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Algebra Colloquium

Recognition by Spectrum of $L_{16}(2^m)$ *

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Abstract. In this paper we prove that the simple linear groups $L_{16}(2^m)$ $(m \geq 1)$ over fields of characteristic 2 are recognizable by the sets of their element orders.

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Given a finite group G, denote by $\omega(G)$ the *spectrum* of G, i.e., the set of its element orders. A group G is said to be *recognizable by spectrum* (briefly, *recognizable*) if every finite group H with $\omega(H) = \omega(G)$ is isomorphic to G. Since a finite group with a non-trivial normal soluble subgroup is not recognizable [8, Corollary 4], the recognition problem for simple and almost simple groups is of prime interest.

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At present there is a vast list of finite and almost finite groups with solved recognition problem. The most recent version of this list is presented in [5, Table 1], and references to some new results can be found in [10]. We mention some results concerning the recognition of simple linear groups over fields of characteristic 2. The following groups were proved to be recognizable: $L_2(2^m)$ with $m \geq 2$ (see [7]) and $L_3(2^m)$ with $m \geq 1$ (see [6]). All these groups have disconnected prime graphs and a certain property of these groups, called quasi-recognizability, was proved with applying the Gruenberg–Kegel theorem on groups with disconnected prime graphs (see [13]). A finite non-abelian simple group S is said to be *quasi-recognizable* if every finite group H with the same spectrum as S contains a unique non-abelian composition factor and this factor is isomorphic to S.

Recent results [10] and [12] allow considering of recognition problem for groups with connected prime graphs. In this way in a recent paper [11], the simple linear groups $L_n(2^m)$, where $n = 2^l \geq 32$, were proved to be recognizable. The cases when n is a power of 2 and equals 4, 8, and 16 were left out of consideration since the corresponding groups require special methods. The point is that the less rank of a Lie type group we investigate, the more simple groups we have to consider proving quasi-recognizability. In this paper, we establish recognizability for the case $n = 16$.

Theorem. *The simple linear groups* $L = L_{16}(2^m)$ $(m \geq 1)$ *are recognizable by spectrum.*

1 Preliminaries

Let G be a finite group, and $\omega(G)$ be its spectrum. The set $\omega(G)$ is ordered by the divisibility relation and we denote by $\mu(G)$ the set of its elements that are maximal under this relation. If p is a prime, then the p-period of G is the maximal power of p that belongs to $\omega(G)$.

Let $\pi(G)$ be the set of all prime divisors of the order of G. On the set $\pi(G)$, we define a graph with the following adjacency relation: vertices p and r in $\pi(G)$ are joined by an edge if and only if $pr \in \omega(G)$. This graph is called the *Gruenberg–Kegel graph* or *prime graph* of G and denoted by $\mathrm{GK}(G)$ (see [13]). Guided by the given graph conception, we say that prime divisors p and r of the order of G are *adjacent* if vertices p and r are joined by an edge in $\mathrm{GK}(G)$. Otherwise, primes p and r are said to be *non-adjacent*.

The set of vertices of a graph is called *independent* if vertices of this set are pairwise non-adjacent. The cardinality of an independent set with maximal number of vertices is usually called the *independence number* of the graph. Denote by $t(G)$ the independence number of the graph $\mathrm{GK}(G)$ of G. By analogy, we denote by $t(2, G)$ the maximal number of vertices in independent sets of $\mathrm{GK}(G)$ containing the vertex 2. We call this number the 2-*independence number*.

The following result concerning connection between the structure of a finite group and the properties of its prime graph is proved in [10].

Lemma 1. [10] *Let G be a finite group satisfying two conditions*:

(a) *There exist three primes in $\pi(G)$ which are pairwise non-adjacent in* $\mathrm{GK}(G)$*, that is, $t(G) \geq 3$.*

(b) *There exists an odd prime in $\pi(G)$ which is non-adjacent to 2 in $\mathrm{GK}(G)$, that is, $t(2,G) \geq 2$.*

Then there exists a finite non-abelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for a maximal normal soluble subgroup K of G. Furthermore, $t(S) \geq t(G) - 1$ and one of the following statements holds:

(1) *$S \simeq \mathrm{Alt}_7$ or $L_2(q)$ for some odd q and $t(S) = t(2,S) = 3$.*

(2) *For every prime p in $\pi(G)$ non-adjacent to 2 in $\mathrm{GK}(G)$, the Sylow p-subgroup of G is isomorphic to the Sylow p-subgroup of S. In particular, $t(2,S) \geq t(2,G)$.*

To apply this result, we use the values of independence and 2-independence numbers of finite simple groups calculated in [12].

