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Recognition of the finite almost simple groups $\mathrm{PGL}_2(q)$ by their spectrum

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1 Introduction

For a finite group G, denote by $\omega(G)$ the spectrum of G, i.e., the set of orders of elements in G. This set is closed under divisibility and hence is uniquely determined by the subset $\mu(G)$ of elements in $\omega(G)$ which are maximal under the divisibility relation.

A group G is said to be *recognizable* by $\omega(G)$ (for short, *recognizable*) if every finite group H with $\omega(H) = \omega(G)$ is isomorphic to G. In other words, G is recognizable if $h(G) = 1$ where $h(G)$ is the number of pairwise non-isomorphic groups H with $\omega(H) = \omega(G)$. It is known that $h(G) = \infty$ for every group G that has a non-trivial soluble normal subgroup, and so the recognizability problem is interesting only for groups with trivial soluble radical, and first of all for simple and almost simple groups.

The goal of this paper is to resolve the recognizability problem for the groups $\mathrm{PGL}_2(q)$, i.e., to find $h(\mathrm{PGL}_2(q))$ for all q.

Theorem. *Let $H = \mathrm{PGL}_2(q)$ be the projective general linear group of dimension* 2 *over a finite field of order q. Then $h(H)$ is infinite if q is a prime or $q = 9$. In all other cases H is recognizable, i.e.*, $h(G) = 1$.

It was previously known that $h(\mathrm{PGL}_2(2^m)) = 1$ for $m \geqslant 2$ ([2], [16]), $h(\mathrm{PGL}_2(7)) = h(\mathrm{PGL}_2(9)) = \infty$ ([15]) and $h(\mathrm{PGL}_2(q)) \in \{1, \infty\}$ for prime q and for $q = p^n$ where p is a prime of the form $2^a 3^b + 1$ ([14], [15]).

2 Preliminary results

In this section we state without proof the results needed later in the paper. The set $\omega(H)$ of a finite group H defines *the Gruenberg–Kegel graph* $\mathrm{GK}(H)$ whose vertices are prime divisors of the order of H, and two primes p, q are adjacent if H has an element of order pq. Denote by $s = s(H)$ the number of connected components in

$\mathrm{GK}(H)$ and by $\pi_i = \pi_i(H)$ the ith connected component for $i = 1, \ldots, s$. For a group H of even order, we assume that $2 \in \pi_1$. Denote by $\mu_i = \mu_i(H)$ (resp. by $\omega_i = \omega_i(H)$) the set of all $n \in \mu(H)$ (resp. all $n \in \omega(H)$) such that every prime divisor of n lies in π_i.

Lemma 1 (Gruenberg–Kegel Theorem; see [17]). *If G is a finite group with disconnected graph $\mathrm{GK}(G)$ then one of the following holds*:

(a) *$s(G) = 2$ and G is a Frobenius group*;

(b) *$s(G) = 2$, G is a 2-Frobenius group, i.e., $G = ABC$ where $A, AB \lhd G$, $B \lhd BC$, and AB, BC are Frobenius groups*;

(c) *there exists a non-abelian simple group P such that $P \leqslant \bar{G} = G/N \leqslant \mathrm{Aut}(P)$ for some nilpotent normal $\pi_1(G)$-subgroup N of G and $\bar{G}/P$ is a $\pi_1(G)$-group. Moreover, $\mathrm{GK}(P)$ is disconnected, $s(P) \geqslant s(G)$ and for every i with $2 \leqslant i \leqslant s(G)$ there exists j with $2 \leqslant j \leqslant s(P)$ such that $\omega_i(G) = \omega_j(P)$.*

Lemma 2. *Let P be a finite simple group with disconnected graph $\mathrm{GK}(P)$. Then $|\mu_i(P)| = 1$ for $2 \leqslant i \leqslant s(P)$; write $\mu_i(P) = \{n_i\}$ for $i > 1$. Then P, $\pi_1(P)$, n_i for $2 \leqslant i \leqslant s(P)$ are as in Tables* 1 (a)–(c).

Tables 1 (a)–(c) are taken from [6]. They combine results and remove misprints from [7] and [17]. In Tables 1 (a)–(c), p denotes an odd prime.

Lemma 3 (Zsigmondy [19]). *Let p be a prime and s be a natural number, $s \geqslant 2$. Then one of the following holds*:

(a) *there exists a prime q such that q divides $p^s - 1$ and q does not divide $p^t - 1$ for all natural numbers $t < s$*;

(b) *$s = 6$ and $p = 2$*;

(c) *$s = 2$ and $p = 2^t - 1$ for some t.*

A prime q satisfying condition (a) of Lemma 3 is said to be a *primitive* prime divisor of $p^s - 1$.

Lemma 4 (cf. [11, Lemma 1]). *Let G be a finite group and N a normal subgroup such that G/N is a Frobenius group with kernel F and a cyclic complement C. If $(|F|, |N|) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \omega(G)$ for some prime divisor p of $|N|$.*

Lemma 5 (Mazurov [11]). *Let G be a finite group having a non-trivial soluble normal subgroup. Then $h(G) = \infty$.*

