THE STRUCTURE OF A FINITE GROUP AND THE MAXIMUM π-INDEX OF ITS ELEMENTS

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Abstract. Given a set of primes π, the *π-index* of an element x of a finite group G is the π-part of the index of the centralizer of x in G. If $\pi = \{p\}$ is a singleton, we just say the *p-index*. If the π-index of x is equal to $p_1^{k_1} \dots p_s^{k_s}$, where $p_1, \dots, p_s$ are distinct primes, then we set $\epsilon_\pi(x) = k_1 + \dots + k_s$. In this short note, we study how the number $\epsilon_\pi(G) = \max\{\epsilon_\pi(x) : x \in G\}$ restricts the structure of the factor group $G/Z(G)$ of G by its center. First, for a finite group G, we prove that $\phi_p(G/Z(G)) \leq \epsilon_p(G)$, where $\phi_p(G/Z(G))$ is the Frattini length of a Sylow p-subgroup of $G/Z(G)$. Second, for a π-separable finite group G, we prove that $l_\pi(G/Z(G)) \leq \epsilon_\pi(G)$, where $l_\pi(G/Z(G))$ is the π-length of $G/Z(G)$.
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1. Introduction

Given a finite group G and $x \in G$, we refer to the size $|x^G|$ of the conjugacy class x^G as the *index* of x, motivating by the well-known fact that it is equal to the index of the centralizer $C_G(x)$ of x in G. At the very beginning of the 20th century, Burnside proved that a group of composite order, having an element whose index is a prime power, cannot be simple; it was the key ingredient in his famous theorem on the solvability of finite $\{p, q\}$-groups [2]. Since then, the analysis of connection between the structure of a group and the properties of the indices of its elements has become a popular topic in finite group theory. The presented short note is within this research field. We are particulary interested in those properties of a group that follow from the restrictions on π-parts of the indices, where π is some set of primes. All groups below are assumed to be finite.

To proceed, we need the following notation. For a natural number n and a set of primes π, let us denote by n_π the π-part of n and by $n_{\pi'}$ the π'-part of n, that is, if $n = p_1^{k_1} \dots p_s^{k_s} q_1^{m_1} \dots q_t^{m_t}$, where $p_1, \dots, p_s \in \pi$ and $q_1 \dots, q_t \in \pi'$, then $n_\pi = p_1^{k_1} \dots p_s^{k_s}$ and $n_{\pi'} = q_1^{m_1} \dots q_t^{m_t}$. The *$\pi$-exponent* of n is the number $\exp_\pi(n) = k_1 + \dots + k_s$. If the set π contains only one prime p, then we say the p-part, p'-part, and *p-exponent* of n.

For a group G, element $x \in G$, and set of primes π, we refer to the number $|x^G|_\pi$ as the *π-index* of x in G. Put $\epsilon_\pi(x) = \exp_\pi(|x^G|)$ and $\epsilon_\pi(G) = \max\{\epsilon_\pi(x) : x \in G\}$. In the case $\pi = \{p\}$, the π-index is called *p-index* of x in G, $\epsilon_p(x) = \exp_p(|x^G|)$, and $\epsilon_p(G) = \max\{\epsilon_p(x) : x \in G\}$.

As usual, $\Phi(G)$ stands for the Frattini subgroup of G, that is the intersection of all maximal subgroups of G. Put $\Phi_0(G) = G$, and for every positive integer m, define $\Phi_m(G) = \Phi(\Phi_{m-1}(G))$, the *mth Frattini subgroup* of G. Set $\phi_p(G)$ for the least integer m such that $\Phi_m(P) = 1$ for a Sylow p-subgroup P of G.

We are ready to formulate the first main result of the paper.

Theorem 1.1. *If G is a finite group, then $\phi_p(G/Z(G)) \leq \epsilon_p(G)$.*

It is worth noting that Theorem 1.1 generalizes [11, Lemma 4]. See some other generalizations of that lemma in [7, 12].