We use the following number-theoretic notation. If n is a natural number, then $\pi(n)$ is the set of prime divisors of n. If $p \in \pi(n)$, then $n|_p$ is the maximal p-power that divides n. By $[x]$ we denote the integer part of x. If q is a natural number, r is an odd prime and $(q,r) = 1$, then by $e(r,q)$ we denote the smallest natural number m such that $q^m \equiv 1 \pmod r$. Given an odd q, put $e(2,q) = 1$ if $q \equiv 1 \pmod 4$ and put $e(2,q) = 2$ if $q \equiv -1 \pmod 4$.

The following number-theoretic result is of fundamental importance for investigations on the structure of prime graphs of finite simple groups of Lie type.

Lemma 2. [14] *Let q be a natural number greater than 1. Then for every natural number l, there exists a prime r such that $e(r,q) = l$, except for the following cases:*

(1) *$l = 6$ and $q = 2$;*

(2) *$l = 2$ and $q = 2^m - 1$ for some natural number m.*

The prime r with $e(r,q) = l$ is called a *primitive prime divisor* of $q^l - 1$. If q is fixed, we denote by r_l any primitive prime divisor of $q^l - 1$ (obviously, $q^l - 1$ can have more than one primitive prime divisor).

Lemma 3. [4, Lemma 1] *Let G be a finite group, $K \lhd G$, and G/K be a Frobenius group with kernel F and a cyclic complement C. If $(|F|, |K|) = 1$ and F does not lie in $KC_G(K)/K$, then $r \cdot |C| \in \omega(G)$ for some prime divisor r of $|K|$.*

Lemma 4. *Let q be a power of a prime p and let r_{2n-2} be a primitive prime divisor of $q^{2n-2} - 1$. The group ${}^2D_n(q)$, where $(n,q) \neq (4,2)$, contains a Frobenius subgroup whose kernel is an elementary abelian p-group and complement is cyclic of order r_{2n-2}.*

Proof. By [3, Part 8, A], there is a parabolic subgroup in ${}^2D_n(q)$ whose Levi radical U is an elementary abelian p-group and whose Levi subgroup contains ${}^2D_{n-1}(q)$. The group ${}^2D_{n-1}(q)$ contains an element x of order r_{2n-2}. Since $pr_{2n-2} \notin \omega({}^2D_n(q))$ (see [12, Proposition 3.1]), the element x acts on U regularly. Thus, $U \cdot \langle x \rangle$ is a desired Frobenius group. □

Lemma 5. [11, Proposition 1] *Let $L = L_n(q)$, where $n = 2^m \geq 4$ and $q = 2^k \geq 2$. Let G be a finite group and K be its non-trivial normal soluble subgroup satisfying $L \leq G/K \leq \mathrm{Aut}(L)$. Then $\omega(G) \not\subseteq \omega(L)$.*

Lemma 6. [11, Proposition 2] *Let* $L = L_n(q)$*, where* $n \geq 10$*,* $q = 2^k \geq 2$*, and* $(q-1, n) = 1$*. Suppose that* $L < G \leq \mathrm{Aut}(L)$*. Then* $\omega(G) \not\subseteq \omega(L)$*.*

2 Proof of the Theorem

Let $L = L_{16}(2^m) = A_{15}(2^m)$, where $m \geq 1$. By [12, §8], we have $t(L) = 8$ and $t(2, L) = 3$. Furthermore, by [9, Proposition 0.5], the 2-period of L is equal to 16.

Let G be a finite group with $\omega(G) = \omega(L)$ and K be the maximal normal soluble subgroup of G. By Lemma 1, there is a finite non-abelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$, moreover, $t(S) \geq t(G) - 1$ and either $t(S) = t(2, S) = 3$ or $t(2, S) \geq t(2, G)$. Since $t(G) = t(L) \geq 8$ and $t(2, G) = t(2, L) = 3$, the group S must satisfy $t(S) \geq 7$ and $t(2, S) \geq 3$. By using [12, §8], we make a table of all the finite non-abelian simple groups satisfying these conditions. For every group S, the table shows the 2-independence number and some independent set $\rho(2, S)$ of $\mathrm{GK}(S)$ with maximal number of vertices among those containing the vertex 2. Furthermore, for every classical group of Lie type, the table gives the independence number as a function of Lie rank; and for sporadic groups and exceptional groups of Lie type, this number is given explicitly.

