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SURVEY ARTICLE

Recognition by spectrum for finite simple groups of Lie type*

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Abstract The goal of this article is to survey new results on the recognition problem. We focus our attention on the methods recently developed in this area. In each section, we formulate related open problems. In the last two sections, we review arithmetical characterization of spectra of finite simple groups and conclude with a list of groups for which the recognition problem was solved within the last three years.

Keywords Finite simple group, spectrum, prime graph, recognition problem
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0 Introduction

For a group G, the set of element orders of G is called the spectrum of G and denoted by $\omega(G)$. Two groups are isospectral if their spectra coincide. A finite group G is said to be recognizable by spectrum if every finite group H with $\omega(H) = \omega(G)$ is isomorphic to G. Denoting by $h(G)$ the number of isomorphism classes of finite groups isospectral to G, the condition for G to be recognizable is written as $h(G) = 1$. A finite group G is said to be almost recognizable by spectrum if $h(G)$ is finite and more than one, and to be irrecognizable by spectrum if $h(G)$ is infinite. Given a finite group G, the recognition problem for G is to determine whether G is recognizable, or almost recognizable, or irrecognizable; and a stronger form of this problem is to determine the value of $h(G)$.

By Corollary 4 in Ref. [19], every finite group with a nontrivial normal soluble subgroup is irrecognizable. Among finite groups without nontrivial

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normal soluble subgroups, the finite simple groups are of the prime interest. Since Shi showed in 1986 that the alternating group of degree 5 is recognizable by spectrum [18], the recognition problem has been investigated for a numerous number of simple groups. Thus, it has been solved for all sporadic simple groups, for all Ree and Suzuki groups, for simple linear and unitary groups of dimension at most 3, and for alternating groups of degrees p, $p+1$ and $p+2$, where p is prime (see the references in Refs. [13,14]). A few years ago, Mazurov conjectured that almost all finite simple groups are almost recognizable, or, more exactly, that every finite group of Lie type (alternating group) is recognizable or almost recognizable provided its Lie rank (its degree) is sufficiently large.

The goal of this article is to survey new results on the recognition problem. We focus our attention on the methods recently developed in this area (Sections 1–3). In Section 4, we review arithmetical characterization of spectra of finite simple groups. We conclude with a list of groups for which the recognition problem was solved within the last three years. In each section we formulate related open problems.

For the notation of the finite simple groups, we follow Ref. [6].

1 Composition structure of a finite group isospectral to a simple group

Let L be a finite nonabelian simple group. Given a finite group G with $\omega(G) = \omega(L)$, the first question to ask is if L and G have the same nonabelian composition factors. Applying this logic, L is said to be quasirecognizable if every finite group G with $\omega(G) = \omega(L)$ has a unique nonabelian composition factor and this factor is isomorphic to L. The smallest nonabelian simple group that is not quasirecognizable is Alt_6 (see Ref. [2]).

Note that groups with the same spectra have coincident prime graphs. The prime graph $GK(G)$ of a finite group G is defined as follows. The vertex set of this graph is the set $\pi(G)$ of prime divisors of $|G|$; primes r and s in $\pi(G)$ are adjacent if $rs \in \omega(G)$. K. W. Gruenberg and O. Kegel introduced this graph (also called the Gruenberg–Kegel graph) in the middle of the 1970s and gave a characterization of finite groups with disconnected prime graph (we denote the number of connected components of $GK(G)$ by $s(G)$). This deep result and a classification of finite simple groups with $s(G) > 1$ obtained by Williams [27] and Kondratiev [12] implied a series of important corollaries. Concerning the recognition problem, they allow to assert that a finite group G, which is isospectral to a simple group L with disconnected prime graph, contains a unique nonabelian composition factor S and $s(S) \geqslant s(L)$. However, this technique cannot be applied if the prime graph of L is connected. On the other hand, among finite simple groups, those with disconnected prime graph are rather an exception than a rule.