Let $d_p(G)$ denote the derived length of a Sylow p-subgroup P of a group G, and let $e_p(G)$ be the integer such that $p^{e_p(G)}$ is the maximum order of elements in P. Since $d_p(G) \leq \phi_p(G)$ and $e_p(G) \leq \phi_p(G)$ (see Lemma 2.3 below), the following holds.

Corollary 1.2. *If G is a finite group, then $d_p(G/Z(G)) \leq \epsilon_p(G)$ and $e_p(G/Z(G)) \leq \epsilon_p(G)$.*

For a set of primes π, a group G is said to be *π-separable*, if G has a normal series each of whose factors is either a π-group or π'-group. Among all such normal series there is the special one (it is shortest, in particular) called the *upper π-series* and defined as follows:

$$1 = P_0 \leqslant K_0 < P_1 < K_1 < \ldots < P_l \leqslant K_l = G, \tag{1}$$

where $K_i/P_i = O_{\pi'}(G/P_i)$ and $P_{i+1}/K_i = O_\pi(G/K_i)$, $i = 0, \ldots l$, are the largest normal π'-subgroup of G/P_i and π-subgroup of G/K_i, respectively. The number l, the least integer such that $K_l = G$, is called the *π-length* of G and is denoted by $l_\pi(G)$. A π-separable group is called *π-solvable*, if the π-factors P_{i+1}/K_i are solvable groups. If $\pi = \{p\}$, then we say *p-solvable* (there is no any distinction with p-separability in this case), the *upper p-series*, the *p-length* $l_p(G)$ (the definitions and notations from this paragraph originate from the classical Hall–Higman paper [8]).

If G is a p-solvable group, then $l_p(G) \leq d_p(G)$; it was proved in [8] for odd p and in [1] for $p = 2$. Thus, the p-length of a p-solvable group is also bounded from above by $\epsilon_p(G)$.

If G is a π-solvable group, then the complete analog of the famous theorem of P. Hall holds for Hall π-subgroups of G: they exist, are conjugate to each other, and every π-subgroup is contained in some Hall π-subgroup, see, e.g. [6, Subsection 6.3]. Moreover, the inequality $l_\pi(G) \leq d_\pi(G)$, where $d_\pi(G)$ is the derived length of a Hall π-subgroup, was also established provided $2 \notin \pi$, see [5, 9]. However, this inequality cannot be converted in the inequality between the π-length of $G/Z(G)$ and $\epsilon_\pi(G)$, as the following simple example shows.

Example 1.3. Suppose that G is a nonabelian group of order pq, where p and q are primes, and $\pi = \{p, q\}$. Then $d_\pi(G/Z(G)) = d(G) = 2$ but $\epsilon_\pi(G) = 1$.

Nevertheless, as our second main result shows, the π-length of $G/Z(G)$ can be bounded by $\epsilon_\pi(G)$ directly, even if G is just a π-separable group.

Theorem 1.4. *If G is a finite π-separable group, then $l_\pi(G/Z(G)) \leq \epsilon_\pi(G)$.*

It is worth noting that each of the main results of the paper becomes false if one tries to replace $G/Z(G)$ by G in the left part of the corresponding inequality. To see this, it suffices to take an abelian group G whose order has a nontrivial π-part.

2. The Frattini length and maximum p-index

In this section, we prove Theorem 1.1. We begin with two properties of conjugacy classes that are readily seen.

Lemma 2.1. *Let P be a Sylow p-subgroup of a group G, and $x \in G$. If $\epsilon_p(x) = m$, then there is an element $z \in x^G$ such that $|P : P \cap C_G(z)| = p^m$.*

Proof. Let us take a Sylow p-subgroup Q of G maximal with respect to the size of the intersection $Q \cap C_G(x)$. Since $|G|_p = |Q|$ and $|C_G(x)|_p = |Q \cap C_G(x)|$, it follows that $|Q : Q \cap C_G(x)| = |G : C_G(x)|_p = p^m$. By the Sylow theorem, there is $y \in G$ with $P = Q^y$. Therefore, for $z = x^y$, $|P : P \cap C_G(z)| = |Q^y : Q^y \cap C_G(x^y)| = |Q : Q \cap C_G(x)| = p^m$. □

