the $i$th cyclotomic polynomial and $r$ is the largest prime dividing $i$, and if $i_{r'}$ does not divide $r - 1$ then $(r, \Phi_{i_{r'}}(n)) = 1$.

The *Gruenberg–Kegel graph* $GK(G)$, or the *prime graph*, of $G$ is the graph with vertex set $\pi(G)$ in which two distinct vertices $p$ and $q$ are adjacent if and only if $pq \in \omega(G)$. The number of connected components of $GK(G)$ is denoted by $s(G)$, and the connected components are denoted by $\pi_i(G)$ with $1 \leq i \leq s(G)$. If $G$ has even order then by default $2 \in \pi_1(G)$. According to this partition, $\omega_i(G)$ is the subset of $\pi_i(G)$-numbers of $\omega(G)$ for every $1 \leq i \leq s(G)$. The structure of finite groups with disconnected prime graph is described by Gruenberg and Kegel.

**Lemma 1.2** [19]. *If* $G$ *is a finite group with* $s(G) > 1$ *then one of the following holds:*

(1) $s(G) = 2$, $G$ *is a Frobenius group;*

(2) $s(G) = 2$, $G = ABC$, *where* $A$ *and* $AB$ *are normal subgroups of* $G$, $B$ *is a normal subgroup of* $BC$, *and* $AB$ *and* $BC$ *are Frobenius groups;*

(3) *there is a nonabelian simple group* $S$ *such that* $S \leq \overline{G} = G/K \leq \mathrm{Aut}\, S$ *for some nilpotent normal subgroup* $K$ *of* $G$; *moreover,* $K$ *and* $\overline{G}/S$ *are* $\pi_1(G)$-*groups,* $s(S) \geq s(G)$, *and for every* $1 < i \leq s(G)$ *there is* $1 < j \leq s(S)$ *such that* $\omega_i(G) = \omega_j(S)$.

Finite simple groups with disconnected prime graph were described by Williams [19] and Kondrat′ev [20]. The complete list of these groups, with corrected inaccuracies, can be found in [2, Tables 1a–1c].

It follows from the results of Williams and Kondrat′ev that if $S$ is a simple group and $s(S) > 1$, then for every $1 < i \le s(S)$ the set $\omega_i(S)$ has the unique element maximal under divisibility [21, Lemma 4]. In the tables mentioned above and in the present paper, this maximal element is denoted by $n_i(S)$.

Recall that an *independent set of vertices*, or *coclique*, in a graph $\Gamma$ is a set of vertices that are pairwise nonadjacent to each other in $\Gamma$. We write $t(\Gamma)$ to denote the independence number of $\Gamma$, i.e., the maximal number of vertices in its cocliques. Given a group $G$, put $t(G) = t(GK(G))$. By analogy, for each prime $r$, define the $r$-independence number $t(r, G)$ to be the maximal number of vertices in cocliques of $GK(G)$ containing the vertex $r$. In [22, 23], for every finite nonabelian simple group, an adjacency criterion of its prime graph is developed and all cocliques of maximal size in this graph are found.

**Lemma 1.3** [24, 25]. *Let $L$ be a finite nonabelian simple group such that $t(L) \ge 3$ and $t(2, L) \ge 2$, and let $G$ be a finite group with $\omega(G) = \omega(L)$. Then the following hold:*

(1) *There exists a nonabelian simple group $S$ such that $S \le \overline{G} = G/K \le \operatorname{Aut} S$, where $K$ is the maximal normal soluble subgroup of $G$.*

(2) *For every coclique $\rho$ of $GK(G)$ of size at least 2, at most one prime of $\rho$ lies in $\pi(K) \cup \pi(\overline{G}/S)$. In particular, $t(S) \ge t(G) - 1$.*

(3) *Every prime $r \in \pi(G)$ not adjacent to 2 in $GK(G)$ does not divide the product $|K| \cdot |\overline{G}/S|$. In particular, $t(2, S) \ge t(2, G)$.*

If $\Gamma$ is a prime graph and $\pi$ is a set of natural numbers, we write $\Gamma \setminus \pi$ to denote the maximal subgraph of $\Gamma$ all whose vertices do not lie in $\pi$. Observe that (2) of Lemma 1.3 yields, alongside the inequality $t(S) \ge t(G) - 1$, the inequality $t(GK(S) \setminus \pi) \ge t(GK(G) \setminus \pi) - 1$ for every set of primes $\pi$.

**Lemma 1.4.** *Let $G$ be a finite group, let $K$ be a normal $r$-subgroup of $G$, and let $G/K$ be a noncyclic abelian $p$-group, where $r$ and $p$ are different primes. Then $rp \in \omega(G)$.*

Proof. This is a direct corollary of [26, Chapter 5, Theorem 3.16].

## § 2. Finite Groups Isospectral to Symplectic and Orthogonal Groups

**Lemma 2.1.** *Let $L$ be one of the simple groups $B_n(q)$ with $n \ge 2$ and $(n, q) \ne (2, 3)$, $C_n(q)$ with $n \ge 3$, $D_n(q)$, ${}^2D_n(q)$ with $n \ge 4$, and let $G$ be a finite group with $\omega(G) = \omega(L)$. Then there exists a simple nonabelian group $S$ such that*

$$S \le \overline{G} = G/K \le \operatorname{Aut} S,$$

*where $K$ is the soluble radical of $G$. Furthermore, $G$, $K$ and $S$ satisfy* (2) *and* (3) *of Lemma* 1.3.

Proof. If $n > 2$ and $L \ne D_4(2)$ then, as is shown in [22], $L$ satisfies $t(L) \ge 3$ and $t(2, L) \ge 2$. So the assertion follows from Lemma 1.3. Let $n = 2$ or $L = D_4(2)$. Then $L$ has prime graph disconnected. Therefore, the claim follows from the Gruenberg–Kegel theorem and the main result of [27] together with the fact that in this case $t(L) = 2$. The lemma is proved.

Some adjacency criterion for prime graphs of symplectic and orthogonal groups is formulated in [22, Propositions 3.1, 4.3, and 4.4] and [23, Propositions 2.4 and 2.5]. The formulation involves the function $\eta : \mathbb{N} \to \mathbb{N}$ defined as

$$\eta(n) = \begin{cases} n, & \text{if } n \text{ is odd;} \\ n/2, & \text{if } n \text{ even.} \end{cases}$$

**Lemma 2.2.** *Let $L$, $G$, $S$, and $K$ be as in Lemma* 2.1.

(1) *Suppose that $L = B_n(q)$ or $L = C_n(q)$, where $n \ge 3$ and $(n, q) \ne (3, 2)$. If there exists $i$ such that $n/2 < \eta(i) \le n$ and $k_i(q) \notin \omega(S)$, then for every $j \ne i$ with $n/2 \le \eta(j) \le n$ the number $k_j(q)$ is coprime to $|K| \cdot |\overline{G}/S|$ and lies in $\omega(S)$.*

(2) *Suppose that $L = D_n(q)$, where $n \ge 4$ and $(n, q) \ne (4, 2)$. If there exists $i \ne 2n$ such that $n/2 < \eta(i) \le n$ and $k_i(q) \notin \omega(S)$, then for every $j \notin \{i, 2n\}$ with $n/2 \le \eta(j) \le n$ the number $k_j(q)$ is coprime to $|K| \cdot |\overline{G}/S|$ and lies in $\omega(S)$.*

(3) *Suppose that $L = {}^2D_n(q)$, where $n \geq 4$ and $(n,q) \neq (4,2), (5,2)$. If there exists $i \neq n$ such that $n/2 < \eta(i) \leq n$ and $k_i(q) \notin \omega(S)$, then for every $j \notin \{i,n\}$ with $n/2 \leq \eta(j) \leq n$ the number $k_j(q)$ is coprime to $|K| \cdot |\overline{G}/S|$ and lies in $\omega(S)$.*

Proof. Since $k_i(q) \notin \omega(S)$, some $r_i = r_i(q)$ divides $|K| \cdot |\overline{G}/S|$. Let $r_j = r_j(q)$ be a primitive prime divisor of the difference $q^j - 1$.

(1) Put $r = r_{2n}(q)$ if $2n \notin \{i,j\}$. If $2n \in \{i,j\}$ and $2(n-1) \notin \{i,j\}$ then put $r = r_{2(n-1)}(q)$. If $\{i,j\} = \{2n, 2(n-1)\}$ then put $r = r_n(q)$ for odd $n$ and $r = r_{n-1}(q)$ for even $n$. If $(n,q) \neq (4,2)$, the required divisors exist. By [23, Proposition 2.4], the numbers $r_i, r_j$, and $r$ are pairwise nonadjacent in $GK(G)$. So, it follows from (2) of Lemma 1.3 that $r_j$ does not divide $|K| \cdot |\overline{G}/S|$. Thus, $k_j(q) \in \omega(S)$.

Put $(n,q) = (4,2)$. By the Gruenberg–Kegel theorem, $n_2(L) = k_8(q)$ lies in $\omega(K)$. Since $k_6(q) = 1$, we may assume that $i = 3$ and $j = 2$. The numbers $r_3(q)$, $r_2(q)$, and $r_8(q)$ compose a coclique in $GK(G)$, and hence $r_2(q)$ does not divide $|K| \cdot |\overline{G}/S|$. Thus, $k_2(q) \in \omega(S)$.

(2) Let $n$ be even. Put $r = r_{2(n-1)}(q)$ if $2(n-1) \notin \{i,j\}$, and $r = r_{n-1}(q)$ if $2(n-1) \in \{i,j\}$ and $n-1 \notin \{i,j\}$. If $\{i,j\} = \{2(n-1), n-1\}$ then put $r = r_n(q)$. Since $(n,q) \neq (4,2)$, the required divisors exist.

Let $n$ be odd. Put $r = r_{2(n-1)}(q)$ if $2(n-1) \notin \{i,j\}$, and $r = r_n(q)$ if $2(n-1) \in \{i,j\}$ and $n \notin \{i,j\}$. If $\{i,j\} = \{2(n-1), n\}$ then put $r = r_{2(n-2)}(q)$ for $n > 5$ and $r = r_{n-2}(q)$ for $n = 5$. Since $n \geq 5$, the required divisors exist.

By [23, Proposition 2.5], the specified number $r$ and $r_i$, $r_j$ are pairwise nonadjacent in $GK(G)$. Hence, $r_j$ does not divide $|K| \cdot |\overline{G}/S|$. Thus, $k_j(q) \in \omega(S)$.

(3) Put $r = r_{2n}(q)$ if $2n \notin \{i,j\}$, and $r = r_{2(n-1)}(q)$ if $2n \in \{i,j\}$ and $2(n-1) \notin \{i,j\}$. If $\{i,j\} = \{2n, 2(n-1)\}$ then put $r = r_{2(n-2)}(q)$. Since $(n,q) \notin \{(4,2), (5,2)\}$, the required divisors exist. By [23, Proposition 2.5], the numbers $r_i, r_j$, and $r$ are pairwise nonadjacent in $GK(G)$. So, $r_j$ does not divide $|K| \cdot |\overline{G}/S|$. Therefore, $k_j(q) \in \omega(S)$. The lemma is proved.

As we mentioned in the Introduction, for a number of symplectic and orthogonal groups, the recognition problem was already solved or quasirecognizability was proved. The only previous results that we use are those on a few groups of small order. These groups are listed in the following lemma. Observe that the assertions of Theorems 1–3 are true for all these groups, and so they may be omitted from a proof.