The proof of quasi-recognizability relies on an case by case analysis of all possibilities for S from this table. The cases of alternating groups and classical groups over fields of characteristic 2 have been considered in [11]; all of them except for the case $S \simeq L$ lead to a contradiction. We examine only the rest cases. Through this paragraph r_l denotes a prime such that $e(r_l, 2^m) = l$.

Let $S = A^{\varepsilon}_{n-1}(q)$ with odd q. Then $n|_2 = (q - \varepsilon 1)|_2 > 2$ and $t(S) = n/2$. Since $t(S) \geq t(G) - 1$ and $t(G) = 8$, we have $n'/2 \geq 7$, whence $n \geq 14$. Therefore, S contains a cyclic subgroup of order $q^8 - 1$. In view of

$$(q^8 - 1)|_2 = (q - 1)|_2(q + 1)|_2(q^2 + 1)|_2(q^4 + 1)|_2 \geq 4 \cdot 2^3 = 32,$$

we have $32 \in \omega(S)$; a contradiction.

Let $S = D^{\varepsilon}_n(q)$ with odd q. Then $q - \varepsilon 1 \equiv 4 \pmod 8$, $n' \equiv 1 \pmod 2$ and $t(S) \leq [(3n + 4)/4]$. Since $t(S) \geq t(G) - 1$ and $t(G) = 8$, we have $(3n + 3)/4 \geq 7$, which implies $n \geq 8$. Actually, $n \geq 9$ since n' is odd. Suppose that $S \neq {}^2D_9(q)$. Then S contains the universal covering of $A_8(q)$ and thus S contains an element of order $q^8 - 1$. By repeating the above argumentation, we have $32 \in \omega(S)$; a contradiction.

Let $S = {}^2D_9(q)$, where $q = p^k$ and p is odd. Since $\rho(2, L) = \{2, r_{15}, r_{16}\}$, it follows from Lemma 1 that r_{15} and r_{16} divide $|S|$, and they are not adjacent to 2 in $\mathrm{GK}(S)$. Therefore, by [12, Proposition 6.7] and Table 1 below, we have $\{e(r_{15}, q), e(r_{16}, q)\} = \{16, 18\}$. Let $r'_{16} \in \{r_{15}, r_{16}\}$ and $e(r'_{16}, q) = 16$.

Denote by r a primitive prime divisor r_{14} of $q^{14} - 1$. Suppose that r divides $|S|$. Since the primes r, r_{15}, r_{16} are pairwise non-adjacent in $\mathrm{GK}(L)$, they are pairwise non-adjacent in $\mathrm{GK}(S)$ as well. Hence, $e(r, q) \notin \{16, 18\}$. As one can verify using [1, Proposition 10], the last condition implies that $4r \in \omega(S)$. On the other hand, it follows from [1, Proposition 7] that $4r \notin \omega(L)$. Since $\omega(S) \subseteq \omega(L)$, we have a contradiction. Thus, r does not divide $|S|$.

Table 1. Simple groups S with $t(S) \geq 7$ and $t(2,S) \geq 3$

S	Conditions	$t(2,S)$	$\rho(2,S) \setminus \{2\}$	$t(S)$
J_4		6	$\{23, 29, 31, 37, 43\}$	7
F_1	none	5	$\{29, 41, 59, 71\}$	11
F_2		3	$\{31, 47\}$	8
Alt_n	n, $n-2$ are prime	3	$\{n,\, n-2\}$	—
$n \geq 47$	$n-1$, $n-3$ are prime	3	$\{n-1,\, n-3\}$	
$A_{n-1}(q)$	$2 < (q-1)\|_2 = n\|_2$	3	$\{r_{n-1}, r_n\}$	$\left[\frac{n+1}{2}\right]$
$n \geq 13$	q even	3	$\{r_{n-1}, r_n\}$	
$^2A_{n-1}(q)$	$2 < (q+1)\|_2 = n\|_2$	3	$\{r_{2n-2},\, r_n\}$	$\left[\frac{n+1}{2}\right]$
$n \geq 13$	q even, $n \equiv 0 \pmod 4$	3	$\{r_{2n-2},\, r_n\}$	
	$n \equiv 1 \pmod 4$	3	$\{r_{n-1},\, r_{2n}\}$	
	$n \equiv 2 \pmod 4$	3	$\{r_{2n-2},\, r_{n/2}\}$	
	$n \equiv 3 \pmod 4$	3	$\{r_{(n-1)/2},\, r_{2n}\}$	
$B_n(q)$, $n \geq 8$	q even	3	$\{r_n, r_{2n}\}$	$\left[\frac{3n+5}{4}\right]$
$D_n(q)$	$q \equiv 5 \pmod 8$, $n \equiv 1 \pmod 2$	3	$\{r_n,\, r_{2n-2}\}$	$\left[\frac{3n+1}{4}\right]$
$n \geq 9$	q even, $n \equiv 0 \pmod 2$	3	$\{r_{n-1},\, r_{2n-2}\}$	
	$n \equiv 1 \pmod 2$	3	$\{r_n,\, r_{2n-2}\}$	
$^2D_n(q)$	$q \equiv 3 \pmod 8$, $n \equiv 1 \pmod 2$	3	$\{r_{2n-2},\, r_{2n}\}$	$\left[\frac{3n+4}{4}\right]$
$n \geq 8$	q even, $n \equiv 0 \pmod 2$	4	$\{r_{n-1},\, r_{2n-2},\, r_{2n}\}$	
	$n \equiv 1 \pmod 2$	3	$\{r_{2n-2},\, r_{2n}\}$	
$E_7(q)$	$q \equiv 1 \pmod 4$	3	$\{r_{14},\, r_{18}\}$	8
	$q \equiv 3 \pmod 4$	3	$\{r_7,\, r_9\}$	
	q even	5	$\{r_7,\, r_9,\, r_{14},\, r_{18}\}$	
$E_8(q)$	none	5	$\{r_{15},\, r_{20},\, r_{24},\, r_{30}\}$	11