Lemma 2.2. *If $\{z_1, \dots, z_s\}$ is a complete set of representatives of conjugacy classes of a group G, then $G = \langle z_1, \dots, z_s \rangle$.*

Proof. It was noted by Burnside in [3, § 26]. □

Now we turn to the necessary properties of the Frattini series. Recall that for a p-group G, the Frattini subgroup $\Phi(G)$ of G can be also defined as $\Phi(G) = [G, G]G^p$, where $[G, G]$ is the derived subgroup of G and $G^p = \langle x^p : x \in G \rangle$, see, e.g., [10, Defintion 4.6] or [6, Theorem 5.1.3].

Lemma 2.3. *$d_p(G) \leq \phi_p(G)$ and $e_p(G) \leq \phi_p(G)$.*

Proof. Suppose that P is a Sylow p-subgroup of a group G. If $P^{(m)} = [P^{(m-1)}, P^{(m-1)}]$ is mth member of the derived series of P, where $P^{(0)} = P$, and $P^{p^m} = \langle x^{p^m} : x \in P \rangle$, then the later definition of the Frattini subgroup of a p-group implies inductively that $P^{(m)}P^{p^m} \leq \Phi_m(P)$, and we are done. □

Lemma 2.4. *If H is a subgroup of a p-group G, then $\Phi(H)$ is a subgroup of $\Phi(G)$.*

Proof. It is clear that $[H, H] \leq [G, G]$ and $H^p \leq G^p$, so $\Phi(H) = [H, H]H^p \leq [G, G]G^p \leq \Phi(G)$, as required. □

Lemma 2.5. *If H is a subgroup of a p-group G with $|G : H| = p^m$, then $\Phi_m(G) \leq H$.*

Proof. Induction on m. The claim is clear when $m = 0$ and $H = G$. So we may assume that $m > 0$ and H is a proper subgroup of G. Since G is a p-group, there is a subgroup K of G such that H is a subgroup of K and $|K : H| = p$. Then $|G : K| = p^{m-1}$ and the inductive hypothesis implies that $\Phi_{m-1}(G) \leq K$. Since H is a maximal subgroup of K, it follows that $\Phi(K) \leq H$. Now Lemma 2.4 yields $\Phi_m(G) = \Phi(\Phi_{m-1}(G)) \leq \Phi(K) \leq H$, and we are done. □

Lemma 2.6. *If H is a p-subgroup of a group G, and $Z = Z(G)$, then $\Phi(HZ/Z) = \Phi(H)Z/Z$.*

Proof. Since $[HZ, HZ] = [H, H]$ and $(HZ)^p = H^pZ^p$, it follows that $[HZ/Z, HZ/Z] = [H, H]Z/Z$ and $(HZ/Z)^p = H^pZ/Z$. Therefore,

$$\Phi(HZ/Z) = [HZ/Z, HZ/Z](HZ/Z)^p = [H, H]H^pZ/Z = \Phi(H)Z/Z,$$

as required. □

Lemma 2.7. *Let P be a p-subgroup of a group G, and $Z = Z(G)$. Then $\Phi_m(PZ/Z) = \Phi_m(P)Z/Z$.*

Proof. Induction on m. It is clear for $m = 0$. Now $m > 0$ and, by the inductive hypothesis, $\Phi_{m-1}(PZ/Z) = \Phi_{m-1}(P)Z/Z$. Applying Lemma 2.6 for $H = \Phi_{m-1}(P)$, we have

$$\Phi_m(PZ/Z) = \Phi(\Phi_{m-1}(PZ/Z)) = \Phi(\Phi_{m-1}(P)Z/Z) = \Phi(\Phi_{m-1}(P))Z/Z = \Phi_m(P)Z/Z,$$

as required. □

We are in position to finish the proof.