**Lemma 2.3.** *Let $L$ be one of the simple groups $B_n(q)$, $C_n(q)$, $D_n(q)$, ${}^2D_n(q)$ and let $G$ be a finite group with $\omega(G) = \omega(L)$.*

(1) *If $L = B_3(2)$ then $G \simeq B_3(2)$ or $G \simeq D_4(2)$* [2, 5].

(2) *If $L = B_3(3)$ then $G \simeq B_3(3)$ or $G \simeq D_4(3)$* [5].

(3) *If $L = B_4(2)$ then $G$ has a unique nonabelian composition factor $S$ and $S \in \{B_4(2), {}^2D_4(2)\}$* [6].

(4) *If $L = B_4(3)$ then $G$ has a unique nonabelian composition factor $S$ and $S \in \{B_4(3), {}^2D_4(3)\}$* [11].

(5) *If $L = C_3(3)$ then $G \simeq C_3(3)$* [4].

(6) *If $L = C_4(3)$ then $G$ has a unique nonabelian composition factor $S$ and $S \in \{C_4(3), {}^2D_4(3)\}$* [11].

(7) *If $L = D_4(2)$ then $G \simeq B_3(2)$ or $G \simeq D_4(2)$* [2, 5].

(8) *If $L = D_4(3)$ then $G \simeq B_3(3)$ or $G \simeq D_4(3)$* [5].

(9) *If $L = D_5(2)$ then $G \simeq D_5(2)$* [8].

(10) *If $L = {}^2D_4(2)$ then $G \simeq {}^2D_4(2)$* [5].

(11) *If $L = {}^2D_4(3)$ then $G$ has a unique nonabelian composition factor $S$ and $S \simeq {}^2D_4(3)$* [11].

(12) *If $L = {}^2D_4(4)$ then $G$ has a unique nonabelian composition factor $S$ and $S \simeq {}^2D_4(4)$* [7].

(13) *If $L = {}^2D_5(2)$ then $G \simeq {}^2D_5(2)$* [4].

**Lemma 2.4.** *If $L \in \{B_3(4), D_4(4)\}$ then the assertion of Theorem 3 holds for $L$.*

Proof. Let $G$ be a finite group with $\omega(G) = \omega(L)$. By Lemma 2.1, we have $S \leq \overline{G} = G/K \leq \operatorname{Aut} S$, where $K$ is the soluble radical of $G$. The numbers 7 and 13 are in $\pi(L)$ and not adjacent to 2 in $GK(L)$, thus they lie in $\pi(S)$, and also $\pi(S) \subseteq \pi(L) = \{2, 3, 5, 7, 13, 17\}$. According to [28, Table 1], the only groups of Lie type over fields of characteristic 2 satisfying these conditions are $A_1(64)$, $A_2(16)$, $B_2(8)$, $B_3(4)$, $D_4(4)$, $G_2(4)$, $F_4(2)$, ${}^3D_4(2)$, and $Sz(8)$. The group $A_2(16)$ has an element of order 91, and the groups $B_2(8)$, $F_4(2)$, and ${}^3D_4(2)$ have elements of order 14. Therefore, to complete the proof, it is sufficient to show that $S \not\simeq Sz(8)$.

Suppose that $S \simeq Sz(8)$. Then $17 \notin \pi(\operatorname{Aut} S)$, and thus $17 \in \pi(K)$. In $S$, there is a Frobenius subgroup with kernel of order $2^6$ and cyclic complement of order 7 (see [16]). Applying [29, Lemma 3], we conclude that $17 \cdot 7 \in \omega(G)$. However, $17 \cdot 7 \notin \omega(L)$; a contradiction.

## § 3. Proof of Theorem 1

Let $L$ be one of the groups in the statement of Theorem 1, and let $q$ be a power of a prime $p$. Let $G$ be a finite group isospectral to $L$. Assume to the contrary that there is an alternating group among nonabelian composition factors of $G$. Then $S \leq \overline{G} = G/K \leq \operatorname{Aut} S$ by Lemma 2.1, where $K$ is the soluble radical of $G$ and $S \simeq \mathrm{Alt}_m$ with $m \geq 5$.

Suppose that for $L$, there is a set $M$ of three natural numbers satisfying the following conditions:

(1) for every $i \in M$, we have $n/2 < \eta(i) \leq n$, and $i \neq 2n$ if $L = D_n(q)$, $i \neq n$ if $L = {}^2D_n(q)$;

(2) for every $i \in M$, the number $k_i(q)$ is not equal to one.

Consider the numbers $k_i(q)$, where $i$ runs over $M$. It follows from Lemma 2.2 that at least two of these three numbers are coprime to $|K| \cdot |\overline{G}/S|$ and lie in $\omega(S)$. Denote them by $a$ and $b$. Suppose that there is a prime divisor $r$ of $a$ such that $r \leq m/2$. Since all prime divisors of $b$ are not adjacent to $r$ in $GK(G)$, they all are larger than $m/2$. Therefore, either all prime divisors of $a$ or all prime divisors of $b$ are larger than $m/2$. Denote by $k$ that of the numbers $a$ and $b$ for which this is true.

Let $r'$ and $r''$ be two different prime divisors of $k$. Then $r' + r'' > m$. Thus, $r'r'' \notin \omega(S)$ and hence $r'r'' \in \omega(L) \setminus \omega(G)$, which is impossible. Let $k$ be a power of a prime $r$ larger than $r$. Then $r^2 > (m/2)^2 > m$ and hence $r^2 \in \omega(L) \setminus \omega(G)$; a contradiction. Therefore, $k$ is a prime and from $k \in \omega(S)$ it follows that $m \geq k$. Thus, $m \geq k_i(q)$ for some $i \in M$.

The idea of the further proof is as follows: Choosing a set $M$ in a special way, we bound $m$ from below in terms of $n$ and $q$. Then we derive a contradiction by showing that the $p$-period of $S$ is strictly larger than the $p$-period of $L$ with a few exceptions which are analyzed one by one.

**Lemma 3.1.** *Let $i$ be an odd prime and let $q$ be a power of a prime $p$. Then $k_i(q) > q^{i-2}$. If $(i, q) \neq (3, 2)$ then $k_{2i}(q) > q^{i-2}/p$ and if $i \neq q + 1$ then $k_{2i}(q) > q^{i-2}$.*

Proof. Since $k_i(q) = (q^i - 1)/(q - 1)(i, q - 1)$, it follows that

$$k_i(q) \geq \frac{q^i - 1}{(q-1)^2} = \frac{q^{i-1} + \cdots + 1}{q - 1} > \frac{q^{i-1}}{q} = q^{i-2}.$$

Suppose that $q > 2$. Then $(q + 1)^2 < 2q^2 \leq pq^2$, and so the equality $k_{2i}(q) = (q^i + 1)/(q + 1)(i, q + 1)$ yields

$$k_{2i}(q) \geq \frac{q^i + 1}{(q+1)^2} > \frac{q^i}{pq^2} = \frac{q^{i-2}}{p},$$

and

$$k_{2i}(q) \geq 2\frac{q^i + 1}{(q+1)^2} > \frac{q^i}{q^2} = q^{i-2}$$

provided that $i \neq q + 1$.

Let $q = 2$. Then $i \neq 3$ and so $(i, q + 1) = 1$. Thus, $k_{2i}(q) = (2^i + 1)/3 > 2^{i-2}$. The lemma is proved.

Case $L = B_n(q)$ or $L = C_n(q)$, where $n \geq 3$.

Denote the $p$-period of $L$ by $p^l$. It follows from [30, Proposition 0.5] that $l$ satisfies the inequalities

$$(p^{l-1} + 1)/2 \leq n < (p^l + 1)/2. \qquad (*)$$

Suppose that $n \geq 11$. Denote by $i$ the largest prime in the interval $(n/2, n]$. Since $n \geq 11$, there are at least two different primes in this interval. Thus, $i \geq (n+5)/2$. Put

$$k_i = k_i(q) = \frac{q^i - 1}{(q-1)(i, q-1)} \quad \text{and} \quad k_{2i} = k_{2i}(q) = \frac{q^i + 1}{(q+1)(i, q+1)}.$$

Denote by $j$ the unique power of 2 lying in $(n/2, n]$. Put

$$k_{2j} = k_{2j}(q) = \frac{q^j + 1}{(2, q-1)}.$$

Since $M = \{i, 2i, 2j\}$ satisfies (1) and (2), at least one of the numbers $k_i$, $k_{2i}$, and $k_{2j}$ is a prime at most $m$. Therefore, $m \geq \min\{k_i, k_{2i}, k_{2j}\}$.

Since $i \geq \max\{(n+5)/2, 11\}$, it follows from Lemma 3.1 that each of the numbers $k_i$ and $k_{2i}$ is at least $\max\{q^{(n+1)/2}/p, q^9/p\}$. Furthermore, $j \geq \max\{(n+1)/2, 8\}$ and hence $k_j > \max\{q^{(n+1)/2}/p, q^8/p\}$. Thus,

$$m \geq \min\{k_i, k_{2i}, k_{2j}\} > \max\{q^{\frac{n+1}{2}}/p, q^8/p\}.$$

Since $m > q^8/p \geq p^6+1$, the group $S$ has an element of order $p^6$. Thus, $p^6 \in \omega(L)$ and $l \geq 6$. For $l \geq 6$, we have $l+2 < (2^{l-1}+3)/4$, and so $l+2 < (p^{l-1}+3)/4$. It follows from $(*)$ that $(p^{l-1}+3)/4 \leq (n+1)/2$. Thus, $l+2 < (n+1)/2$ and so $m > q^{(n+1)/2}/p > p^{l+2}/p = p^{l+1}$, which implies that $p^{l+1} \in \omega(G) \setminus \omega(L)$; a contradiction.

Suppose that $n = 9, 10$. The set $M = \{9, 18, 16\}$ satisfies (1) and (2). Hence, $m \geq \min\{k_9, k_{18}, k_{16}\}$. Therefore, $m > q^6/q$ for $q \neq 2$, and $m \geq 19$ for $q = 2$. If $p \neq 2$ then $m > p^5$, and so $l \geq 5$. But then $n \geq (p^{l-1}+1)/2 \geq (3^4+1)/2 = 41$; a contradiction. If $p = 2$ and $q > 2$ then $m > 4^6/4 = 1024$ and so $2^9 \in \omega(G) \setminus \omega(L)$. Finally, let $q = 2$. If $n = 9$ then the equality $e(73, 2) = 9$ implies that 73 lies in $\pi(L)$ and is not adjacent to 2 in $GK(L)$. This yields, by (3) of Lemma 1.3, that $73 \in \omega(S)$, and so $m \geq 73$. Similarly, if $n = 10$ then 41 is not adjacent to 2 in the prime graph of $L$ and so 41 lies in $\omega(S)$, and thus $m \geq 41$. In both cases $29 \in \omega(G) \setminus \omega(L)$; a contradiction.

Suppose that $n = 8$. Then $GK(L)$ has two connected components, and $n_2(L) = (q^8+1)/(2, q-1)$. By the Gruenberg–Kegel theorem, $GK(S)$ is disconnected either and $n_2(L) = n_2(S)$. Hence, $m \geq (q^8+1)/(2, q-1)$. Then $m > q^7+1$, and so $l \geq 7$ and $n \geq (p^6+1)/2 \geq (2^6+1)/2 > 32$; a contradiction.

Suppose that $n = 5, 6, 7$. The set $M = \{5, 10, 8\}$ satisfies (1) and (2). Hence, $m > q^3$ if $q \neq 4$, and $m \geq 41$ if $q = 4$. For $p > 3$, we have $n \geq (p^2+1)/2 \geq (5^2+1)/2 = 13$, which is impossible. If $p \in \{2, 3\}$, $q > p$, and $q \neq 4$ then $m > q^3 \geq p^6$, and so $n \geq (p^5+1)/2 \geq (2^5+1)/2 > 16$. If $q = 3$ then $m > 27$, and if $q = 4$ then $m \geq 41$. In both cases $19 \in \omega(S)$. Since $e(19, 2) = e(19, 3) = 18$, the number 19 can divide the order of $L$ only for $n \geq 9$; a contradiction.