Suppose first that $r \in \pi(\overline{G}/S)$ and α is the element in $\overline{G} \setminus S$ of the corresponding order. We may assume that α is a product of a diagonal automorphism δ and a field automorphism φ. The group of diagonal automorphisms of S is cyclic of order 4. If $\varphi = 1$, then $|\alpha|$ divides 4, but r is odd. Thus, $\varphi \neq 1$. The element φ normalizes the subgroup of diagonal automorphisms. Since this subgroup is cyclic, we have $\overline{\delta}^{\varphi} = \overline{\delta}^{\,l}$, where $\overline{\delta}$ is the image of δ in $\overline{G}/S$ and l is a natural number. As $|\delta|$ divides 4, the number l must equal 1, i.e., φ centralizes δ and $|\alpha| = |\delta| \cdot |\varphi|$. Thus, $\delta = 1$. The centralizer C of α in S contains the group $^2D_9(q_0)$, where $q_0 = p^{k/r}$. Since $^2D_9(q_0)$ contains an element of order 4, the group $\overline{G}$ contains an element of order $4r$; a contradiction.

Now suppose that $r \in \pi(K)$. Let $\widetilde{G} = G/O_{r'}(K)$ and $\widetilde{K} = K/O_{r'}(K)$. Then $R = O_r(\widetilde{K}) \neq 1$. Suppose that $\widetilde{K} = R$. The group S acts faithfully on $\widetilde{K}$. Otherwise, in view of its simplicity, S centralizes $\widetilde{K}$, therefore G contains an element of order $r'_{16} \cdot r$. By Lemma 4(1), the group S contains a Frobenius group F whose kernel is an elementary abelian p'-group and complement is a cyclic group of order

r'_{16}. By applying Lemma 3 to the preimage of F in $\widetilde{G}$, we obtain $r'_{16} \cdot r \in \omega(G)$; a contradiction. Let $\widetilde{K} \neq R$. There is a prime t such that $T = O_t(\widetilde{K}/R)$ is non-trivial. Since $O_{r'}(\widetilde{K}) = 1$, the group T acts faithfully on R. Then T acts faithfully on $\widehat{R} = R/\Phi(R)$ as well, where $\Phi(R)$ is the Frattini subgroup of R. Denote by $\widehat{G}$ the factor group $\widetilde{G}/\Phi(R)$. By [11, Lemma 4(3)], at least one of the primes r_{16} and r_{15} is non-adjacent to t in $\omega(G)$. Denote this prime by s. Let x be an element of order s in $\widehat{G}/\widehat{R}$. Then $H = T\langle x\rangle$ is a Frobenius subgroup in $\widehat{G}/\widehat{R}$. The preimage of H in $\widehat{G}$ satisfies conditions of Lemma 3, hence G contains an element of order $r \cdot s$; a contradiction.

Let $S = E_7(q)$, where $q = p^k$ is odd. Recall that r_{15} and r_{16} divide $|S|$, and they are not adjacent to 2 in $\mathrm{GK}(S)$. Therefore, by [12, Proposition 6.7] and Table 1, we have that the set $\{e(r_{15}, q), e(r_{16}, q)\}$ coincides with $\{14, 18\}$ if $q \equiv 1 \pmod 4$, or $\{7, 9\}$ if $q \equiv 3 \pmod 4$.