Lemma 6. *Let $L = L_2(p)$ where p is a prime, $p > 3$.*

Table 1 (a)
Finite simple groups P with $s(P) = 2$

P	Restrictions on P	$\pi_1(P)$	n_2
A_n	$6 < n = p,\ p+1,\ p+2$; $n,\ n-2$ not both prime	$\pi((n-3)!)$	p
$A_{p-1}(q)$	$(p,q) \neq (3,2), (3,4)$	$\pi(q\prod_{i=1}^{p-1}(q^i-1))$	$\frac{q^p-1}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1) \mid (p+1)$	$\pi(q(q^{p+1}-1)\prod_{i=1}^{p-1}(q^i-1))$	$\frac{q^p-1}{q-1}$
${}^2A_{p-1}(q)$		$\pi(q\prod_{i=1}^{p-1}(q^i-(-1)^i))$	$\frac{q^p+1}{(q+1)(p,q+1)}$
${}^2A_p(q)$	$(q+1) \mid (p+1)$, $(p,q) \neq (3,3), (5,2)$	$\pi(q(q^{p+1}-1)\prod_{i=1}^{p-1}(q^i-(-1)^i))$	$\frac{q^p+1}{q+1}$
${}^2A_3(2)$		$\{2,3\}$	5
$B_n(q)$	$n = 2^m \geqslant 4$, q odd	$\pi(q\prod_{i=1}^{n-1}(q^{2i}-1))$	$\frac{1}{2}(q^n+1)$
$B_p(3)$		$\pi(3(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1))$	$\frac{1}{2}(3^p-1)$
$C_n(q)$	$n = 2^m \geqslant 2$	$\pi(q\prod_{i=1}^{n-1}(q^{2i}-1))$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$	$q = 2,3$	$\pi(q(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1))$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$	$p \geqslant 5$, $q = 2,3,5$	$\pi(q\prod_{i=1}^{p-1}(q^{2i}-1))$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$	$q = 2,3$	$\pi(q(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1))$	$\frac{q^p-1}{(2,q-1)}$
${}^2D_n(q)$	$n = 2^m \geqslant 4$	$\pi(q\prod_{i=1}^{n-1}(q^{2i}-1))$	$\frac{q^n+1}{(2,q+1)}$
${}^2D_n(2)$	$n = 2^m+1 \geqslant 5$	$\pi(2(2^n+1)\prod_{i=1}^{n-2}(2^{2i}-1))$	$2^{n-1}+1$
${}^2D_p(3)$	$5 \leqslant p \neq 2^m+1$	$\pi(3\prod_{i=1}^{p-1}(3^{2i}-1))$	$\frac{1}{4}(3^p+1)$
${}^2D_n(3)$	$9 \leqslant n = 2^m+1 \neq p$	$\pi(3(3^n+1)\prod_{i=1}^{n-2}(3^{2i}-1))$	$\frac{1}{2}(3^{n-1}+1)$
$G_2(q)$	$2 < q \equiv \varepsilon(3)$, $\varepsilon = \pm 1$	$\pi(q(q^2-1)(q^3-\varepsilon))$	$q^2-\varepsilon q+1$
${}^3D_4(q)$		$\pi(q(q^6-1))$	q^4-q^2+1
$F_4(q)$	q odd	$\pi(q(q^6-1)(q^8-1))$	q^4-q^2+1
${}^2F_4(2)'$		$\{2,3,5\}$	13
$E_6(q)$		$\pi(q(q^5-1)(q^8-1)(q^{12}-1))$	$\frac{q^6+q^3+1}{(3,q-1)}$
${}^2E_6(q)$	$q > 2$	$\pi(q(q^5+1)(q^8-1)(q^{12}-1))$	$\frac{q^6-q^3+1}{(3,q+1)}$
M_{12}		$\{2,3,5\}$	11
J_2		$\{2,3,5\}$	7
Ru		$\{2,3,5,7,13\}$	29
He		$\{2,3,5,7\}$	17
McL		$\{2,3,5,7\}$	11
Co_1		$\{2,3,5,7,11,13\}$	23
Co_3		$\{2,3,5,7,11\}$	23
Fi_{22}		$\{2,3,5,7,11\}$	13
HN		$\{2,3,5,7,11\}$	19

(a) (Burkhardt [4]). *L has an irreducible module V over $\mathbb{C}$ of degree $p-1$ such that all elements of order p in L act on V fixed-point-freely and an element of order $(p+1)/2$ has a fixed point in V.*

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Table 1 (b)
Finite simple groups P with $s(P) = 3$

P	Restrictions on P	$\pi_1(P)$	n_2	n_3
A_n	$n > 6$, $n = p$, $p - 2$ are primes	$\pi((n-3)!)$	p	$p - 2$
$A_1(q)$	$3 < q \equiv \varepsilon(4)$, $\varepsilon = \pm 1$	$\pi(q - \varepsilon)$	$\pi(q)$	$\frac{1}{2}(q + \varepsilon)$
$A_1(q)$	$q > 2$, q even	$\{2\}$	$q - 1$	$q + 1$
${}^2A_5(2)$		$\{2, 3, 5\}$	7	11
${}^2D_p(3)$	$p = 2^m + 1$	$\pi(3(3^{p-1} - 1)\prod_{i=1}^{p-2}(3^{2i} - 1))$	$\frac{1}{2}(3^{p-1} + 1)$	$\frac{1}{4}(3^p + 1)$
$G_2(q)$	$q \equiv 0(3)$	$\pi(q(q^2 - 1))$	$q^2 - q + 1$	$q^2 + q + 1$
${}^2G_2(q)$	$q = 3^{2m+1} > 3$	$\pi(q(q^2 - 1))$	$q - \sqrt{3q} + 1$	$q + \sqrt{3q} + 1$
$F_4(q)$	q even	$\pi(q(q^4 - 1)(q^6 - 1))$	$q^4 + 1$	$q^4 - q^2 + 1$
${}^2F_4(q)$	$q = 2^{2m+1} > 2$	$\pi(q(q^3 + 1)(q^4 - 1))$	$q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1$	$q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1$
$E_7(2)$		$\{2, 3, 5, 7, 11, 13, 17, 19, 31, 43\}$	73	127
$E_7(3)$		$\{2, 3, 5, 7, 11, 13, 19, 37, 41, 61, 73, 547\}$	757	1093
M_{11}		$\{2, 3\}$	5	11
M_{23}		$\{2, 3, 5, 7\}$	11	23
M_{24}		$\{2, 3, 5, 7\}$	11	23
J_3		$\{2, 3, 5\}$	17	19
HiS		$\{2, 3, 5\}$	7	11
Suz		$\{2, 3, 5, 7\}$	11	13
Co_2		$\{2, 3, 5, 7\}$	11	23
Fi_{23}		$\{2, 3, 5, 7, 11, 13\}$	17	23
F_3		$\{2, 3, 5, 7, 13\}$	19	31
F_2		$\{2, 3, 5, 7, 11, 13, 17, 19, 23\}$	31	47

(b) (Burichenko [3]). *Let W be a reduction of V modulo 2. If $(p-1)/2$ is odd then there exists a non-split extension E of W by L.*

Lemma 7 (Moghaddamfar and Shi [15]). *Let G be a finite group such that*

$$\mu(G) = \mu(\mathrm{PGL}(2, p^n)) = \{p^n - 1, p, p^n + 1\},$$

where p is an odd prime and $n \geqslant 2$.