Let $q = 2$. If $n = 5$ then 31 is not adjacent to 2 in$GK(L)$ and so 31 lies in $\omega(S)$. Similarly, if $n = 7$, then 127 is not adjacent to 2 in $GK(L)$. In both cases $m \geq 31$, and so $19 \in \omega(S) \setminus \omega(L)$; a contradiction. Let $n = 6$. In this case 13 is not adjacent to 2 in $GK(L)$. Thus, 13 is coprime to $|K| \cdot |\overline{G}/S|$ and lies in $\omega(S)$. Therefore, $13 \leq m \leq 16$. Since 65 and 31 lie in $\omega(L)$ but not in $\omega(\mathrm{Aut}\, S)$, the order of the soluble radical $K$ is divisible by 5 and 31. We write $T$ to denote the preimage of a Sylow 11-subgroup of $\overline{G}$ in $G$. The group $T$ is soluble, and thus, by [24, Proposition 1], its prime graph cannot include cocliques of three and more elements. Since $\sigma = \{5, 11, 31\} \subseteq \pi(T)$, at least two primes of $\sigma$ are adjacent in $GK(T)$, and so in $GK(G) = GK(L)$ as well. This is a contradiction since $L$ contains no elements of order $5 \cdot 11$, $5 \cdot 31$, or $11 \cdot 31$.

Suppose that $n = 4$. By Lemma 2.3, we may assume that $q > 3$. The graph $GK(L)$ has two connected components and $n_2(L) = (q^4+1)/(2, q-1)$. By the Gruenberg–Kegel theorem, $GK(S)$ is disconnected either and $n_2(L) = n_2(S)$. Thus, $m \geq (q^4+1)/(2, q-1)$. If $q > p$ then $m \geq (q^4+1)/(2, q-1) > q^4/p \geq p^7$, and so $n \geq (p^6+1)/2 \geq (2^6+1)/2 > 32$; a contradiction. If $q = p > 3$ then $m > p^3$ and hence $n \geq (p^2+1)/2 \geq 5$, which is impossible.

Suppose that $n = 3$. By Lemma 2.3, we may assume that $q > 3$, and so $M = \{3, 6, 4\}$ satisfies (1) and (2).

Let $q = 2^\alpha > 2$. Since the 2-period of $L$ is 8; therefore, $m \leq 17$. If $\alpha$ is odd then $m \geq \min\{k_3, k_4, k_6\} = k_6 = (q^2 - q + 1)/3 \geq (8^2 - 8 + 1)/3 = 19$; a contradiction. If $\alpha$ is even then $m \geq \min\{k_3, k_4, k_6\} = k_3 = (q^2 + q + 1)/3$. For $q > 4$, we have $m \geq 91$. It remains to consider $C_3(4)$. Since $k_6 = 13$ is not adjacent to 2 in $GK(L)$, it lies in $\omega(S)$. Thus, $m \geq 13$. Then $11 \in \omega(S) \setminus \omega(L)$; a contradiction.

Let $q = 3^\alpha > 3$. Then the 3-period of $L$ is 9, and so $m \leq 26$. On the other hand, $m \geq \min\{k_3, k_4, k_6\} = k_4 = (q^2 + 1)/2 \geq 41$; a contradiction.

Let $q = p^\alpha$, where $p \notin \{2, 3\}$. In this case $\min\{k_3, k_4, k_6\} = (q^2 + \varepsilon q + 1)/3$, where $q \equiv \varepsilon 1 \pmod 3$. If $p > 5$ then the $p$-period of $L$ is $p$, and if $p = 5$ then it is equal to $5^2$. If $q > p > 5$ then $m \geq (p^4 - p^2 + 1)/3 > p^2$, which is impossible. If $q = 5^\alpha > 5$ then $m \geq (5^4 + 5^2 + 1)/3 > 5^3$; a contradiction. Thus we may assume that $q = p$. We now examine four possibilities that depend on the remainder of $p$ modulo 12.

If $p \equiv 11 \pmod{12}$ then every prime divisor $r$ of $k_3 = p^2 + p + 1$ is not adjacent to 2 in $GK(L)$. Thus, $r$ is coprime to $|K| \cdot |\overline{G}/S|$ and $r \in \omega(S)$. This is impossible unless $r = k_3$ is a prime and $m - 3 \leq k_3 \leq m$. Thus, $p^2 \in \omega(S) \setminus \omega(L)$.

If $p \equiv 1 \pmod{12}$ then $p \geq 13$ and every prime divisor $r$ of $k_6 = p^2 - p + 1$ is not adjacent to 2 in $GK(L)$. Thus, $r$ is coprime to $|K| \cdot |\overline{G}/S|$ and $r \in \omega(S)$. This is impossible unless $r = k_6$ is a prime and $m - 3 \leq k_6 \leq m$. Let $s$ be a prime divisor of $k_3 = (p^2 + p + 1)/3$. Then $sp \notin \omega(L)$. However, for $p \geq 13$ we have $s + p \leq (p^2 + p + 1)/3 + p = (p^2 + 4p + 1)/3 < p^2 - p + 1 \leq m$. Thus, $sp \in \omega(S) \setminus \omega(L)$.

If $p \equiv 7 \pmod{12}$ then $p \geq 7$ and every prime divisor $r$ of $k_3 = (p^2 + p + 1)/3$ is not adjacent to 2 in $GK(L)$. Then $r$ is coprime to $|K| \cdot |\overline{G}/S|$ and $r \in \omega(S)$. This is impossible unless $r = k_3$ is a prime and $m - 3 \leq k_3 \leq m$. Since $p \geq 7$, we derive that $m > 2p$, and so a Sylow $p$-subgroup of $S$ includes an elementary abelian subgroup of order $p^2$. Thus, by Lemma 1.4, each prime divisor of $|K|$ other than $p$ is adjacent to $p$ in $GK(G)$. Each prime divisor of $k_6 = p^2 - p + 1$ is not adjacent to $p$ in $GK(L)$, and so $k_6 = p^2 - p + 1 \in \omega(S)$. Since $k_6 > k_3 + 3 \geq m$, $k_6$ must be a composite number. Therefore, there is a prime divisor $s$ of $k_6$ at most $\sqrt{p^2 - p + 1}$. We have $s + p \leq \sqrt{p^2 - p + 1} + p < 2p < m$; a contradiction.

If $p \equiv 5 \pmod{12}$ then every prime divisor $r$ of $k_6 = (p^2 - p + 1)/3$ is not adjacent to 2 in $GK(L)$. Thus, $r$ is coprime to $|K| \cdot |\overline{G}/S|$ and $r \in \omega(S)$. This is impossible unless $r = k_6$ is a prime and $m - 3 \leq k_6 \leq m$. Suppose that $p \neq 5$, and hence $p \geq 17$. Then $m > 2p$, and so a Sylow $p$-subgroup of $S$ includes an elementary abelian subgroup of order $p^2$. Therefore, by Lemma 1.4, each prime divisor of $|K|$ other than $p$ is adjacent to $p$ in $GK(G)$. Each prime divisor of $k_3 = p^2 + p + 1$ is not adjacent to $p$ in $GK(L)$; hence, $k_3 = p^2 + p + 1 \in \omega(S)$. Since $k_3 > k_6 + 3 \geq m$; therefore, $k_3$ must be a composite number. Consequently, there is a prime divisor $s$ of $k_3$ that does not exceed $\sqrt{p^2 + p + 1}$. We have $s + p \leq \sqrt{p^2 + p + 1} + p < 2p + 1 < m$; a contradiction. Finally, let $p = 5$. Then $7 \leq m \leq 10$. Thus, $31 = k_3$ does not divide the order of $\operatorname{Aut} S$. Therefore, $31 \in \omega(K)$. However, $S$ includes an elementary abelian group of order $3^2$. So, $31 \cdot 3 \in \omega(G) \setminus \omega(L)$ by Lemma 1.4.

Case $L = D_n(q)$ or $L = {}^2D_n(q)$, where $n \geq 4$.

By [30, Proposition 0.5], if the $p$-period of $L$ is $p^l$ then $(p^{l-1} + 3)/2 \leq n < (p^l + 3)/2$. In particular, $n \geq (p^{l-1} + 3)/2 > (p^{l-1} + 1)/2$. This implies that the Lie rank of a group of type $D_n$ or ${}^2D_n$ is at least the rank of a group of type $B_n$ or $C_n$ provided that $p$-periods of these groups are equal.

Suppose that $n \geq 12$. Denote by $i$ the largest prime in the interval $(n/2, n)$. Since for $n \geq 12$ there are at least two different primes in this interval, $i \geq (n + 5)/2$. Put

$$k_i = k_i(q) = \frac{q^i - 1}{(q - 1)(i, q - 1)} \quad \text{and} \quad k_{2i} = k_{2i}(q) = \frac{q^i + 1}{(q + 1)(i, q + 1)}.$$

We write $j$ to denote the power of 2 such that $j \in (n/2, n]$ with the following exception: if $L = D_n(q)$ and $n$ is a power of 2 then $j$ denotes $n/2$. Put

$$k_{2j} = k_{2j}(q) = \frac{q^j + 1}{(2, q - 1)}.$$

Since the three-element set $M = \{i, 2i, 2j\}$ satisfies (1) and (2), at least one of the numbers $k_i, k_{2i}, k_{2j}$ is a prime not exceeding $m$. Thus, $m \geq \min\{k_i, k_{2i}, k_{2j}\}$.

Since $i \geq \max\{(n+5)/2, 11\}$ and $j \geq \max\{(n+1)/2, 8\}$, we derive that

$$\min\{k_i, k_{2i}, k_{2j}\} > \max\{q^{\frac{n+1}{2}}/p, q^8/p\}.$$

Reasoning by analogy to the case of groups of types $B_n$ and $C_n$, we infer that $l+2 < (n+1)/2$, and so $m \geq q^{(n+1)/2}/p > q^{l+2}/p \geq p^{l+1}$. But then $p^{l+1} \in \omega(G) \setminus \omega(L)$; a contradiction.

Suppose that $n = 10, 11$. Since $M = \{9, 18, 16\}$ satisfies (1) and (2), the proof is similar to that for groups $B_n(q)$ and $C_n(q)$ with $n = 9, 10$. Indeed, all estimations remain valid, and so in all cases but $q = 2$, we immediately infer that the $p$-period of $L$ is strictly less than the $p$-period of $S$, which is impossible. Let $q = 2$. In the prime graphs of ${}^2D_{10}(2)$, ${}^2D_{11}(2)$, and $D_{11}(2)$, the number 41 is not adjacent to 2. Thus, $m \geq 41$ for these groups. For $D_{10}(2)$ the number 73 is not adjacent to 2, and so $m \geq 73$. In all cases $29 \in \omega(G) \setminus \omega(L)$; a contradiction.

Suppose that $n = 9$. The set $M = \{7, 14, 16\}$ satisfies (1) and (2), thus $m \geq \min\{k_7, k_9, k_{16}\} > q^5/p$. If $p \neq 2$ then $p^4 \in \omega(L)$, and so $n \geq (p^3+3)/2 \geq (3^3+3)/2 = 15$; a contradiction. If $p = 2$ and $q > 2$ then $p^8 \in \omega(L)$, and so $n \geq (2^7+3)/2 > 9$; a contradiction. If $q = 2$ then 257 is not adjacent to 2 in $GK(L)$. Thus, $m \geq 257$. Therefore, $2^8 \in \omega(L)$ and $n \geq (2^7+3)/2 > 9$; a contradiction.