Proof of Theorem 1.1. Let $\epsilon_p(G) = m$. Then for every $x \in G$, $|x^G|_p = |G : C_G(x)|_p \leq p^m$. Let us fix some Sylow p-subgroup P of G. By Lemma 2.1, there is $z \in x^G$ with

$$|P : P \cap C_G(z)| \leq p^m.$$

By Lemma 2.5,

$$\Phi_m(P) \leq P \cap C_G(z) \leq C_G(z).$$

Since this is true for all $x \in G$, Lemma 2.2 implies that one can take elements $z_1, \dots, z_s$ of G such that $G = \langle z_1, \dots, z_s \rangle$ and $\Phi_m(P) \leq C_G(z_i)$ for every $i = 1, \dots, s$. It follows that $\Phi_m(P) \leq Z$, where $Z = Z(G)$. Lemma 2.7 yields

$$\Phi_m(PZ/Z) = \Phi_m(P)Z/Z = 1,$$

and we are done.

3. The π-length and maximum π-index

The goal of this section is to prove Theorem 1.4.

Lemma 3.1. *Let N be a normal subgroup of a group G, and $\overline{G} = G/N$. If $\overline{x}$ is an image of an element $x \in G$ in $\overline{G}$, then $|\overline{x}^{\overline{G}}|$ divides $|x^G|$. Moreover, if $C_G(x) \cap N$ is a proper subgroup of N, then $|\overline{x}^{\overline{G}}|$ is a proper divisor of $|x^G|$.*

Proof. It is clear that $C_{\overline{G}}(\overline{x})$ includes $C_G(x)N/N$. Therefore,

$$|\overline{x}^{\overline{G}}| = |\overline{G} : C_{\overline{G}}(\overline{x})| \text{ divides } |G/N : C_G(x)N/N = |G : C_G(x)|/|N : C_G(x) \cap N| \text{ divides } |x^G|,$$

and $|\overline{x}^{\overline{G}}|$ is a proper divisor of $|x^G|$ provided $C_G(x) \cap N$ is a proper subgroup of N. □

Lemma 3.2. *Let N be a normal subgroup of a group G, and $\overline{G} = G/N$. Then $\epsilon_\pi(\overline{G}) \leq \epsilon_\pi(G)$. If N is also a proper π-subgroup of G with $C_G(N) \leq N$, then $\epsilon_\pi(\overline{G}) < \epsilon_\pi(G)$.*

Proof. The first statement follows from the first statement of Lemma 3.1 directly. Suppose that N is also a proper π-subgroup of G with $C_G(N) \leq N$. Then for every $x \in G \setminus N$, the intersection $N \cap C_G(x)$ is a proper subgroup of N. Hence

$$|\overline{x}^{\overline{G}}|_\pi \text{ divides } |G : C_G(x)|_\pi/|N : C_G(x) \cap N|_\pi,$$

which is a proper divisor of $|x^G|_\pi$. Since $N < G$, it follows that $\epsilon_\pi(\overline{G}) < \epsilon_\pi(G)$, as claimed. □

The next two statements are the famous Hall–Higman Lemma 1.2.3 [8] reformulated for π-separable groups, see, e.g., [6, Theorem 6.3.2], and the direct corollary of this lemma.

Lemma 3.3. *If G be a π-separable group and $\overline{G} = G/O_{\pi'}(G)$, then $C_{\overline{G}}(O_\pi(\overline{G})) \leq O_\pi(\overline{G})$. In particular, if $O_{\pi'}(G) = 1$, then $C_G(O_\pi(G)) \leq O_\pi(G)$.*

Let $O_{\pi',\pi}(G)$ be the preimage in G of $O_\pi(G/O_{\pi'}(G))$.

Lemma 3.4. *If G is a π-separable group, then $Z(G) \leq C_G(O_{\pi',\pi}(G)) \leq O_{\pi',\pi}(G)$. Moreover, if $Z(G) = O_{\pi',\pi}(G)$, then $G = Z(G)$ is abelian.*

Proof. The first claim follows from Lemma 3.3 and the fact that

$$C_G(O_{\pi',\pi}(G))O_{\pi'}(G)/O_{\pi'}(G) \leq C_{G/O_{\pi'}(G)}(O_{\pi',\pi}(G)/O_{\pi'}(G)).$$

Suppose that $Z(G) = O_{\pi',\pi}(G)$. Then $G = C_G(Z(G)) = C_G(O_{\pi',\pi}(G)) \leq O_{\pi',\pi}(G) = Z(G)$, as required. □

We are ready to prove the theorem now.