(1) *If $(p, n) \neq (3, 2)$, then case* (c) *of Lemma* 1 *applies. Moreover, P is not isomorphic to any of the following simple groups*:

(a) *alternating groups on $n \geqslant 5$ letters,*

(b) *sporadic simple groups,*

(c) *$L_2(p^k)$ where $k \neq n$, or*

(d) *$L_2(2p^m \pm 1)$, $m \geqslant 1$, where $2p^m \pm 1$ is a prime.*

Table 1 (c)
Finite simple groups P with $s(P) > 3$

$s(P)$	P	Restrictions on P	$\pi_1(P)$	n_2	n_3	n_4	n_5	n_6
4	$A_2(4)$		$\{2\}$	3	5	7		
	${}^2B_2(q)$	$q = 2^{2m+1} > 2$	$\{2\}$	$q-1$	$q-\sqrt{2q}+1$	$q+\sqrt{2q}+1$		
	${}^2E_6(2)$		$\{2,3,5,7,11\}$	13	17	19		
	$E_8(q)$	$q \equiv 2,3(5)$	$\pi(q(q^8-1) \times (q^{14}-1) \times (q^{12}-1) \times (q^{18}-1) \times (q^{20}-1))$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1		
	M_{22}		$\{2,3\}$	5	7	11		
	J_1		$\{2,3,5\}$	7	11	19		
	O′N		$\{2,3,5,7\}$	11	19	31		
	LyS		$\{2,3,5,7,11\}$	31	37	67		
	Fi_{24}'		$\{2,3,5,7,11,13\}$	17	23	29		
	F_1		$\{2,3,5,7,11,13,17,19,23,29,31,47\}$	41	59	71		
5	$E_8(q)$	$q \equiv 0,1,4(5)$	$\pi(q(q^8-1) \times (q^{10}-1) \times (q^{12}-1) \times (q^{14}-1) \times (q^{18}-1))$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1	$\frac{q^{10}+1}{q^2+1}$	
6	J_4		$\{2,3,5,7,11\}$	23	29	31	37	43

(2) *If* $(p,n) = (3,2)$, *then there exists a soluble group* G *such that* $\mu(G) = \mu(\mathrm{PGL}(2,3^2))$.

3 Proof of the main results

Let $q = p^m$ with p a prime and $H = \mathrm{PGL}_2(q)$. Then $\mu(H) = \{p, q-1, q+1\}$.

Lemma 8. *Suppose that* $q = p > 3$. *Then there exists an extension* E *of the* $L_2(p)$-*module* W *from Lemma* 6 *by* $L = L_2(p)$ *with* $\omega(E) = \omega(H)$.

Proof. By Lemma 6 we have $2p \notin \omega(E)$ for every extension E of W by L and, since L contains a Frobenius subgroup of order $p(p-1)/2$, we have $(p-1) \in \omega(E)$. If $(p+1)/2$ is odd, then $p+1 \in \omega(E)$ by Lemma 6, and since W is an elementary abelian 2-group and $\mu(L) = \{p, (p-1)/2, (p+1)/2\}$, we conclude that $\mu(E) = \mu(H)$ for every such E, in particular, for the natural semidirect product of W and L. Suppose that $(p+1)/2$ is even. Then $(p-1)/2$ is odd and, by Lemma 6, there exists a non-split extension E of W by L. We only have to prove that E has an element of order $p+1$. Suppose that this is false and choose Wt in E/W of order $(p+1)/2$. Let $u = t^{(p+1)/4}$. Then u is an involution and hence there exists an element $e \in E$ such that Wuu^e has order $(p+1)/2$ in E/W. By assumption, uu^e is of order $(p+1)/2$, which implies that $U = \langle u, u^e \rangle$ intersects W trivially. Since WU/W has a Sylow 2-subgroup S/W of E/W, S splits over W and hence E splits over W, and this is a contradiction. $\square$

By [2] and [16], $h(H) = 1$ if $p = 2$. By Lemma 5 and Lemma 8, $h(H) = \infty$ if q is a prime. By [14], $h(H)$ is infinite if $q = 9$. Therefore it remains to prove that $h(H) = 1$ if $m > 1$, q is odd and $q > 9$.

From here on we assume the following hypothesis:

G is a finite group with $\mu(G) = \mu(\mathrm{PGL}_2(q)) = \{p, q+1, q-1\}$, where p is an odd prime, $q = p^m > 9$ and $m > 1$.

By Lemma 7, case (c) of Lemma 1 applies, i.e., there exists a non-abelian simple group P such that $P \leqslant \bar{G} = G/N \leqslant \mathrm{Aut}(P)$ for some nilpotent normal $\pi_1(G)$-subgroup N of G and $\bar{G}/P$ is a $\pi_1(G)$-group. Moreover, $\mathrm{GK}(P)$ is disconnected, $s(P) \geqslant s(G)$, and for every i with $2 \leqslant i \leqslant s(G)$, there exists j with $2 \leqslant j \leqslant s(P)$ such that $\omega_i(G) = \omega_j(P)$. Furthermore, by [14], P cannot be isomorphic to an alternating group or a sporadic group. Thus P is isomorphic to one of the groups of Lie type in Table 1. We shall prove that $P \cong L_2(q)$ by examining in turn every possibility for P.