The proof of the Gruenberg–Kegel Theorem relies substantially upon the fact that in disconnected prime graph of an insoluble group there is an odd prime which is disconnected with 2. It turned out that disconnectedness could be successfully replaced in most cases by a weaker condition for the prime 2 to be nonadjacent to at least one odd prime. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise non-adjacent in $GK(G)$. In other words, $t(G)$ is a maximal number of vertices in cocliques, i.e., independent sets, of $GK(G)$. In graph theory, this number is usually called the independence number of the graph. By analogy we denote by $t(r, G)$ the maximal number of vertices in cocliques of $GK(G)$ containing a prime r. We call this number the r-independence number. The following structure theorem was proved by Vasilev [22] (here we give a refine version of this result from Ref. [24]).

Theorem 1 *Let G be a finite group with $t(G) \geqslant 3$ and $t(2, G) \geqslant 2$. Then*

(1) *There exists a finite simple nonabelian group S, such that*

$$S \leqslant \overline{G} = G/K \leqslant \mathrm{Aut}(S)$$

for maximal soluble normal subgroup K of G.

(2) *For every independent subset ρ of $\pi(G)$ with $|\rho| \geqslant 3$, at most one prime in ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geqslant t(G) - 1$.*

(3) *One of the following holds:*

(a) *every prime $r \in \pi(G)$ non-adjacent in $GK(G)$ to 2 does not divide the product $|K| \cdot |\overline{G}/S|$; in particular, $t(2, S) \geqslant t(2, G)$;*

(b) *there exists a prime $r \in \pi(K)$ non-adjacent in $GK(G)$ to 2; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \simeq Alt_7$ or $A_1(q)$ for some odd q.*

The following result by Vasilev and Gorshkov [24] shows that the exceptional case (b) of the statement (3) of Theorem 1 can be omitted when we apply the theorem to recognition of finite nonabelian simple groups.

Theorem 2 *Let L be a finite nonabelian simple group with $t(L) \geqslant 3$ and $t(2, L) \geqslant 2$, and let G be a finite group with $\omega(G) = \omega(L)$. Then*

(1) *There exists a finite simple nonabelian group S, such that*

$$S \leqslant \overline{G} = G/K \leqslant \mathrm{Aut}(S)$$

for maximal soluble normal subgroup K of G.

(2) *For every independent subset ρ of $\pi(G)$ with $|\rho| \geqslant 3$, at most one prime in ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geqslant t(G) - 1$.*

(3) *Every prime $r \in \pi(G)$ non-adjacent in $GK(G)$ to 2 does not divide the product $|K| \cdot |\overline{G}/S|$. In particular, $t(2, S) \geqslant t(2, G)$.*

The answer to the question which simple groups satisfy the conditions of Theorem 2 was provided by Vasilev and Vdovin. In Ref. [26], they gave arithmetic criteria of adjacency in prime graphs of finite simple groups; using the criteria they described maximal independent sets and obtained the independent and 2-independent numbers for all these groups. It turned out that $t(2, L) \geqslant 2$ for all nonabelian simple groups excepting the alternating groups of degree n, where there is no primes among $n,\ n-1, n-2, n-3$; and $t(L) \geqslant 3$ for all nonabelian simple groups with connected prime graph excepting the alternating group of degree 10.

If a nonabelian simple group L satisfies the conditions $t(L) \geqslant 3$ and $t(2, L) \geqslant 2$, then so does a group G with the same spectrum. Item (1) of Theorem 2 asserts that G contains a unique nonabelian composition factor S, while items (2) and (3) impose restrictions on S. These restrictions together with information from Ref. [26] serve as the basis for proving $S \simeq L$, i.e., for proving L to be quasirecognizable. Examples of employing this technique can be found in Refs. [8–10,15,25].

We formulate the question that seems crucial for proving quasirecognizability of simple groups of Lie type.

Problem 1 Let L be a finite simple classical group of Lie rank at least 24 over field of characteristic p, and let G be a finite group with $\omega(G) = \omega(L)$. Is it true that the unique nonabelian composition factor of G is a group of Lie type over field of characteristic p?

We conclude with the following rather interesting problem.