Since the prime graph of $L = {}^2D_8(q)$ has two connected components and $n_2(L) = (q^8+1)/(2, q-1)$, this case can be examined in the same manner as in the case of $B_8(q)$ and $C_8(q)$.

Suppose that $n = 6, 7$ for $L = {}^2D_n(q)$ and $n = 6, 7, 8$ for $L = D_n(q)$. Then $M = \{5, 10, 8\}$ satisfies (1) and (2). So, for $q \neq 2$, the argument is analogous to that for $B_n(q)$ and $C_n(q)$ with $n = 5, 6, 7$. Let $q = 2$. In $GK(L)$, the number 2 is not adjacent to 31 if $L = D_6(2)$ or $L = {}^2D_6(2)$, to 127 if $L = D_7(2)$ or $L = D_8(2)$, and to 43 if $L = {}^2D_7(2)$. This yields that $m \geq 31$. But then $19 \in \omega(S) \setminus \omega(L)$; a contradiction.

Suppose that $n = 5$. By Lemma 2.3, we may assume that $q > 2$. Put

$$i = \begin{cases} 5, & \text{if } L = D_5(q), \\ 10, & \text{if } L = {}^2D_5(q), \end{cases} \qquad j = \begin{cases} 3, & \text{if } (3, q-1) = 1, \\ 6 & \text{otherwise} \end{cases}$$

and consider $k_8 = k_8(q)$, $k_i = k_i(q)$, and $k_j = k_j(q)$.

The three-element set $M = \{8, i, j\}$ satisfies (1) and (2). Thus, $m \geq q^2 - q + 1$. Since for $q > 2$ we have $q^2 - q + 1 > 2p$, a Sylow $p$-subgroup of $S$ includes an elementary abelian $p$-subgroup of order $p^2$. Therefore, by Lemma 1.4 each prime divisor of $|K|$ other than $p$ is adjacent to $p$ in $GK(G)$. All prime divisors of $k_8$ and $k_i$ are not adjacent to $p$ in $GK(L)$. Hence, $k_8$ and $k_i$ are in $\omega(S)$. Thus at least one of these numbers must be a prime between $m/2$ and $m$. Denote this number by $k$. Then $m \geq k \geq \min\{k_8, k_i\}$. Therefore, $m > p^3$ if $q \neq 4$, and $m \geq 41$ if $q = 4$. If $p > 3$ then the $p$-period of $L$ is at most $p^2$, which is an immediate contradiction. If $p = 3$ then the 3-period of $L$ is $3^3$, which again leads to a contradiction for $q > 3$. Finally, if $p = 2$ then the 2-period of $L$ is $2^4$, and we derive a contradiction for $q > 4$. Thus we are left with the groups $D_5(q)$ and ${}^2D_5(q)$, where $q = 3, 4$. In this case $m > 27$ and so $19 \in \omega(S) \setminus \omega(L)$, which is impossible.

The prime graph of $L = {}^2D_4(q)$ has two connected components and $n_2(L) = (q^4+1)/2$. So the argument leading to a contradiction just repeats that for $B_4(q)$ and $C_4(q)$.

It remains to consider the case of $L = D_4(q)$. By Lemma 2.3 we may assume that $q > 3$. Then, by analogy to the case of $L = B_3(q)$ or $L = C_3(q)$, the set $M = \{3, 6, 4\}$ satisfies (1) and (2). The further argument completely coincides with that for $B_3(q)$ and $C_3(q)$, including the detailed analysis of the four cases modulo 12 for $q = p > 3$.

Case $L = B_2(q)$, where $q > 3$.

The prime graph of $L$ has two connected components. We write $k$ to denote $n_2(L) = (q^2+1)/(2, q-1)$. By the Gruenberg–Kegel theorem, we deduce that $k$ is a prime and $m - 2 \leq k \leq m$.

Suppose first that $q > p$. Then $m \geq (p^4+1)/(2, p-1) > p^3 + 1$. However, the $p$-period of $L$ is $p$ for $p > 3$ and is $p^2$ for $p \in \{2, 3\}$. Therefore, $p^3 \in \omega(S) \setminus \omega(L)$.

Thus, $q = p \geq 5$. In this case each prime divisor of the order of $L$ not dividing $k$ either equals $p$ or divides $p-1$ or divides $p+1$. In all cases it does not exceed $p+1$. On the other hand, $2(p+1) < (p^2+1)/2 = k$ and $p+1 \geq 6$. Thus, the interval $(p+1, k)$ contains at least one prime $r$. We derive a contradiction since $r \in \omega(S) \setminus \omega(L)$.

Theorem 1 is proved.

## § 4. Proof of Theorem 2

Let $L$ be one of the groups in the statement of Theorem 2, and let $q$ be a power of a prime $p$. Let $G$ be a finite group isospectral to $L$. Assume that the assertion of the theorem is false. Then by Lemma 2.1 $S \leq \overline{G} = G/K \leq \operatorname{Aut} S$, where $K$ is the soluble radical of $G$ and $S$ is either a sporadic group or the Tits group. Observe that all elements of $\omega(S)$ do not exceed 119.

Suppose that $L = B_2(q)$. Then $GK(L)$ is disconnected and by the Gruenberg–Kegel theorem, the number $u = (q^2+1)/(2, q-1)$ belongs to the spectrum of $S$. It follows from [3, 31] that

$$\mu(L) = \begin{cases} \left\{\frac{q^2+1}{(2,q-1)}, \frac{q^2-1}{(2,q-1)}, p(q+1), p(q-1)\right\}, \text{ if } p > 3, \\ \left\{\frac{q^2+1}{(2,q-1)}, \frac{q^2-1}{(2,q-1)}, p(q+1), p(q-1), p^2\right\}, \text{ if } p \in \{2,3\}. \end{cases}$$

Both numbers $p(q+1)/3$ and $p(q-1)/2$ are less than $u$; if $p = 2$ then $p^2 < u$; and if $p = 3$ then $q > 3$ and $p^2 < u$. Thus,

(1) $(q^2+1)/(2, q-1) \in \omega(S)$;

(2) one of the numbers $(q^2+1)/(2, q-1)$, $p(q+1)$, $p(q+1)/2$, and $p(q-1)$ is the largest element of $\omega(S)$.

The condition $(q^2+1)/(2, q-1) \leq 119$ forces that $q \leq 13$. The case-by-case check for all $q \leq 13$ shows that none of sporadic group nor the Tits group satisfies (1) and (2). Therefore, $n > 2$.

Suppose that $n \geq 12$. In much the same way as in the proof of Theorem 1 in the case of $L \in \{D_n(q), {}^2D_n(q)\}$ we choose the triple $\{k_i(q), k_{2i}(q), k_{2j}(q)\}$. By Lemma 2.2, at least two of these three numbers belong to the spectrum of $S$. Trivial estimations show that for $n \geq 12$ all numbers of the triple are larger than 119, and so none of them belongs to $\omega(S)$; a contradiction.

Suppose that $n = 11$. By analogy to the proof of Theorem 1, we establish that two of the numbers $k_9(q)$, $k_{18}(q)$, and $k_{16}(q)$ lie in $\omega(S)$. For $q > 2$, both of $k_9(q)$ and $k_{16}(q)$ are larger than 19, and hence $q = 2$. Since $k_9(2) = 73$ and $k_{16}(2) = 257$, neither of these numbers can belong to $\omega(S)$; a contradiction.

**Lemma 4.1.** *If $3 \leq i \leq 20$, $q$ is a power of a prime, $k_i(q)$ lies in the spectrum of a sporadic group or the Tits group, and, moreover, prime divisors of $k_i(q)$ are not adjacent to 2 in the prime graph if this group, then a triple $(i, q, k_i(q))$ is contained in Table 1.*

Table 1

| | $i$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $q$ | 3 | 4 | 5 | 6 | 8 | 10 | 12 | 14 | 18 | 20 |
| 2 | 7 | 5 | 31 | 1 | 17 | 11 | 13 | 43 | 19 | 41 |
| 3 | 13 | 5 | | 7 | 41 | | | | | |
| 4 | 7 | 17 | | 13 | | 41 | | | | |
| 5 | 31 | 13 | | 7 | | | | | | |
| 7 | 19 | | | 43 | | | | | | |
| 8 | | | | 19 | | | | | | |
| 9 | | 41 | | | | | | | | |
| 11 | | | | 37 | | | | | | |

Proof. Since $119 \geq k_i(q) \geq (q^2-q+1)/3$, we infer that $q \leq 19$. Direct calculations show now that the lemma holds.

Suppose that $3 \leq n \leq 10$. By the criterion for adjacency to 2 and Lemma 1.3, there is at least one $i$ in $\{n-1, n, 2n-2, 2n\}$ such that prime divisors of $k_i(q)$ are not adjacent to 2 in $GK(L)$ and so $k_i(q) \in \omega(S)$. Then it follows from Lemma 4.1 that $q \leq 11$. For these values of $q$, $\pi(L)$ does not contain the numbers 23, 59 and 67, and so $S$ is distinct from $F_1$, $F_2$, $LyS$, $Fi_{24}'$, $Fi_{23}$, $J_4$, $M_{24}$, $M_{23}$, and the Convey groups. Therefore, $S$ has no elements of order 37, 41, and 43. Hence, $q \leq 8$.

All groups $L$ with $3 \leq n \leq 10$ and $q \leq 8$ but those listed in Lemma 2.3 can be checked by the following procedure.

Let $L = B_5(2)$. Then by Lemma 1.3, the numbers $k_5(2) = 31$ and $k_{10}(2) = 11$ lie in $\omega(S)$, and also $\pi(S) \subseteq \pi(L) = \{2, 3, 5, 7, 11, 17, 31\}$. Table 1 in [28] shows that any sporadic group $S$ does not satisfy these conditions.

A similar check leaves the pairs $(L, S)$ of $\{(B_6(2), Suz), (B_6(2), Fi_{22}), (D_4(5), J_2)\}$ as the only possibilities. The case of $L = B_6(2)$ is not possible since 13, 17, and 31 compose a coclique in $GK(L)$; and so one of the numbers 17 and 31 must lie either in the spectrum of $Suz$ or in the spectrum of $Fi_{22}$, which is false. Similarly, in the case where $L = D_4(5)$, a coclique can be composed by 7, 13, and 31. Therefore, one of the numbers 13 and 31 must lie in the spectrum of $S = J_2$, which is false.

Theorem 2 is proved.

## § 5. Proof of Theorem 3

Let $L$ be one of the groups in the statement of Theorem 3, $q = p^\alpha$, and let $G$ be a finite group isospectral to $L$. By Lemma 2.1 and the hypothesis, $S \leq G/K \leq \mathrm{Aut}(S)$, where $S$ is a simple nonabelian group isomorphic to a group of Lie type over a field of characteristic $p$. Since the assertion of the theorem holds for the groups $L$ in Lemmas 2.3 and 2.4, below these groups are not considered. The proof uses the adjacency criteria in prime graphs from [22, 23], Tables 4–7 of [22], and Tables 2–4 of [23].

**Lemma 5.1.** *Suppose that $S \not\simeq A_1(p)$. Then each number $r$ of $\pi(L)$ not adjacent to $p$ in $GK(L)$ is coprime to $|K| \cdot |\overline{G}/S|$; in particular, $t(p, S) \geq t(p, L)$.*

Proof. Let $L$ be one of the groups $B_n(q)$ and $C_n(q)$ with $n$ even. Then $r$ fails to be adjacent not only to $p$, but to 2 in $GK(L)$ either, and the claim of the lemma follows from (3) of Lemma 1.3.