Let $\varepsilon = \pm 1$ be such that $q - \varepsilon$ is divisible by 4. Consider the graph $\Gamma(G)$ with vertex set V consisting of elements of $\omega(G)$ which are distinct from 1 and are either odd or divisible by 4. Two vertices a and b are adjacent in $\Gamma(G)$ (and we write $a \sim b$) if $\mathrm{lcm}(a,b) \in \omega(G)$. It is obvious that $\Gamma(G)$ has exactly three connected components: $\{p\}$, $\omega^- = \{a \in V \mid a \text{ divides } q - \varepsilon\}$ and $\omega^+ = \{a \in V \mid a \text{ divides } q + \varepsilon\}$. Each of these components is a complete graph, and so $\sim$ is an equivalence relation. Notice that $\mathrm{lcm}(\omega^-) = q - \varepsilon$, $\mathrm{lcm}(\omega^+) = (q+\varepsilon)/2$. If $q - \varepsilon$ is a power of 2, then either q is a Mersenne or Fermat prime, or $q = 9$. In each case, we obtain a case that we have considered before. Thus we may assume that ω^- contains an odd number.

It is obvious that N is equal to the Fitting subgroup $F(G)$, and hence $N = O_2(G) \times O(G)$. Let f be the unique element in $\mu(O(G))$, and f_2 the unique element in $\mu(O_2(G))$.

For $v \in \{+,-\}$ denote by $L_n^v(r)$ (resp. $\mathrm{SL}_n^v(r)$) the group $L_n(r)$ (resp. $\mathrm{SL}_n(r)$) if $v = +$ and $U_n(r)$ (resp. $\mathrm{SU}_n(r)$) if $v = -$.

Lemma 9. *$P \ncong L_n^v(r)$ for $n \geqslant 4$.*

Proof. Suppose false. Let $P \cong L_n^v(r)$ where $r = t^u$ and t is a prime. Then P contains cyclic tori $T_2, \ldots, T_{n-2}, T_{n-1}, T_n$ of orders $r^2 - 1, \ldots, r^{n-2} - (v1)^{n-2}$, $(r^{n-1} - (v1)^{n-1})/(n, r - v1)$, $(r^n - (v1)^n)/(r - v1)(n, r - v1)$ respectively, a torus T_1 which is the direct product of two cyclic groups of order $r - v1$, and also a subgroup isomorphic to the direct product of a group of order t and $\mathrm{SL}_2(r)$.

Suppose first that t is odd. Since P has an elementary abelian subgroup of order t^2 we have $t \sim f$ if $f \neq 1$. On the other hand, $t \sim s$ for $s = 4$ or s equal to any odd prime divisor of $(r^2 - 1) = (r-1)(r+1)$, and hence, for any odd prime divisor of $|T_i|$, for $i \leqslant n-2$. Since $|\mathrm{Out}(P)|$ divides $2(r^2-1)u$ we have $t \sim s$ for every odd prime divisor s of $\overline{G}/P$, which implies that

$$\{p, (q+\varepsilon)/2\} = \{(r^{n-1} - (v1)^{n-1})/(n, r - v1), (r^n - (v1)^n)/(r - v1)(n, r - v1)\}. \quad (1)$$

We shall prove that (1) is also true for $t = 2$.

If $n = 4$ then, by Lemma 2 we have $P \cong L_4(2) \cong A_8$, which is excluded, or $P \cong U_4(2) \cong S_4(3)$. In this case, $4 \sim 3$ and $3 \sim f \sim 5$ if $f \neq 1$, so that $\Gamma(G)$ cannot have three connected components.

Thus we may assume that $n \geqslant 5$. Then P has a subgroup isomorphic to $L_2(r) \times \mathrm{SL}_3^v(r)$, so that $4 \sim r^2 - 1$. Since P has a subgroup isomorphic to the direct product of two groups of order $r+1 \neq 1$, we conclude that $r+1 \sim f$ if $f \neq 1$. On the other hand, $|\mathrm{Out}(P)|$ divides $2(r^2-1)u$, so that $4 \sim s$ for every odd divisor s of $\overline{G}/P$, and (1) holds in all cases.

It is easy to see that $p < (q+\varepsilon)/2$. If $|T_{n-1}| < |T_n|$ then $p = |T_{n-1}|$ and hence n is even. On the other hand, p is a connected component in $\mathrm{GK}(P)$, so that $r - v1$ divides n, by Lemma 2. Therefore $p = (r^{n-1} - (v1)^{n-1})/(r - v1)$ and for every $n \geqslant 4$ we have

$$p^2 = (r^{n-1} - (v1)^{n-1})2/(r - v1)2 > 1 + 2(r^n - 1)/(r - v1)2 > q,$$

which contradicts our assumption. So $|T_n| = p < (q+\varepsilon)/2 = |T_{n-1}|$. This is possible only for $v = -$. Furthermore, n must be a prime and so $n \geqslant 5$. But in this case it is easy to calculate that $p^2 > q$, which is again impossible. $\square$

Lemma 10. $P \ncong L_3^v(r)$.

Proof. Suppose false. Let $r = t^u$ where t is a prime. Then, by [1], [13] and [18], $\omega(P)$ consists of all divisors of the members of the set

$$\{r - v1, t(r - v1)/(3, r - v1), (r^2 - 1)/(3, r - v1), (r^2 + vr + 1)/(3, r - v1)\}$$

if r is odd and of the set

$$\{r - v1, t(r - v1)/(3, r - v1), (r^2 - 1)/(3, r - v1), (r^2 + vr + 1)/(3, r - v1), 4\}$$

if r is even.

Consider first some particular cases using the information from [7].

If $P = L_3(2)$ then $\mu(P) = \{3, 4, 7\}$, $|\mathrm{Out}(P)| = 2$ and $p = 7$. Since P contains a Frobenius group of order 21 we have $3 \sim f$ if $f \neq 1$, so that ω^- consists only of powers of 2, which is impossible.

If $P = L_3(3)$ then $\mu(P) = \{3, 8, 13\}$. Since $3 \sim f$ if $f \neq 1$ and $|\mathrm{Out}(P)| = 4$, $q - \varepsilon$ is a power of 2, which is excluded.