Proof of Theorem 1.4. First, observe that the statement of the theorem is true for all sufficiently small groups G, even for all abelian groups, so we may proceed by induction on the order of G and assume that G is nonabelian.

Consider the upper π-series (1) of G. First, we suppose that $K_0 = O_{\pi'}(G) = 1$. Then $P_1 = O_\pi(G) \neq 1$. Lemmas 3.3 and 3.4 imply that $C_G(P_1) \leq P_1$ and $Z(G) < P_1$. In particular, $\epsilon_\pi(G) \geq 1$ and $l_\pi(G/Z(G)) = l_\pi(G)$. If $G = P_1$, then $l_\pi(G/Z(G)) = 1$ and we are done. Thus, $G > P_1$.

Now, on the one hand, Lemma 3.2 yields $\epsilon_\pi(G) > \epsilon_\pi(G/P_1)$, i.e.,

$$\epsilon_\pi(G/P_1) + 1 \leq \epsilon_\pi(G).$$

On the other hand, $P_2/P_1 = O_{\pi',\pi}(G/P_1)$, so Lemma 3.4 guarantees that $Z(G/P_1) < P_2/P_1$ or G/P_1 is abelian. If $Z(G/P_1) < P_2/P_1$, then $Z(G/P_1) \leq K_1/P_1$, so $l_\pi((G/P_1)/Z(G/P_1)) = l_\pi(G/K_1) = l_\pi(G/P_1) = l_\pi(G) - 1 = l_\pi(G/Z(G)) - 1$. If G/P_1 is abelian, then it must be a π'-group, because $P_1 \neq 1$. Hence $l_\pi((G/P_1)/Z(G/P_1)) = 0 = l_\pi(G/Z(G)) - 1$. Thus, in both cases,

$$l_\pi(G/Z(G)) \leq l_\pi((G/P_1)/Z(G/P_1)) + 1.$$

Applying the inductive hypothesis to G/P_1, we obtain

$$l_\pi(G/Z(G)) \leq l_\pi((G/P_1)/Z(G/P_1)) + 1 \leq \epsilon_\pi(G/P_1) + 1 \leq \epsilon_\pi(G),$$

as required.

Thus, to complete the proof of the theorem it suffices to consider the case, where $K_0 \neq 1$. Put $\overline{G} = G/K_0$ and note that $\overline{G}$ is nontrivial, because the theorem holds for every π'-group G. Observe also that $O_{\pi'}(\overline{G}) = 1$. We may suppose that $l_\pi(\overline{G}/Z(\overline{G})) < l_\pi(G/Z(G))$. Otherwise, by induction on the order of G, $l_\pi(G/Z(G)) = l_\pi(\overline{G}/Z(\overline{G})) \leq \epsilon(\overline{G}) \leq \epsilon(G)$, where the last inequality follows from Lemma 3.2.

Put $\overline{P}_1 = P_1/K_0 = O_\pi(\overline{G})$. Lemma 3.4 yields $Z(\overline{G}) \leq \overline{P}_1$. If $Z(\overline{G}) < \overline{P}_1$, then $l_\pi(\overline{G}/Z(\overline{G})) = l_\pi(G/Z(G))$, a contradiction. Hence $Z(\overline{G}) = \overline{P}_1$. Again by Lemma 3.4, it is possible only if $Z(\overline{G}) = \overline{P}_1 = \overline{G}$. In particular, $G = P_1$. Since $l_\pi(G/Z(G)) \leq 1$ in this case, it suffices to show that $\epsilon_\pi(x) > 0$ for some element $x \in G$. Otherwise, $|G : C_G(x)|_p = 1$ for every $x \in G$ and $p \in \pi$. Then, by virtue of [4, Lemma 1], Sylow p-subgroups of G lie in $Z(G)$ for all $p \in \pi$. It follows that $G/Z(G)$ is a π'-group, so $l_\pi(G/Z(G)) = 0$, which completes the proof.

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