Suppose now that $L$ is distinct from $B_n(q)$ and $C_n(q)$ with $n$ even. Then $GK(G)$ has a coclique of size 3 and form $\{r, s, p\}$. It is not hard to check that both $r$ and $s$ are larger than 3. Assume that $r \notin \pi(S)$. Then either $r \in \pi(\overline{G}/S)$ or $r \in \pi(K)$. Furthermore, it follows from (2) of Lemma 1.3 that $s, p \notin \pi(K) \cup \pi(\overline{G}/S)$.

Let $r \in \pi(\overline{G}/S)$. Since $r \notin \pi(S)$ and $r > 3$, $G$ contains a field automorphism of $S$ of order $r$. In every group of Lie type, the centralizer of a field automorphism contains an element of order $p$. Thus, $rp \in \omega(G)$; a contradiction.

Let $r \in \pi(K)$. As in the previous case $rp \in \omega(G)$, which yields a contradiction. Let $R$ be a Sylow $r$-subgroup of $K$ and $N = N_G(R)$. Then by the Frattini argument $N/(N \cap K) \simeq G/K \simeq S$ and so without loss of generality, we may assume that $R$ is normal in $G$ and a Sylow $p$-subgroup $P$ of $G$ acts on $R$ by conjugation. Since $p \notin \pi(K) \cup \pi(\overline{G}/S)$, the group $P$ is isomorphic to a Sylow $p$-subgroup of $S$. By hypothesis, $S$ is other than $A_1(p)$, and so its Sylow $p$-subgroup includes an elementary abelian subgroup of order $p^2$. For $A_1(p^\beta)$ with $\beta > 1$, this a subgroup of an elementary abelian Sylow $p$-subgroup; for other Chevalley groups, this subgroup is generated by a root element associated with the highest root of the root system together with any other nonidentity root element associated with a positive root. For twisted groups, this subgroup can be easily constructed on using [32, Proposition 13.6.3]. Thus, an abelian noncyclic group acts on $R$. By Lemma 1.4, $G$ has an element of order $pr$. The proof is complete.

**Lemma 5.2.** *If $S$ is isomorphic to a group of type $A_1$, then either $L = B_2(q)$ with $q > 3$ and $S \simeq A_1(q^2)$, or $L \in \{B_3(q), C_3(q), D_4(q)\}$ and $S \simeq A_1(q^3)$.*

Proof. Let $S \simeq A_1(p^\beta)$. Then $t(GK(S) \setminus \{p\}) = 2$, and hence $t(GK(L) \setminus \{p\}) \leq 3$. Therefore, $L \in \{B_2(q), B_3(q), C_3(q), D_4(q)\}$.

If $L = B_2(q)$ then $GK(L)$ is disconnected. By the Gruenberg–Kegel theorem $(q^2 + 1)/(2, p - 1) = (p^\beta \pm 1)/(2, p - 1)$, and so $p^\beta = q^2$. Thus in this case $S \simeq A_1(q^2)$.

Suppose that $L \in \{B_3(q), C_3(q), D_4(q)\}$. By Lemma 2.3, we may assume that $q > 3$. Thus, $r_6 = r_{6\alpha}(p)$ and $r_3 = r_{3\alpha}(p)$ are well-defined and $\{p, r_3, r_6\}$ is a coclique in $GK(L)$.

Let $\beta > 1$. By Lemma 5.1 $r_6$ lies in $\pi(S)$. Therefore, $r_6$ divides $p^{2\beta} - 1$, and so $6\alpha$ divides $2\beta$. If $2\beta > 6\alpha$ then $r_{2\beta}(q) \in \pi(S) \setminus \pi(L)$. Thus, $2\beta = 6\alpha$ and $S \simeq A_1(q^3)$.

Let $\beta = 1$. By Lemma 1.3, at least one of the numbers $r_6$ and $r_3$ lies in $\pi(S)$ and therefore divides $p^2 - 1$; this contradicts the definition of primitive divisor. The lemma is proved.

**Lemma 5.3.** *Let $S$ be a group over a field of order $p^\beta$.*

(1) *If $r_i(p^\beta) \in \pi(S)$ then $i\beta \leq 2n\alpha$ for $L \neq D_n(q)$ and $i\beta \leq 2(n-1)\alpha$ for $L = D_n(q)$.*

(2) *If $(i\beta, p) \neq (6,2)$ and $k_{i\beta}(p) \in \omega(S)$ then $i\beta$ divides $2j\alpha$, where $j \in \{1, \dots, n\}$ for $L \neq D_n(q)$ and $j \in \{1, \dots, n-1\}$ for $L = D_n(q)$.*

Proof. (1) Assume that $L \neq D_n(q)$ and $i\beta > 2n\alpha$. Then $i\beta > 6$ for $p = 2$, and so there exists $r = r_{i\beta}(p)$. By hypothesis, $r_i(p^\beta)$ divides the order of $S$. Hence, $k_i(p^\beta)$ divides the order of $S$. Therefore, $r$ divides the order of $S$. Thus, $r \in \pi(L)$, and the required assertion follows from the definition of primitive divisor. The case where $L = D_n(q)$ can be handled in a similar way.

(2) If $i\beta \leq 2$ then the assertion holds. If $i\beta > 2$ then $k_{i\beta}(p) \neq 1$, and the claim follows from the definition of primitive divisor. The lemma is proved.

In what follows, $S$ is a group over a field of order $p^\beta$. By Lemma 5.2, we may assume that $S$ is not of type $A_1$.

We let $e(p,S)$ and $e(p,S)'$ to stand for

$$\{e(r,p^\beta) \mid r \in \pi(S) \setminus \{p\}, pr \notin \omega(S)\}$$

and

$$\{e(r,p^\beta) \mid r \in \pi(S) \setminus \{2,3,p\}, pr \notin \omega(S)\}$$

respectively. Observe that for all $S$ but the groups of types $A_2$ and $^2A_2$, the sets $e(p,S)$ and $e(p,S)'$ coincide. By (1) of Lemma 5.3 $e\beta \leq 2n\alpha$ for every $e$ of $e(p,S)$.

Suppose that $L$ is one of the groups $B_n(q)$ and $C_n(q)$, where $n$ is even. The set of primes that are not adjacent to $p$ in $GK(L)$ coincides with the set of primes that are not adjacent to 2 in $GK(L)$ and equal to the set of primes dividing $k_{2n}(q)$. Among these divisors, choose a number $r_{2n}$ to satisfy not only the condition $e(r_{2n},p^\alpha) = 2n$ but also a stronger condition $e(r_{2n},p) = 2n\alpha$. In other words, $r_{2n}$ is of the form $r_{2n\alpha}(p)$. Since $L \neq B_3(2)$, this number exists. By Lemma 5.1 $r_{2n}$ lies in $\pi(S)$. Furthermore, $r_{2n} > 3$. Thus if $e = e(r_{2n},p^\beta)$ then $e \in e(p,S)'$. By the definition of primitive divisor, $e\beta$ is divisible by $2n\alpha$. On the other hand, $e\beta \leq 2n\alpha$ by (1) of Lemma 5.3. Therefore, we obtain the equation $e\beta = 2n\alpha$, where $e$ is the maximal element in $e(p,S)'$.

Suppose that $L$ is one of the groups $B_n(q)$ and $C_n(q)$, where $n$ is odd. Then $t(p,L) = 3$. Since $L \notin \{B_3(2), B_3(4)\}$, there exist primitive prime divisors $r_{2n} = r_{2n\alpha}(p)$ and $r_n = r_{n\alpha}(p)$. Both these numbers are not adjacent to $p$ in $GK(L)$ and, therefore, divide the order $S$ by Lemma 5.1. Put $e_2 = e(r_{2n},p^\beta)$ and $e_1 = e(r_n,p^\beta)$. Then $e_2, e_1 \in e(p,S)'$, and if $S$ is not a Ree or Suzuki group then $e_2 \neq e_1$. By the definition of primitive divisor, $e_2\beta$ is divisible by $2n\alpha$. On the other hand, $e_2\beta \leq 2n\alpha$. Therefore, $e_2\beta = 2n\alpha$. By exactly the same reason, $e_1\beta$ is divisible by $n\alpha$ and $e_1\beta \leq 2n\alpha$, and so $e_1\beta \in \{n\alpha, 2n\alpha\}$. Thus we derive the equation $2n\alpha = e_2\beta$, where $e_2$ is the maximal element in $e(p,S)'$, and the condition $n\alpha \in \{e_1\beta, e_1\beta/2\}$, where $e_1$ is some element of $e(p,S)'$. If $S$ is other than Ree and Suzuki groups then the condition turns into the equation $n\alpha = e_1\beta$ and, in particular, $e_2/e_1 = 2$.

Suppose that $L = D_n(q)$, where $n$ is even. Since $L \notin \{D_4(2), D_4(4)\}$, there exist primitive prime divisors $r_{2(n-1)} = r_{2(n-1)\alpha}(p)$ and $r_{n-1} = r_{(n-1)\alpha}(p)$. Repeating the previous argument and putting $e_2 = e(r_{2(n-1)},p^\beta)$, $e_1 = e(r_{n-1},p^\beta)$, we infer that $e_2, e_1 \in e(p,S)'$ and $e_2\beta = 2(n-1)\alpha$. Furthermore, if $S$ is other than Ree and Suzuki groups then $e_1\beta = (n-1)\alpha$ and, in particular, $e_2/e_1 = 2$.

Suppose that $L = D_n(q)$, where $n$ is odd. Then there exist primitive prime divisors $r_{2(n-1)} = r_{2(n-1)\alpha}(p)$ and $r_n = r_{n\alpha}(p)$. Putting $e_2 = e(r_{2(n-1)},p^\beta)$ and $e_1 = e(r_n,p^\beta)$, we infer that $e_2, e_1 \in e(p,S)'$ and $e_2\beta = 2(n-1)\alpha$. Furthermore, $e_1\beta$ is divisible by $n\alpha$ and is at most $2(n-1)\alpha$. Thus, $e_1\beta = n\alpha$ and, in particular, $e_2/e_1 = 2(n-1)/n < 2$.

Suppose that $L = {}^2D_n(q)$, where $n$ is odd. Then there exist primitive prime divisors $r_{2n} = r_{2n\alpha}(p)$ and $r_{2(n-1)} = r_{2(n-1)\alpha}(p)$. Putting $e_2 = e(r_{2n},p^\beta)$ and $e_1 = e(r_{2(n-1)},p^\beta)$, we deduce that $e_2, e_1 \in e(p,S)'$ and $e_2\beta = 2n\alpha$. Furthermore, $e_1\beta$ is divisible by $2(n-1)\alpha$ and is at most $2n\alpha$. Therefore, $e_1\beta = 2(n-1)\alpha$ and, in particular, $e_2/e_1 = n/(n-1) < 2$.

Finally, suppose that $L = {}^2D_n(q)$, where $n$ is even. Then $t(p,L) = 4$. By Lemma 5.1 $t(p,S) \geq 4$. Therefore, $S$ is isomorphic either to one of the groups ${}^2D_m(p^\beta)$ with $m$ even, $E_8(p^\beta)$, $E_7(p^\beta)$, $E_6(p^\beta)$ or to one of the Ree and Suzuki groups. Since $L \notin \{^2D_4(2), {}^2D_4(4)\}$, there exist divisors $r_{2n} = r_{2n\alpha}(p)$, $r_{2(n-1)} = r_{2(n-1)\alpha}(p)$, and $r_n = r_{(n-1)\alpha}(p)$. Put $e_2 = e(r_{2n}, p^\beta)$, $e_1 = e(r_{2(n-1)}, p^\beta)$ and $e_0 = e(r_n, p^\beta)$. Then $\{e_2, e_1, e_0\} \subseteq e(p, S)'$ with $e_2\beta = 2n\alpha$ and $e_1\beta = 2(n-1)\alpha$ and, in particular, $e_2/e_1 = n/(n-1) < 2$.