If $P = L_3(4)$ then $\mu(P) = \{3, 4, 5, 7\}$ and $|\mathrm{Out}(P)| = 12$. If $f \neq 1$ then, as P contains $2^4 : A_5$, we have $3 \sim f \sim 5$ and ω^- consists only of powers of 2, which is impossible. Thus $p^{2m} - 1$ is a $\{2, 3, 5, 7\}$-number which is not divisible by 25. This is impossible by Lemma 3.

Let $P = L_3(7)$. Then $\mu(P) = \{6, 14, 16, 19\}$, so that $p = 19$ and $q - 1$ is divisible by 18, and hence $18 \in \omega(G)$. If $|\bar{G}/P|$ is divisible by 3 then G has an element of order $19 \cdot 3$, which is impossible, and so 3 divides f. Since a Sylow 7-subgroup of P is non-cyclic, $7 \sim f$. Furthermore, P contains a Frobenius subgroup of order $49 \cdot 4$, so that $4 \sim f \sim 3$ by Lemma 4, and the equality $|\mathrm{Out}(P)| = 6$ implies that $4 \sim s$ for every odd prime $s \neq 19$ dividing G, which is impossible.

Let $P = U_3(3)$. Then $\mu(P) = \{7, 8, 12\}$, $|\mathrm{Out}(P)| = 2$ and $p = 7$. If f is a power of 3 then ω^- does not contain an odd prime: otherwise, as P contains a Frobenius subgroup of order $32 \cdot 4$ we have $4 \sim f \sim 3$ by Lemma 4, and ω^+ is empty, and this is a contradiction.

Let $P = U_3(5)$. Then $\mu(P) = \{6, 7, 8, 10\}$, $\mu(U_3(5).3) = \{21, 24, 30\}$, $\mu(U_3(5).2) = \{7, 8, 12, 20\}$. Thus $p = 7$ and hence $|\bar{G}/P| \leqslant 2$. If $f \neq 1$ then, as Sylow 3-subgroups and 5-subgroups of P are non-cyclic, one of the sets ω^+, ω^- does not contain an odd prime, which is impossible. So $f = 1$ and hence $\bar{G} = P$, since otherwise $5 \sim 4 \sim 3$ and ω^+ is empty. By Lemma 3 we have $m \leqslant 4$. If $m = 2$ then $q + 1 = 50$ and G has an element of order 25; if $m = 3$ then $q - 1 \in \omega(G)$ is divisible by 57. If $m = 4$ then $q - 1$ is divisible by 25. All of these cases are impossible.

Suppose that r is odd, $3 \neq r \neq 7$ for $v = +$ and $3 \neq r \neq 5$ for $v = -$. Then there exists a divisor s of $(r - v1)/(r - v1, 3) \geqslant 3$ which is an odd prime or 4. Then

$$t \sim s \sim s(r - v1, 3) \sim \frac{r^2 - 1}{(r - v1, 3)} \sim 4.$$

Since a Sylow t-subgroup of P is non-cyclic, $t \sim f$ if $f \neq 1$. On the other hand, if $|\bar{G}/P|$ is divisible by an odd prime $a \neq 3$ then $\bar{G}$ contains a field automorphism of P of order a and hence $a \sim t$. Thus ω^+ is empty, which is impossible.

Suppose that r is even. If $s = (r - v1)/(r - v1, 3) = 1$ then, since $U_3(2)$ is soluble, $v = +$ and r is equal to 2 or 4. Both cases are excluded and so $s = (r - v1)/(r - v1, 3) \neq 1$. Then, as in the previous paragraph,

$$s \sim r - v1 \sim \frac{r^2 - 1}{(r - v1, 3)} \sim 3$$

and hence $s \sim v$ for every prime divisor v of $|O(G)|$ or $|\bar{G}/P|$. If $s \sim 4$ then ω^+ is empty. If $s \not\sim 4$ then ω^- consists only of powers of 2. Every case is impossible. □

Lemma 11. *$P \not\cong L_2(r)$, where $r = t^u$ and t is a prime distinct from p.*

Proof. Suppose false. Then $\mathrm{GK}(P)$ has at least two connected components, one of which is $\{p\}$.

Suppose that $t = 2$. If $f \neq 1$ then, as P contains a Frobenius subgroup of order $r(r - 1)$, by Lemma 4 we have $r - 1 \sim f$. It follows that $p = r + 1$, so that u is a power of 2. In particular, $\bar{G}/P$ is a 2-group. If $\bar{G}/P \neq 1$ then, for a subgroup V of order p in G, the quotient $N_G(V)/V$ is a 2-group of order divisible by 4. By Lemma 4 we have $4 \sim f \sim r-1$ and ω^+ is empty, a contradiction.

Let $\bar{G} = P$. Since $p-1 = r \in \omega(G)$, either $4 \in \omega(N)$, and one can obtain a contradiction as in the previous paragraph, or $r \leqslant 8$ which implies that $r = 4$, $\mu(P) = \{2, 3, 5\}$, and one of the connected components of $\Gamma(G)$ does not contain an odd prime. Hence $f = 1$.

Now, suppose that $\bar{G}/P$ contains an odd prime s. Then $r = (2^h)^s$ where h is a natural number and $\bar{G}$ contains a field automorphism of P of order s. Since $r+1$ is not a prime, $p = r-1$ and hence $h = 1$. Thus $|\bar{G}/P| = s$ and $s \sim 3 \sim r+1$. Since $N \neq 1$ and $\bar{G}$ contains a Frobenius subgroup of order $p \cdot 2s$, by Lemma 4 we have $s \sim 4$. It follows that ω^+ is empty, and this is a contradiction.

Thus $\bar{G}/P$ is a 2-group and hence one of the sets ω^+, ω^- does not contain an odd prime. This is impossible, and t is odd.

Since $p \neq t$, we have $p = (r+\lambda)/2$ where $\lambda = \pm 1$. First we prove that

$$2t < q + \varepsilon, \tag{2}$$

$$r - \lambda < q + \varepsilon. \tag{3}$$

Suppose that $2t \geqslant q + \varepsilon$. Then $4p - 2\lambda = 2r \geqslant 2t \geqslant q + \varepsilon = p^m + \varepsilon$ where $m \geqslant 2$. Since p is an odd prime, this inequality is possible only for $p = 3$, $m = 2$, but this case is excluded by assumption.