Suppose that $q \equiv 1 \pmod 4$. Let $t \in \pi(S)$ and x be an element of order t in S. If $e(t, q) = 14$, then x lies in the unique (up to conjugation) maximal torus of maximal period $n_{14} = (q^7 + 1)/2$; if $e(t, q) = 18$, then x lies in the unique maximal torus of maximal period $n_{18} = (q^6 - q^3 + 1)(q + 1)/2(3, q + 1)$ (see [2]). The numbers n_{14} and n_{18} have a common prime divisor. Denote this divisor by s. Then s is adjacent in $\mathrm{GK}(S)$ to every prime divisor of n_{14} and to every prime divisor of n_{18}. Hence, both numbers r_{15} and r_{16} are adjacent to s in $\mathrm{GK}(S)$. However, by [11, Lemma 4], there is no number in $\pi(L)$ adjacent to both primes r_{15} and r_{16} in $\mathrm{GK}(L)$; a contradiction.

Suppose that $q \equiv 3 \pmod 4$. Let $t \in \pi(S)$ and x be an element of order t in S. If $e(t, q) = 7$, then x lies in the unique (up to conjugation) maximal torus of maximal period $n_7 = (q^7 - 1)/2$; if $e(t, q) = 9$, then x lies in the unique maximal torus of maximal period $n_9 = (q^6 + q^3 + 1)(q - 1)/2(3, q - 1)$ (see [2]). The numbers n_7 and n_9 have a common prime divisor except when $q = 3$. Hence, if $q > 3$, then we proceed as in the previous paragraph.

Let $S = E_7(3)$. The unique primitive prime divisors of $3^7 - 1$ and $3^9 - 1$ are 1093 and 757, respectively. Therefore, for any primitive prime divisors r_{15} and r_{16}, the set $\{r_{15}, r_{16}\}$ must coincide with $\{1093, 757\}$. Since $e(757, 2) = 756$, either $15m$ or $16m$ is divisible by 756. Hence, $m \geq 189$. The set $\pi(L)$ contains a prime r with $e(r, 2) = 16m$. Since $e(r, 2) \leq r - 1$, we have $r > 16m \geq 3024$. On the other hand, $r \in \{1093, 757\}$; a contradiction.

Let $S = E_8(q)$, where q is odd. Since S contains a torus of order $q^8 - 1$, we have $32 \in \omega(S)$; a contradiction.

Let S be $E_7(2^k)$ or $E_8(2^k)$. Choose primitive prime divisors r_{16} and r_{15} of $q^{16} - 1$ and $q^{15} - 1$ such that $e(r_{16}, 2) = 16k$ and $e(r_{15}, 2) = 15k$, respectively. By Lemma 1, the primes r_{16} and r_{15} divide the order of S. Put $e_{16} = e(r_{16}, 2^k)$ and $e_{15} = e(r_{15}, 2^k)$. Suppose that $e_{16}k > 16m$. Then a prime r with $e(r, 2) = e_{16}k$ divides the order of S and does not divide the order of L. So $r \in \omega(S) \setminus \omega(G)$, which is impossible. Thus, $e_{16}k = 16m$. Suppose that $e_{15}k > 15m$. Then $e_{15}k \geq 30m > 16m$ and the similar argumentation leads us to a contradiction. Thus, $e_{15}k = 15m$. Therefore, $e_{16}/e_{15} = 16/15$. On the other hand, since r_{16} and r_{15} are non-adjacent to 2 in $\mathrm{GK}(S)$, by [12, Proposition 3.2], we have that e_{16} and e_{15} belong to $\{7, 9, 14, 18\}$ in the case of $E_7(2^k)$, and to $\{15, 20, 24, 30\}$ in the case of $E_8(2^k)$; an easy contradiction.

Let S be a sporadic group. Choose r_{16} and r_{15} as above. By Lemma 1, the primes r_{16} and r_{15} divide the order of S and are non-adjacent to 2 in $\mathrm{GK}(S)$. All primes non-adjacent to 2 in $\mathrm{GK}(S)$ belong to the set $\rho(2, S)$ from Table 1. Therefore, $16k, 15k \in e(S) = \{e(l, 2) \mid l \in \rho(2, S)\}$. Since $e(S)$ is equal to $\{5, 11, 14, 28, 36\}$ if $S = J_4$, to $\{20, 28, 35, 58\}$ if $S = F_1$, and to $\{5, 23\}$ if $S = F_2$, we have a contradiction.

Thus, $S \simeq L$ and quasi-recognizability is proved. Applying Lemma 5 and Lemma 6, we complete the proof of the theorem. □

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