Problem 2 Finite groups G and H are said to be section-free if neither of them contains a subgroup isomorphic to a section of another. Determine whether or not there exist two section-free finite isospectral groups G and H, such that $h(G) = h(H)$ is finite?

2 Spectra of covers

If L is a quasirecognizable nonabelian simple group and G is a finite group with $\omega(G) = \omega(L)$, then
$$S \leqslant G/K \leqslant \mathrm{Aut} S,$$
where K is the soluble radical of G; therefore, G includes a cover of L. We say that a group G is a cover for a group L, or that G covers L, if L is a homomorphic image of G. A finite group L is called recognizable (by spectrum) from its covers if every finite cover G of L with $\omega(G) = \omega(L)$ is isomorphic to L. Obviously, every recognizable group is recognizable from its covers.

Mazurov and Zavarnitsine [29] proved that all simple alternating groups are recognizable from covers. Since the recognition problem is solved for all sporadic simple groups, the simple groups of Lie type remain to be considered. Let L be a finite simple group of Lie type over field of characteristic p and let G be a cover for L with $\omega(G) = \omega(L)$. To prove that $G \simeq L$ using induction on $|G|$, it suffices to consider the case when G is of the form $V \rtimes L$, where V is an elementary abelian r-group for some prime r. It is natural to distinguish between the modular case $r = p$ and the non-modular case.

Recently, Mazurov and Zavarnitsine [30,32] obtained a powerful result concerning the modular case of covers of linear and unitary groups. Here we give the final statement from Ref. [32]. For convenience, we put

$$L_n^+(q) = L_n(q), \quad L_n^-(q) = U_n(q),$$

and use this convention throughout the paper.

Theorem 3 *Let $\varepsilon \in \{+,-\}$, $q = p^m$, $n \geqslant 4$ and $L = L_n^\varepsilon(q)$. Assume that q is either prime or even if $n = 4$. If L acts on a vector space V of characteristic p, then $\omega(V \leftthreetimes L) \neq \omega(L)$.*

As observed in Ref. [32], the action of $L_4^\varepsilon(q)$ in the defining characteristic turned out to be a more subtle issue.

Problem 3 For which q is the simple group $L_4^\varepsilon(q)$ recognizable from its covers?

Since for linear groups the non-modular case was done earlier by Zavarnitsine [31], Theorem 3 results in

Theorem 4 *Let $L = L_n(q)$ be a simple linear group. If either $n \neq 4$, or q is prime, or q is even then L is recognizable by spectrum from its covers.*

Keeping in mind Theorem 3, it is natural to pose the following problem.

Problem 4 Let $L = U_n(q)$ and $n \geqslant 5$. Is it true that L is recognizable from its covers?

3 Spectra of automorphic extensions

Let L be a finite nonabelian simple group. If L is proved to be both quasirecognizable and recognizable from its covers, then every finite group G with $\omega(G) = \omega(L)$ may be assumed satisfying $L \leqslant G \leqslant \mathrm{Aut}L$. Therefore, the recognition problem substantially includes studying orders of elements in extensions of simple groups by automorphisms.

If L is an alternating or sporadic group and $L < G \leqslant \mathrm{Aut}L$ then, using only number-theoretic considerations and Ref. [6], one can verify that $\omega(G) \neq \omega(L)$. One of the most fruitful ways of studying spectra of finite groups of Lie type, as well as spectra of their automorphic extensions, is to consider these groups as centralizers of Frobenius endomorphisms in algebraic groups (see Chapter 1 in Ref. [5]). In Ref. [28], Zavarnitsine derived a general formula which expresses the spectrum of some automorphic extensions of a finite group of Lie type in terms of spectra of its certain subgroups.