Suppose that $r - \lambda \geqslant q + \varepsilon$. Then $2p - 2\lambda \geqslant q + \varepsilon = p^m + \varepsilon$, and one obtains a contradiction as before. Thus (2) and (3) are proved.

Let $\lambda = 1$. Then $r = t^{2^u}$ where $u \geqslant 0$. In particular, $|\bar{G}/P|$ is a power of 2. If $P \neq \bar{G}$ then, for a subgroup R of order p in $\bar{G}$, the normalizer of R in $\bar{G}$ is a Frobenius group whose complement contains an element of order 4. By Lemma 4, either $f \sim 4$ or $f = 1$. If additionally $t \sim 4$, then $(r-1)/2 \nsim 4$ and hence $(r-1)/2 = q + \varepsilon$, which contradicts (3). Thus $t \nsim 4$ and hence either $f = 1$ or $t \nsim f$, so that $2t = q + \varepsilon$, which is impossible by (2). It follows that $P = \bar{G}$.

If $f \sim 4$ or $f = 1$ then one obtains a contradiction as in the previous paragraph. If $4 \nsim f \neq 1$ then $O_2(G)$ is an elementary 2-group and hence either $2t = q + \varepsilon$, or $(r-1)/2 = q - \varepsilon$, or $r - 1 = q - \varepsilon$. All cases are impossible by (2) and (3).

Let $\lambda = -1$. Then either $r = t$, or $t = 3$ and $r = 3^s$ where s is an odd prime.

Suppose that $|\bar{G}/P|$ is divisible by an odd prime s. Then $r = 3^s$ and for a subgroup R of order p in $\bar{G}$, the normalizer of R in $\bar{G}$ is a Frobenius group whose complement contains an element of order $2|\bar{G}/P|$. By Lemma 4, either $f \sim s$, or $f = 1$. On the other hand, $3 = t \sim s$, so that if $3 \sim 4$ then $(r+1)/2 = q + \varepsilon$ which contradicts (3). It follows that $s \sim 3 \nsim 4$. In particular, $|\bar{G}/P| = s$.

If $f \neq 1$ then, since a Sylow 3-subgroup of P is non-cyclic, $3 \sim f$ and hence $O_2(G)$ is an elementary abelian 2-group. It follows that $(r+1)/2 = q + \varepsilon$ or $r + 1 = q + \varepsilon$, which is impossible by (3). Thus $f = 1$ and hence $2ts = q + \varepsilon$. Since $p = (3^s - 1)/2$, one obtains $6s = ((3^s - 1)/2)^m + \varepsilon$, which is impossible for an odd prime s and $m \geqslant 2$.

It follows that $|\bar{G}/P| \leqslant 2$. If $t \sim 4$ and either $f = 1$ or $4 \sim f$ then $(r+1)/2 = q + \varepsilon$, which is impossible. If $t \sim 4$ and $4 \nsim f \neq 1$ then $O_2(G)$ is elementary abelian and G cannot contain an element of order $4t$, and we have a contradiction. If $t \not\sim 4$ and $f = 1$ then $2t = q + \varepsilon$, which is impossible. Thus $t \not\sim 4$ and $f \neq 1$. If $t \sim f$ then $f \not\sim 4$, so that $O_2(G)$ is elementary abelian and hence $(r+1)/2 = q + \varepsilon$ or $r + 1 = q + \varepsilon$, which is impossible by (3). If $t \neq f$ then $2t = q + \varepsilon$, which contradicts (2). $\square$

Lemma 12. *If $P \cong L_2(r)$, where $r = p^u$, then $G \cong H$.*

Proof. It is proved in [15] that if $t = p$ then $m = u$ and so $P \cong L_2(q)$. So we only need to prove that $N = 1$ and $\overline{G} = \mathrm{PGL}_2(q)$. By Lemma 1, p does not divide $|N|$. Since P contains an elementary abelian p-subgroup of order p^2, for any prime divisor s of $|N|$ we have $s \sim p$, and we have a contradiction. Thus $N = 1$ and $\overline{G} = G$.

If there exists an odd divisor s of $|G/P|$ then G/P contains a field automorphism of order s. If $s \neq p$ then $s \sim p$, a contradiction. If $s = p$ then $p \sim 4$, which is impossible. Thus $|G/P|$ is a power of 2. Furthermore, since $\mu(\mathrm{PSL}_2(q)) \neq \mu(H)$, the quotient G/P is non-trivial. If G/P contains a field automorphism then $2p \in \omega(G)$, a contradiction. So G/P is subgroup of order 2. Let φ generate G/P. If φ is a diagonal-field automorphism then one can check that $\omega(G)$ does not contain $q+1$ so that φ is a diagonal automorphism and $G \cong H$. $\square$

Lemma 13. $P \not\cong S_{2n}(r)$, $O_{2n}^{\varepsilon}(r)$, $O_{2n+1}(r)$.

Proof. Suppose first that $P \cong S_{2n}(r)$. Then $n \geqslant 2$ and, for $i = 1, 2, \dots, n-1$, P contains a subgroup P_i isomorphic to a central product of $\mathrm{Sp}_{2i}(r)$ and $\mathrm{Sp}_{2n-2i}(r)$. Let $r = t^u$ where t is a prime. If t is odd then, since

$$|\mathrm{Sp}_{2i}(r)| = r^{i2}(r^2-1)(r^4-1)\dots(r^{2i}-1)/(2, r-1) \quad \text{and} \quad \mathrm{Sp}_2(r) = \mathrm{SL}_2(r),$$

we have $4 \sim t \sim f$ if $f \neq 1$ and also $t \sim r^{2i} - 1$ for $i = 1, 2, \dots, n-1$. Moreover, $t \sim s$ for every odd prime divisor s of $\overline{G}/P$. So p and ω^+ consist of divisors of cyclic tori of orders $(r^n-1)/2$ and $(r^n+1)/2$. If n is even then $(r^n-1)/2 \sim 4$ and $\Gamma(G)$ has at most two connected components, which contradicts the assumption. If n is odd then either $(r^n+1)/2 \sim r+1$ which is again impossible, or $r + 1 = 4$. In this last case, $p = (3^n-1)/2$, $p^m + \varepsilon = (3^n+1)/2$ which is impossible since $m \geqslant 2$.