The following proof consists in consecutively considering all simple groups of Lie type as $S$. If there are (a) and (b) then (a) concerns the case where $L = B_n(q)$ or $L = C_n(q)$ with $n$ even, and (b) concerns the remaining cases.

**1.** Let $S \simeq A_{m-1}(p^\beta)$, where $m \geq 3$. Then $e(p, S) = \{m, m-1\}$ for $m \neq 3$ and $m \in e(p, S)' \subseteq \{m, m-1\}$ for $m = 3$.

(a) Recall that $e$ is the maximal element in $e(p, S)'$ and $e\beta = 2n\alpha$. Thus, $e = m$ and $m\beta = 2n\alpha$. In particular, $(m-1)\beta > 2$.

Assume that $m > n$. Then $\beta < 2\alpha$ and so $2(n-1)\alpha < (m-1)\beta < 2n\alpha$. Therefore, $(m-1)\beta$ divides none of the numbers $2i\alpha$, where $i \in \{1, 2, \dots, n\}$. On the other hand, $S$ includes a cyclic torus of order $(p^{(m-1)\beta} - 1)/(m, p^\beta - 1)$ (for example, see [33, Theorem 2.1]), and hence $k_{(m-1)\beta}(p) \in \omega(S)$. This contradicts (2) of Lemma 5.3 provided that $((m-1)\beta, p)$ is not equal to $(6, 2)$. Let $((m-1)\beta, p) = (6, 2)$. Since $m\beta = 2n\alpha$, it follows that $m = 4$, $\beta = 2$ and $n = 2$, $\alpha = 2$. Thus, $S \simeq A_3(4)$ and $L = B_2(4)$. In this case $7 \in \omega(S) \setminus \omega(L)$, which is impossible.

Let $m \leq n$. Then $n \geq 4$ and since $L \neq B_4(2)$, $GK(L)$ includes a coclique of size 4 that contains neither $p$ nor 2 nor 3 [23, Table 3]. In other words, $t(GK(L) \setminus \{p, 2, 3\}) \geq 4$. By (2) of Lemma 1.3 we have $t(GK(S) \setminus \{p, 2, 3\}) \geq 3$, and by [23, Table 2] this yields $m \geq 5$. Thus, $n \geq m \geq 5$. Then $t(L) - 1 \geq (3n+2)/4 - 1 > (m+1)/2 \geq t(S)$, which is impossible.

(b) Recall that $e_2, e_1 \in e(p, S)'$ and $e_2 > e_1$. Thus, $e_2/e_1 = m/(m-1)$. If $e_2/e_1 = 2$ then $m = 2$; a contradiction. If $e_2/e_1 = 2(n-1)/n$, where $n \geq 5$ is odd, then $m = 2(n-1)/(n-2)$, and so $n = 3$; a contradiction.

Let $e_2/e_1 = n/(n-1)$, where $n \geq 5$ is odd. Then $m = n$ and $\beta = 2\alpha$. Therefore, $L = {}^2D_n(q)$ and $S \simeq A_{n-1}(q^2)$. In this case $r_n(q) \in \pi(S) \setminus \pi(L)$; a contradiction.

**2.** Let $S \simeq {}^2A_{m-1}(p^\beta)$, where $m \geq 3$. Then $e(p, S)'$ is one of the sets $\{2m-2, m\}$ and $\{2m-2, m/2\}$ for even $m$, and one of the sets $\{2m, m-1\}$ and $\{2m, (m-1)/2\}$ for odd $m \neq 3$. If $m = 3$ then $2m \in e(p, S)' \subseteq \{2m, (m-1)/2\}$.

(a) Since $e$ is a maximal element in $e(p, S)'$, it follows that $2(m-1)\beta = 2n\alpha$ for even $m$ and $2m\beta = 2n\alpha$ for odd $m$.

Consider the case of $m$ even. Let $m - 1 > n$. Then $\beta < \alpha$, and so $n\alpha < m\beta < (n+1)\alpha$. Therefore, $m\beta$ divides none of the numbers $2i\alpha$, where $i \in \{1, 2, \dots, n\}$. On the other hand, $k_{m\beta}(p) \in \pi(S)$. If $((m-1)\beta, p) \neq (6, 2)$ then this contradicts (2) of Lemma 5.3. If $m\beta = 6$ then $m = 6$, $\beta = 1$, and so $n = 5$. However, $n$ is even; a contradiction.

Let $m - 1 \leq n$. Since $m \geq 4$, we have $n \geq 4$. Thus, $t(GK(L) \setminus \{2, 3, p\}) \geq 4$. Therefore, $t(GK(S) \setminus \{2, 3, p\}) \geq 3$, and so $m \geq 5$. Taking it into consideration that $m$ is even, we infer that $n \geq m \geq 6$. Just as in the case of linear groups, these inequalities yield $t(S) < t(L) - 1$; a contradiction.

We now handle the case where $m$ is odd. Let $m > n$. Then $\beta < \alpha$ and thus we have

$$(n-1)\alpha < (m-1)\beta < n\alpha, \;\; 2(n-1)\alpha < 2(m-1)\beta < 2n\alpha, \;\; 2n\alpha \leq 4(n-1)\alpha < 4(m-1)\beta.$$

On the other hand, $k_{(m-1)\beta} \in \omega(S)$, and since $m\beta = n\alpha$ is even, $\beta$ is even as well, and so $(m-1)\beta \neq 6$. By (2) of Lemma 5.3, some multiple of $(m-1)\beta$ is equal to $2i\alpha$ where $i \in \{1, \dots, n\}$. The above inequalities imply that the only possibility for this multiple is $3(m-1)\beta$, and it is equal to $2n\alpha$ since it is larger than $2(n-1)\alpha$. Therefore, $2n\alpha = 3(m-1)\beta > 3(n-1)\alpha$. So $n = 2$, $m = 3$, and $\beta = 3\alpha/2$. Thus, $L = B_2(q)$ and $S \simeq {}^2A_2(q^{2/3})$. The prime graphs of $B_2(q)$ and ${}^2A_2(q^{2/3})$ both have two connected components, and by Gruenberg–Kegel theorem

$$\frac{q^2+1}{(2, q-1)} = n_2(B_2(q)) = n_2({}^2A_2(q^{2/3})) = \frac{q^2+1}{(q^{2/3}+1)(3, q^{2/3}+1)};$$

a contradiction.

Let $m \le n$. Then $n \ge 4$ and $t(GK(L) \setminus \{2,3,p\}) \ge 4$, therefore, $t(GK(S) \setminus \{2,3,p\}) \ge 3$, and so $m \ge 5$ and $n \ge 6$. Then $t(L) - 1 > t(S)$; a contradiction.

(b) Recall that $e_2, e_1 \in e(p,S)'$ and $e_2/e_1 \le 2$. Since $2(m-1)/m < 2$ and each of the numbers $2m/(m-1)$, $4(m-1)/m$, $4m/(m-1)$ are larger than 2, $e_2/e_1 = 2(m-1)/m$ and $m$ is even. If $e_2/e_1 = 2(n-1)/n$ where $n$ is odd then $m = n$ contrary to the evenness of $m$. If $e_2/e_1 = n/(n-1)$ where $n \ge 4$ is odd then $m = 2(n-1)/(n-2)$, and so $n = 3$; a contradiction.

**3.** Let $S \simeq B_m(p^\beta)$ or $S \simeq C_m(p^\beta)$. If $m$ is even then $e(p,S) = \{2m\}$. If $m$ is odd then $e(p,S) = \{2m, m\}$.

(a) We have $e = 2m$ and $m\beta = n\alpha$. This equation will be handled in (b).

(b) Since $t(p,S) \ge t(p,L) > 2$, the number $m$ is odd and $e(p,S) = \{2m, m\}$. Therefore, $e_2/e_1 = 2$, and so either $L = D_n(q)$, where $n$ is even, or $L \in \{B_n(q), C_n(q)\}$, where $n$ is odd.

Suppose that $L = D_n(q)$, where $n$ is even. Then $m\beta = (n-1)\alpha$. Observe that $k_{2(m-1)\beta}(p) \in \omega(S)$, and also, since $m$ is odd, $2(m-1)\beta \ne 6$.

Let $m > n-1$. Then $\beta < \alpha$, and so

$$n\alpha \le 2(n-2)\alpha < 2(m-1)\beta < 2(n-1)\alpha,$$

contrary to (2) of Lemma 5.3. Therefore, $m \le n-1$. Furthermore, it follows from

$$(3m+5)/4 \ge t(S) \ge t(L) - 1 \ge (3n-2)/4 - 1$$

that $m \ge n-3$. Thus, $m \in \{n-1, n-2, n-3\}$. Moreover, since $n$ is even and $m$ is odd, $m \in \{n-1, n-3\}$.

Let $m = n-3$. Then $n \ge 6$ and $(n-3)\beta = (n-1)\alpha$. Denote $\beta/(n-1) = \alpha/(n-3)$ by $\gamma$. By (2) of Lemma 5.3, the number $2(m-1)\beta = 2(n-4)(n-1)\gamma$ must divide $2i\alpha = 2i(n-3)\gamma$ for some $1 \le i \le n-1$. This is impossible unless $i = n-1$. Thus, $n-4$ divides $n-3$, which is false for $n \ge 6$.

If $m = n-1$ then $\beta = \alpha$ and $S \in \{B_{n-1}(q), C_{n-1}(q)\}$, as stated in (4) of the theorem we prove.

Suppose that $L \in \{B_n(q), C_n(q)\}$, where $n$ is odd. Then $m\beta = n\alpha$. Thus in the case of $L \in \{B_n(q), C_n(q)\}$, the equality $m\beta = n\alpha$ holds for all $n$. However, if $n$ is odd then $m$ is odd as well. We now consider even and odd $n$ together assuming for a while that $(2(m-1)\beta, p) \ne (6,2)$.

Let $m > n$. Then $\beta < \alpha$ and so

$$n\alpha \le 2(n-1)\alpha < 2(m-1)\beta < 2n\alpha.$$

Therefore, $2(m-1)\beta$ divides none of the numbers $2i\alpha$, where $i \in \{1, 2, \dots, n\}$. On the other hand, $k_{2(m-1)\beta}(p) \in \omega(S)$; a contradiction. Thus, $m \le n$. It follows from $(3m+5)/4 \ge t(S) \ge t(L) - 1 \ge (3n+2)/4 - 1$ that $m \ge n-2$. Thus, $m \in \{n, n-1, n-2\}$.

Let $m = n-1$. Then $n \ge 3$ and $(n-1)\beta = n\alpha$. Denote $\beta/n = \alpha/(n-1)$ by $\gamma$. By (2) of Lemma 5.3, the number $2(m-1)\beta = 2(n-2)n\gamma$ must divide $2i\alpha = 2i(n-1)\gamma$ for some $i \in \{1, 2, \dots, n\}$. This is impossible unless $i = n$. Therefore, $n-2$ divides $n-1$, and so $n = 3$. Then $m$ is odd, but $m = 2$; a contradiction.

Let $m = n-2$. Then $n \ge 4$ and $(n-2)\beta = n\alpha$. Repeating the previous argument with the index $2(m-1)\beta$, we infer that $n = 4$, $m = 2$ and $\beta = 2\alpha$. Thus, $L \in \{B_4(q), C_4(q)\}$ and $S \simeq B_2(q^2)$. Then $t(S) = 2$, while the independence number for $L$ when $q > 2$ is equal to 4; a contradiction.

If $m = n$ then $\alpha = \beta$ and $S \in \{B_n(q), C_n(q)\}$, as stated in (3) of the theorem we prove.

It remains to consider the case where $(2(m-1)\beta, p) = (6,2)$. In this case either $m = 2$ and $\beta = 3$, or $m = 4$ and $\beta = 1$. Therefore, $n$ is even. Now the equation $m\beta = n\alpha$ implies that either $S \simeq L$, as required, or $S \simeq B_4(2)$ and $L = B_2(4)$. The latter situation is impossible since $7 \in \omega(B_4(2)) \setminus \omega(B_2(4))$.

**4.** Let $S \simeq D_m(p^\beta)$. Then $e(p,S) = \{2m-2, m-1\}$ for even $m$ and $e(p,S) = \{2m-2, m\}$ for odd $m$.

(a) We have $e = 2m-2$, and hence $(m-1)\beta = n\alpha$. This equation will be handled in (b).

(b) Let $m$ be odd. Then $e_2/e_1 = 2(m-1)/m < 2$, and so either $e_2/e_1 = n/(n-1)$ or $e_2/e_1 = 2(n-1)/n$. In the first case $m = 2(n-1)/(n-2) < 4$, which is impossible; in the second case $L \simeq D_n(q)$

and $m = n$, $\beta = \alpha$, and thus $S \simeq L$, as required. Therefore, we may assume that $m$ is even and $e_2/e_1 = 2$, and hence either $L = D_n(q)$, where $n$ is even, or $L \in \{B_n(q), C_n(q)\}$, where $n$ is odd.

Suppose that $L = D_n(q)$, where $n$ is even. Then $(m-1)\beta = (n-1)\alpha$. Observe that $k_{2(m-2)\beta}(p) \in \omega(S)$, and since $m$ is even, $2(m-2)\beta \neq 6$.

Let $m > n$. Then $\beta < \alpha$ and

$$n\alpha \leq 2(n-2)\alpha < 2(m-2)\beta < 2(n-1)\alpha,$$

contrary to (2) of Lemma 5.3. Thus, $m \leq n$. On the other hand, it follows from

$$(3m+1)/4 \geq t(S) \geq t(L) - 1 \geq (3n-2)/4 - 1$$

that $m \geq n-2$. Therefore, $m \in \{n, n-1, n-2\}$. Furthermore, since both $n$ and $m$ are even, $m \in \{n, n-2\}$.

Let $m = n-2$. Then $n \geq 6$ and $(n-3)\beta = (n-1)\alpha$. Denote $\beta/(n-1) = \alpha/(n-3)$ by $\gamma$. By (2) of Lemma 5.3, the number $2(m-2)\beta = 2(n-4)(n-1)\gamma$ must divide $2i\alpha = 2i(n-3)\gamma$ for some $1 \leq i \leq n-1$. This is impossible unless $i = n-1$. Therefore, $n-4$ divides $n-3$, which is false for $n \geq 6$.

If $m = n$ then $\beta = \alpha$, and hence $S \simeq L$, as required.

Suppose that $L \in \{B_n(q), C_n(q)\}$, where $n$ is odd. Then $(m-1)\beta = n\alpha$. Thus in the case where $L \in \{B_n(q), C_n(q)\}$, the equality $(m-1)\beta = n\alpha$ holds for all $n$ but if $n$ is odd then $m$ is even. Now we consider even and odd $n$ together assuming for a while that $(2(m-2)\beta, p) \neq (6, 2)$.

Let $m - 1 > n$. Then $\beta < \alpha$ and hence

$$n\alpha \leq 2(n-1)\alpha < 2(m-2)\beta < 2n\alpha,$$

contrary to (2) of Lemma 5.3. Therefore, $m - 1 \leq n$. Furthermore, it follows from

$$(3m+1)/4 \geq t(S) \geq t(L) - 1 \geq (3n+2)/4 - 1$$

that $m \geq n-1$. Thus, $m \in \{n+1, n, n-1\}$.

Let $m = n$. Then $n \geq 4$ and $(n-1)\beta = n\alpha$. Denote $\beta/n = \alpha/(n-1)$ by $\gamma$. By (2) of Lemma 5.3, the number $2(m-2)\beta = 2(n-2)n\gamma$ must divide $2i\alpha = 2i(n-1)\gamma$ for some $1 \leq i \leq n$. This is impossible unless $i = n$. Therefore, $n-2$ divides $n-1$, and hence $n = 3$, but $n \geq 4$; a contradiction.

Let $m = n-1$. Then $n \geq 5$ and $(n-2)\beta = n\alpha$. Repeating the argument with the index $2(m-2)\beta$, we deduce that $n = 4$ and $m = 3$; a contradiction.

Let $m = n+1$. Then $\alpha = \beta$, and thus $S \simeq D_{n+1}(q)$. If $n = 3$ then $S \simeq D_4(q)$ and $L \in \{B_3(q), C_3(q)\}$, as stated in (2) of the theorem we prove. For $n > 3$, we show that $\pi(S) \not\subseteq \pi(L)$ yielding a contradiction. If $n$ is even then $r_{n+1}(q) \in \pi(S) \setminus \pi(L)$. Let $n \geq 5$ be odd and $S \not\simeq D_8(2)$. Then $S$ has an element of order $r_{n+3}(q)r_{n-1}(q)$. On the other hand, $\eta(n+3) + \eta(n-1) = n+1 > n$ and $1 < \eta(n+3)/\eta(n-1) = (n+3)/(n-1) \leq 2$, hence by the adjacency criterion [23, Proposition 2.4], it follows that $r_{n+3}(q)r_{n-1}(q) \notin \omega(L)$. If $S \simeq D_8(2)$ then $L \simeq B_7(2)$ and $99 \in \omega(S) \setminus \omega(L)$ (see [33]).

It remains to consider the case where $(2(m-2)\beta, p) = (6, 2)$. In this case $m = 5$, $\beta = 1$, and $S \simeq D_5(2)$. It follows from $(m-1)\beta = n\alpha$ that $L \in \{B_2(4), B_4(2)\}$. Since $31 \in \pi(S) \setminus \pi(L)$, this is a contradiction.

**5.** Let $S \simeq {}^2D_m(p^\beta)$. Then $e(p, S) = \{2m, 2m-2, m-1\}$ for even $m$ and $e(p, S) = \{2m, 2m-2\}$ for odd $m$.

(a) We have $e = 2m$ and $m\beta = n\alpha$. Repeating the corresponding argument of 3(b), we infer that either $L \in \{B_n(q), C_n(q)\}$, as stated in (3) of the theorem, or $L = B_2(4)$ and $S \simeq {}^2D_4(2)$. In the latter case $7 \in \omega(S) \setminus \omega(L)$, which is impossible.

(b) Since $e_2$ is the maximal element of $e(p, S)$, it follows that $e_2 = 2m$. Furthermore, $e_2/e_1 \neq 2m/(m-1)$ since $e_2/e_1 \leq 2$. Thus, $e_2/e_1 = m/(m-1) < 2$. If $e_2/e_1 = 2(n-1)/n$ then $m = 2(n-1)/(n-2) < 3$, which is false. If $e_2/e_1 = n/(n-1)$ then $L = {}^2D_n(q)$, $m = n$, $\beta = \alpha$ and $S \simeq L$, as required.

**6.** Let $S \simeq E_8(p^\beta)$. Then $e(p, S) = \{30, 24, 20, 15\}$. Also it follows from $(3n-2)/4 \leq t(L) \leq t(S) + 1 = 13$ that $n \leq 18$.

(a) Since $e = 30$, it follow that $15\beta = n\alpha$. This equation will be handled in (b).

(b) Let $e_2/e_1 = 30/24 = 5/4$. Then $e_2/e_1 = n/(n-1)$ and $30\beta = 2n\alpha$, and hence $n = 5$ and $3\beta = \alpha$. This yields $S \simeq E_8(q^{1/3})$ and $L = {}^2D_5(q)$. In this case $r_{5\alpha}(p) \in \omega(S) \setminus \omega(L)$; a contradiction. If $e_2/e_1 = 30/20$ then the equation $e_2/e_1 = n/n-1$ has no solutions larger than 3, and the equation $e_2/e_1 = 2(n-1)/n$ has no odd solutions larger than 3, however in these cases $n \geq 4$. Thus, $e_1 = 15$ and $e_2/e_1 = 2$.

Suppose that $L \in \{B_n(q), C_n(q)\}$, where $n$ is odd. Then $15\beta = n\alpha$. Therefore, this equation holds independently of the parity of $n$. It yields $24\beta = 8n\alpha/5$ and $20\beta = 4n\alpha/3$. By (2) of Lemma 5.3, there are $i, j \in \{1, \dots, n\}$ such that $8n\alpha/5 \mid 2i\alpha$ and $4n\alpha/3 \mid 2j\alpha$, hence $n$ is divisible by both 5 and 3. Thus, $n = 15$ and $\beta = \alpha$.

Suppose that $L = D_n(q)$, where $n$ is even. Then $15\beta = (n-1)\alpha$. By analogy to the previous case we infer that $n-1$ is divisible by both 5 and 3. Therefore, $n = 16$ and $\beta = \alpha$.

Thus, $L \in \{B_{15}(q), C_{15}(q), D_{16}(q)\}$ and $S \simeq E_8(q)$. By Lemma 2.2, at least one of the numbers $r_{13}(q)$ and $r_{26}(q)$ must belong to $\omega(S)$, which is false.

**7.** Let $S \simeq E_7(p^\beta)$. Then $e(p, S) = \{18, 14, 9, 7\}$. Also it follows from $(3n-2)/4 \leq t(L) \leq t(S) + 1 = 9$ that $n \leq 12$. In (a), we infer that $e = 18$ and $9\beta = n\alpha$. In (b), it is easy to check that $e_2/e_1 \notin \{18/14, 18/7\}$, and hence $e_1 = 9$ and $e_2/e_1 = 2$.

Suppose that $L \in \{B_n(q), C_n(q)\}$. Then $9\beta = n\alpha$. Thus, $14\beta = 14n\alpha/9$. Therefore, there is $i \in \{1, \dots, n\}$ such that $14n\alpha/9 \mid 2i\alpha$. Since $i$ must be divisible by 7 and $n \leq 12$, it follows that $i = 7$. Then $n = 9$ and $\beta = \alpha$.

Suppose that $L = D_n(q)$, where $n$ is even. Then $9\beta = (n-1)\alpha$. Just as in the previous case, we deduce that $n - 1 = 9$ and $\beta = \alpha$.

Thus, $L \in \{B_9(q),\ C_9(q), D_{10}(q)\}$ and $S \simeq E_7(q)$. Then $r = r_{16}(q) \in \pi(L) \setminus \pi(S)$. Observe that $r \geq 17$ and $r$ is adjacent in $GK(G)$ to neither of the numbers $r_5(q)$, $r_9(q)$ and $r_{18}(q)$.

Let $r \in \pi(\overline{G}/S)$. Then $G$ contains a field automorphism of $S$ of order $r$. The centralizer of this automorphism in $S$ has an element of order $q_0^5 - 1$, where $q = q_0^r$. Since $r$ and 5 are coprime, if $s = r_5(q_0)$ then $e(s, q) = 5$. Thus, $rs \in \omega(G) \setminus \omega(L)$; a contradiction.