Thus $t = 2$. If $n \geqslant 3$ then obviously $4 \sim r^2 - 1 \sim f$ if $f \neq 1$ and hence $4 \sim s$ for every odd divisor s of G which is impossible. It follows that $n = 2$. In this case, $r^2 - 1 \sim f$ if $f \neq 1$ and $4 \sim s$ for every odd divisor s of $|\overline{G}/P|$. Thus $p = r^2 + 1$. In particular, $|\overline{G}/P|$ is a power of 2, and hence $\Gamma(G)$ has at most two connected components, which is false.

If P is orthogonal group, then $n \geqslant 4$ for $O_{2n}^{\varepsilon}(r)$ and $n \geqslant 3$ for $O_{2n+1}(r)$. Similar arguments show that $\Gamma(G)$ has at most two connected components, which contradicts the assumption. $\square$

Thus we have to consider only cases when P is an exceptional group of Lie type. For $v \in \{+, -\}$, denote by $E_6^v(r)$ the group $E_6(r)$ if $v = +$ and ${}^2E_6(r)$ if $v = -$.

Throughout the rest of the paper we will use information from [6], [8], [9] on the orders of maximal tori and the orders of centralizers of semisimple elements. We will start from the following easy observation. Since the result holds for groups $H = \mathrm{PGL}_2(p)$, $\mathrm{PGL}_2(2^m)$ and $\mathrm{PGL}_2(9)$, we may assume that $p > 2$, $m \geqslant 3$ if $p = 3$, and $m \geqslant 2$ if $p \geqslant 5$. Now if $p \geqslant 5$ then $2p \leqslant (p^2 - 1)/2$ and if $p = 3$ then $2p \leqslant (p^3 - 1)/2$. So we may assume that the inequality

$$2p < (q + \varepsilon)/2 \tag{4}$$

is true for the group $H = \mathrm{PGL}_2(q)$, $q = p^m$.

Lemma 14. $P \not\cong E_6^v(r)$, $E_7(r)$, $E_8(r)$.

Proof. Suppose false. Let $r = t^u$, where t is prime. Since $12 \in \omega(P)$, we have $3 \in \omega^-$. If t is odd then $4t \in \omega(P)$ and $4 \sim t$. Furthermore, since P contains an elementary abelian group of order t^2, we have $f \sim t$ and $f \in \omega^-$. The same is true for $t = 2$, since P contains a Frobenius subgroup of order 12 with a complement of order 3. Let s be a prime divisor of $|\overline{G}/P|$ which is not 2 or 3. Then s divides u and there exists an element $g \in \overline{G} \backslash P$ of order s which can be represented as a product of a diagonal automorphism δ and a field automorphism θ of P. If δ is non-trivial then s is divisible by 3 in case $P \cong E_6^v(r)$; and s is divisible by 2 if $P \cong E_7(r)$. Both cases are impossible. Therefore g is a field automorphism of P. Hence g centralizes in P a subgroup P_0 isomorphic to the group of the same Lie type as P over the field of order r_0, where $r_0^s = r$. In particular, $s \sim 4$. Thus p and $(q + \varepsilon)/2$ divide the orders of maximal tori of P. Moreover, since $n_2(G) = p$ (see Lemma 2), one of the numbers $n_j(P)$ with $j = 2, \dots, s(P)$ is equal to p. Now we consider distinct possibilities for P in turn.

Let $P \cong {}^2E_6(2)$. Then $p \geqslant 13$ and $(q + \varepsilon)/2 \geqslant 84$. Therefore ω^+ is empty, which is impossible.

Let $P \cong E_6^v(r)$, where $r > 2$ for $v = -$. Then $p = (r^6 + vr^3 + 1)/(3, r - v1)$. Let s be a prime in ω^+. One can check that s must be a primitive prime divisor of $r^{12} - 1$. Therefore s divides $r^4 - r^2 + 1$. But in P there exists a torus T of order $(r^4 - r^2 + 1)(r^2 + vr + 1)/(3, r - v1)$. Since $r > 2$ when $v = -$, we have $s \sim z$ for some odd prime divisor z of $r^2 + vr + 1$. Since $z \in \omega^-$, ω^+ is empty, a contradiction.

Let $P \cong E_7(r)$. Since $s(E_7(r)) = 1$ for any $r > 3$, we need to consider only the groups $E_7(2)$ and $E_7(3)$. In both cases we obtain a contradiction using the same arguments as in the case $P = {}^2E_6(2)$.

Let $P \cong E_8(r)$. Then $\{p, (q + \varepsilon)/2\} \subseteq \rho$, where

$$\begin{aligned}\rho = \{&r^8 - r^7 + r^5 - r^4 + r^3 - r + 1, r^8 - r^6 + r^4 - r^2 + 1,\\ &r^8 - r^4 + 1, r^8 + r^7 - r^5 - r^4 - r^3 + r + 1\}.\end{aligned}$$

For all r and distinct numbers x, y from ρ we have $2x > y$, which contradicts the inequality (4). $\square$

Lemma 15. $P \not\cong G_2(r)$.