Let $r \in \pi(K)$. The numbers $r$, $r_9(q)$, and $r_{18}(q)$ compose a coclique in $GK(G)$, and so $r_9(q), r_{18}(q) \in \pi(S) \setminus \pi(K)$. Furthermore, the neighborhoods of $r_9(q)$ and $r_{18}(q)$ in $GK(G)$ disjoint. In $S$, there is a subgroup isomorphic to a group of type $A_6(q)$. Therefore, there is a subgroup isomorphic to $GL_6(q)$. Thus, $S$ includes a Frobenius subgroup with kernel of order $q^5$ and cyclic complement of order $q^5 - 1$. By [29, Lemma 3] we have $r(q^5 - 1) \in \omega(G) \setminus \omega(L)$; a contradiction.

**8.** Let $S$ be isomorphic to one of the groups $E_6(p^\beta)$, $F_4(p^\beta)$, and ${}^3D_4(p^\beta)$. Then

$$\{12\} \subseteq e(p, S) \subseteq \{12, 9, 8\}.$$

It follows from $(3n-2)/4 \leq t(L) \leq t(S) + 1 \leq 6$ that $n \leq 8$.

(a) Since $e = 12$, we have $6\beta = n\alpha$. Then $9\beta = 3n\alpha/2$ and $8\beta = 4n\alpha/3$. There are $i, j \in \{1, \dots n\}$ such that $3n\alpha/2 \mid 2i\alpha$ and $4n\alpha/3 \mid 2j\alpha$. Therefore, $n$ is divisible by both 4 and 3; a contradiction.

(b) It is not hard to check that $e_2/e_1 \neq 12/8$. Thus, $e_2/e_1 = 12/9$. Then $e_2/e_1 = n/(n-1)$ and $6\beta = n\alpha$, and so $n = 4$ and $\beta = 2\alpha/3$. This means that $S \simeq E_6(q^{2/3})$ and $L = {}^2D_4(q)$. In this case $r_{5\beta}(p) \in \pi(S) \setminus \pi(L)$; a contradiction.

**9.** Let $S \simeq {}^2E_6(p^\beta)$. Then $e(p, S) = \{18, 12, 8\}$. As in the previous case, $n \leq 8$.

(a) Since $e = 18$, we have $9\beta = n\alpha$. Then $12\beta = 4n\alpha/3$ and $8\beta = 8n\alpha/9$. There are $i, j \in \{1, \dots, n\}$ such that $4n\alpha/3 \mid 2i\alpha$ and $8n\alpha/9 \mid 2j\alpha$, which is impossible for $n \leq 8$.

(b) It is not hard to check that $e_2/e_1$ cannot lie in $\{18/12, 18/8\}$.

**10.** Let $S \simeq G_2(p^\beta)$. Then $e(p, S) = \{6, 3\}$. Therefore, $e = 6$ and $e_2/e_1 = 2$. It follows from $(3n-2)/4 \leq t(L) \leq t(S) + 1 = 4$ that $n \leq 6$.

Suppose that $L \in \{B_n(q), C_n(q)\}$. Then $3\beta = n\alpha$. If $n = 2, 4$, then the graph $GK(L)$ is disconnected. By the Gruenberg–Kegel theorem, we derive the equation

$$(q^n + 1)/(2, q - 1) = p^{2\beta} \pm p^\beta + 1,$$

and so either $2^{n\alpha} = 2^\beta(2^\beta \pm 1)$ or $q^n = 2p^{2\beta} \pm 2p^\beta + 1$; both these equalities are impossible. If $n \geq 5$ and $(n, q) \neq (5, 2)$ then $t(GK(L)) \geq 5$, contrary to the fact that $t(GK(S)) = 3$. If $(n, q) = (5, 2)$ then $n\alpha$ is not divisible by 3, contrary to the equation $3\beta = n\alpha$. If $n = 3$ then $\beta = \alpha$ and $S \simeq G_2(q)$, as stated in (2) of the theorem we prove.

Suppose that $L = D_n(q)$, where $n$ is even. Then $3\beta = (n - 1)\alpha$. If $n = 4$ then $S \simeq G_2(q)$, as stated in (2).

Let $n = 6$. Then $\beta = 5\alpha/3$. In particular, $\alpha \geq 3$. By Lemma 2.2, at least on the numbers $k_8(q) = (q^4 + 1)/(2, q - 1)$ and $k_3(q) = (q^2 + q + 1)/(3, q - 1)$ must lie in $\omega(S)$, but none of these numbers divides $|S|_{p'} = (q^{10} - 1)(q^{10/3} - 1)$; a contradiction.

It remains to consider the case where $S$ is a simple Ree or Suzuki group. In these groups, in contrast to the previous ones, the numbers $r$ not adjacent to each other nor to $p$ in $GK(S)$ can have the same indices $e(r, p^\beta)$, and so $e_2$ and $e_1$ can be equal.

**11.** Let $S \simeq {}^2F_4(2^\beta)$, where $\beta \geq 3$ is odd. Then $e(2, S) = \{12, 6\}$. Thus, $e = e_2 = 12$ and $e_2/e_1 \in \{2, 1\}$.

Suppose that $L = D_n(q)$, where $n$ is odd. Then $e_2/e_1 = 2(n - 1)/n$; this is a contradiction since $1 < 2(n - 1)/n < 2$ for $n \geq 4$. In a similar manner, we prove that $L \neq {}^2D_n(q)$.

Suppose that $L = B_n(q)$. Then $6\beta = n\alpha$. It follows from $t(L) \leq t(S) + 1 \leq 6$ that $n \leq 7$.

Let $n \in \{2, 4\}$. Then the graph $GK(L)$ is disconnected and, from the Gruenberg–Kegel theorem we infer that

$$2^{n\alpha} + 1 = n_2(L) = 2^{2\beta} \pm 2^{(3\beta+1)/2} + 2^\beta \pm 2^{(\beta+1)/2} + 1.$$

Hence, $2^{6\beta} = 2^{(\beta+1)/2}(2^{(3\beta-1)/2} \pm 2^\beta + 2^{(\beta-1)/2} \pm 1)$; a contradiction.

If $n = 3$ then $\beta = \alpha/2$ and, in particular, $\alpha$ is even. Thus, $S \simeq {}^2F_4(q^{1/2})$ and $L = B_3(q)$. By Lemma 2.2, at least one of the numbers $k_2(q) = q^2 + 1$ and $k_3(q) = (q^2 + q + 1)/3$ must lie in $\omega(S)$. However, none of these numbers divides $|S|_{2'} = (q^3 + 1)(q^2 - 1)(q^{3/2} + 1)(q^{1/2} - 1)$; a contradiction.

Let $n \in \{5, 6, 7\}$. By Lemma 2.2, at least one of the numbers $r_{(n-1)\alpha}(2)$ and $r_{2(n-1)\alpha}(2)$ must lie in $\omega(S)$. However, it is not hard to check that this is false.

Suppose that $L = D_n(q)$, where $n$ is even. Then $6\beta = (n-1)\alpha$. It follows from $t(L) \leq t(S)+1 \leq 6$ that $n \leq 8$. The case of $L = D_4(q)$ is analogous to that of $L = B_3(q)$, and the case where $L \in \{D_6(q), D_8(q)\}$ is analogous to that of $L \in \{B_5(q), B_7(q)\}$.

**12.** Let $S \simeq {}^2B_2(2^\beta)$, where $\beta \geq 3$ is odd. Then $e(2, S) = \{4, 1\}$. Therefore, $e = e_2 = 4$ and $e_2/e_1 \in \{4, 1\}$. As in the previous case, we show that $L \neq D_n(q)$, where $n$ is odd, and $L \neq {}^2D_n(q)$.

Suppose that $L = B_n(q)$. Then $2\beta = n\alpha$. It follows from $t(L) \leq t(S)+1 \leq 5$ that $n \leq 6$. If $n \in \{5, 6\}$ then $t(GK(L) \setminus \{2\}) = 5$; however, $t(GK(S) \setminus \{2\}) = 3$; a contradiction.

Let $n \in \{2, 4\}$. Then $GK(L)$ is disconnected and by the Gruenberg–Kegel theorem we derive one of the equalities

$$2^{n\alpha} + 1 = 2^\beta - 1, \quad 2^{n\alpha} + 1 = 2^\beta \pm 2^{(\beta+1)/2} + 1.$$

The first equality is clearly impossible; the latter yields $2^{2\beta} = 2^{(\beta+1)/2}(2^{(\beta-1)/2} \pm 1)$, which is false either.

Let $n = 3$. Then $\beta = 3\alpha/2$, and in particular $\alpha$ is even. Thus, $S \simeq {}^2B_2(q^{3/2})$ and $L = B_3(q)$. By Lemma 2.2, at least one of the numbers $k_4(q) = q^2 + 1$ and $k_3(q) = (q^2 + q + 1)/3$ must lie in $\omega(S)$. Since $q > 4$ by Lemmas 2.3 and 2.4, none of these numbers divides $|S|_{2'} = (q^3 + 1)(q^{3/2} - 1)$; a contradiction.

Suppose that $L = D_n(q)$, where $n$ is even. Then $2\beta = (n - 1)\alpha$. It follows from $t(L) \leq t(S) + 1 \leq 5$ that $n \leq 6$. The case of $L = D_4(q)$ is similar to that of $L = B_3(q)$.

Let $n=6$. Then $\beta=5\alpha/2$. Thus, $S\simeq {}^2B_2(q^{5/2})$ and $L=B_3(q)$. By Lemma 2.2, at least one of the numbers $k_8(q)=q^4+1$ and $k_5(q)=(q^5-1)/(q-1)$ must lie in $\omega(S)$. However, none of them divides $|S|_{2'}=(q^5+1)(q^{5/2}-1)$; a contradiction.

**13.** Let $S\simeq {}^2G_2(3^\beta)$, where $\beta\geq 3$ is odd. Then $e(3,S)=\{6,2,1\}$. Therefore, $e=e_2=6$ and $e_2/e_1\in\{6,3,2,1\}$. As in the previous cases, we show that $L\neq D_n(q)$, where $n$ is odd, and $L\neq {}^2D_n(q)$.

Suppose that $L\in\{B_n(q),C_n(q)\}$. Then $3\beta=n\alpha$. It follows from $t(L)\leq t(S)+1\leq 6$ that $n\leq 7$. If $n=7$ then $t(GK(L)\setminus\{3\})=6$. However, $t(GK(S)\setminus\{3\})=4$; a contradiction.

Let $n\in\{2,4\}$. Then $GK(L)$ is disconnected and by the Gruenberg–Kegel theorem we infer that

$$\frac{3^{n\alpha}+1}{2}=3^\beta\pm 3^{(\beta+1)/2}+1,$$

and hence $3^{3\beta}=2\cdot 3^\beta\pm 2\cdot 3^{(\beta+1)/2}+1$; a contradiction.

Let $n\in\{3,5,6\}$. Then $S\simeq {}^2G_2(q^{n/3})$. By Lemma 2.2, at least one of the numbers $k_{2(n-1)}(q)$ and $k_n(q)$ must lie in $\omega(S)$. However, none of them divides $|S|_{3'}=(q^n+1)(q^{n/3}-1)$; a contradiction.

Suppose that $L=D_n(q)$, where $n$ is odd. Then $2\beta=(n-1)\alpha$. It follows from $t(L)\leq t(S)+1\leq 6$ that $n\leq 8$. The case of $L=D_8(q)$ is similar to that of $L\in\{B_7(q),C_7(q)\}$, and the case of $L\in\{D_6(q),D_4(q)\}$ is similar to that of $L\in\{B_5(q),B_3(q),C_5(q),C_3(q)\}$.

Theorem 3 is proved.

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