Proof. Suppose false. Let $r = t^u$, where t is a prime. Since $G_2(2)' \cong U_3(3)$ we may assume that $r > 2$. If $t = 2$ then

$$\mu(P) = \{8, 12, r^2 - 1, 2(r-1), 2(r+1), r^2 - r + 1, r^2 + r + 1\};$$

and if t is odd then

$$\mu(P) = \{r^2, r(r^2 - 1)/2, r^2 - 1, r^2 - r + 1, r^2 + r + 1\}.$$

It easy to check that for any r we have 3, 4, $(r^2 - 1) \in \omega^-$. If t is odd then t also lies in ω^-. Since P contains a Frobenius subgroup of order $r^2(r^2 - 1)$ with a cyclic complement of order $r^2 - 1$ (see [5, Lemma 1.4]), for non-trivial f we have $f \sim r^2 - 1$. Using the same argument as in the previous lemma we conclude that every odd divisor s of $|\bar{G}/P|$ lies in ω^-. Thus $\{p, (q+\varepsilon)/2\} = \{r^2 - r + 1, r^2 + r + 1\}$. If $t \neq 3$ then either $r^2 - r + 1$ or $r^2 + r + 1$ is divisible by 3, and this is a contradiction. So $t = 3$, $p = r^2 - r + 1$, $(q+\varepsilon)/2 = r^2 + r + 1$. Using the inequality (4) we obtain that $r^2 - 3r + 1 < 0$ which is impossible for $r > 2$. $\square$

Lemma 16. $P \not\cong {}^3D_4(r)$, $F_4(r)$.

Proof. Suppose false. Let $r = t^u$, where t is prime. Since $G_2(r) < {}^3D_4(r) < F_4(r)$, we can argue as in the proof of the previous lemma that ω^- contains 3, 4, $r^2 - 1$, s, where s is an odd divisor of $|N|$. Furthermore, if t is odd then $t \in \omega^-$. Let s be an odd prime divisor of $|\bar{G}/P|$. We may represent an element of order s in $\bar{G} \backslash P$ as a field automorphism. Since $3 \in \omega^-$ we may assume that $s \neq 3$. Now the argument as in Lemma 14 shows that $s \in \omega^-$. Moreover, using information on the orders of tori in P we obtain the following:

If $P \cong {}^3D_4(r)$ then only divisors of $r^4 - r^2 + 1$ do not lie in ω^-. Therefore $\Gamma(G)$ has only two components, a contradiction.

If $P \cong F_4(r)$ then all divisors of $|G|$ except divisors of $r^4 - r^2 + 1$ and $r^4 + 1$ lie in ω^-. Therefore $p = r^4 - r^2 + 1$; and if r is even then $(q+\varepsilon)/2 = r^4 + 1$, while if r is odd then $q + \varepsilon = r^4 + 1$. In each case this contradicts the inequality (4). $\square$

Lemma 17. $P \not\cong {}^2F_4(r)$.

Proof. Suppose false. First assume that P is a Tits group, that is, $P \cong {}^2F_4(2)'$. Then $A = \mathrm{Aut}(P) \cong {}^2F_4(2)$ and $\mu(P) = \{10, 12, 13, 16\}$, $\mu(A) = \{12, 13, 16, 20\}$. Obviously $3, 4 \in \omega^-$. Furthermore, since P contains a Frobenius subgroup of order $52 \cdot 12$ with a cyclic complement of order 12, each odd prime $s \neq 5$ dividing $|N|$ lies in ω^-. Since $s(G) = 2$, we have $p = 13$. If $\bar{G} = A$ then ω^+ is empty. If $\bar{G} = P$ then $(q+\varepsilon)/2 = 5 < 13 = p$. In both cases we obtain a contradiction.

Thus we may assume that $r = 2^u$, where $u > 1$ and u is odd. Arguments like those in the previous lemmas yield that

$$p = r^2 - \sqrt{2r^3} + r - \sqrt{2r} + 1 \quad \text{and} \quad (q+\varepsilon)/2 = r^2 + \sqrt{2r^3} + r + \sqrt{2r} + 1.$$

Let $3p \geqslant (q+\varepsilon)/2$. Then $m = 2$ and $p = 3$ or $p = 5$. Since the case $q = 32$ was studied earlier, we now have $q = 52$ and

$$13 = (q+\varepsilon)/2 = r^2 + \sqrt{2r^3} + r + \sqrt{2r} + 1.$$

This is impossible for every $r > 2$. Thus $3p < (q+\varepsilon)/2$, and so

$$3p - (q+\varepsilon)/2 = 2r^2 - 4\sqrt{2r^3} + 2r - 4\sqrt{2r} + 1 < 0.$$

However this also is impossible for every $r > 2$. $\square$

Lemma 18. $P \not\cong \mathrm{Sz}(r)$.

Proof. Suppose false. The group ${}^2B(2)$ is a Frobenius group with kernel of order 5 and a cyclic complement of order 4; it is not simple. Thus $\mathrm{Sz}(r) \cong {}^2B(r)$, where $r = 2^u$, $u > 1$ and u is odd. We have

$$\mu(P) = \{4, r-1, r-\sqrt{2r}+1, r+\sqrt{2r}+1\} \quad \text{and} \quad |\mathrm{Out}(P)| = u.$$

Since each outer automorphism of P is a field automorphism, it centralizes ${}^2B(2)$, and so $u \sim 4$. On the other hand, since P contains a subgroup isomorphic to ${}^2B(2)$, any odd prime divisor of $|N|$ not equal to 5 lies in ω^-. Since 5 divides the order of $\mathrm{Sz}(r)$ for every r we have

$$\{p, (q+\varepsilon)/2\} \subset \{r-1, r-\sqrt{2r}+1, r+\sqrt{2r}+1\}.$$

Now the inequality (4) is possible if and only if $r = 8$, $p = r - \sqrt{2r} + 1 = 5$ and $(q+\varepsilon)/2 = r + \sqrt{2r} + 1 = 13$. But $7 \in \omega(\mathrm{Sz}(8)) \setminus \omega(L_2(25))$, and we have a contradiction. $\square$

Lemma 19. $P \not\cong {}^2G_2(r)$.

Proof. Suppose false. The group ${}^2G_3(3)$ is not simple, and ${}^2G_3(3)' \cong L_2(8)$, which has been discussed. Thus $r = 3^u$, where $u > 1$ and u is odd. As in Lemma 18 we argue that $p = r - \sqrt{3r} + 1$ and $(q+\varepsilon)/2 = r + \sqrt{3r} + 1$. For every r this contradicts to the inequality (4). This completes the proof of the lemma and the theorem. $\square$

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