Theorem 5 *Let G be a connected linear algebraic group over an algebraically closed field of characteristic p and τ be a surjective endomorphism of G. For a natural number r, put $G_r = C_G(\tau^r)$. If for some r the group G_r is finite, then τ is an automorphism of G_r of order r and*

$$\omega(G_r\langle\tau\rangle) = \bigcup_{k|r} \frac{r}{k}\omega(G_k).$$

Suppose that the subgroup $Sc(G)$ of G consisting of all scalar matrices is τ-invariant and put

$$\overline{G}_r = G_r/(G_r \cap Sc(G)).$$

Then

$$\omega(\overline{G}_r\langle\tau\rangle) = \bigcup_{k|r} \frac{r}{k}\omega(\overline{G}_k).$$

In the same paper, applying Theorem 5, Zavarnitsine obtained the following result on extensions of finite linear and unitary groups.

Theorem 6 *Let $X \in \{GL_n^\varepsilon, PGL_n^\varepsilon, SL_n^\varepsilon, PSL_n^\varepsilon\}$ and let q be a power of a prime p.*

(1) *Assume that $r \in \mathbb{N}$ is arbitrary if $\varepsilon = +$, and is odd if $\varepsilon = -$. If τ is a field automorphism of $X(q^r)$, then*

$$\omega(X(q^r)\langle\tau\rangle) = \bigcup_{k|r} \frac{r}{k}\omega(X(q^k)).$$

(2) *Let $\varepsilon = +$ and $r \in \mathbb{N}$ be even. If τ is the product of the field automorphism of $X(q^r)$ of order r and the inverse-transpose automorphism of order 2, then*

$$\omega(X(q^r)\langle\tau\rangle) = \bigcup_{k|r} \frac{r}{k}\omega(X^{(-)^k}(q^k)).$$

In view of Theorem 6, one can compare the spectrum of a linear or unitary group with the spectrum of its extension by a field automorphism only by number-theoretical means, providing that spectra of linear and unitary groups are described (see Section 4). Using this approach, Grechkoseeva [8] obtained the following result.

Theorem 7 *Let $L = L_n(q)$ where $n \geqslant 4$, $q = p^k$, and let $d = (n, q-1)$. Suppose that τ is a field automorphism of L and $\pi(|\tau|) \subseteq \pi(d)$. Then $\omega(L\langle\tau\rangle) = \omega(L)$ if and only if $n \neq p^m + 1$ and $|\tau|$ divides $(q-1)/d$.*

The following problem seems to be the main difficulty in investigating spectra of automorphic extension of linear groups.

Problem 5 Suppose that $L = L_n(q)$ where $n \geqslant 4$ and q is odd, and let G be a group such that

$$L \leqslant G \leqslant \mathrm{Aut}L, \quad |G : L| = 2.$$

Give a criterion for G to satisfy $\omega(G) = \omega(L)$.

4 Spectra of finite groups of Lie type

The problem of recognition by spectrum, especially when put for a whole series of groups at once, often results in a new problem of determining whether a given natural number arises as the order of an element in a given simple group. Since the latter problem is easy for the alternating groups, and Ref. [6] provides an exhaustive description of spectra of the sporadic ones, only groups of Lie type are to be investigated.

Let G be a finite group of Lie type over field of characteristic p. It is known that G can be represented as $O^{p'}(C_{\overline{G}}(\sigma))$, where $\overline{G}$ is a suitable connected simple linear algebraic group over an algebraically closed field of characteristic p, and σ is a suitable Frobenius map. The Jordan decomposition for $\overline{G}$ states that for each $x \in \overline{G}$, there exists a unique pair of semisimple element $x_s \in \overline{G}$ and unipotent element $x_u \in \overline{G}$, such that

$$x = x_s x_u = x_u x_s$$

(see Chapter 1 in Ref. [5]). One of the ways to describe the spectrum of G is to find orders of its semisimple and unipotent elements, and then, having obtained this information, to find 'mixed orders', i.e., orders of elements that are neither semisimple nor unipotent. Since semisimple elements of G are exactly p'-elements, while its unipotent elements are exactly p-elements, we single out two special subsets in $\omega(G)$: the subset $\omega_p(G)$ of p-powers and the subset $\omega_{p'}(G)$ of numbers coprime to p. Also, for brevity, we put

$$\omega_{\mathrm{mix}}(G) = \omega(G) \setminus (\omega_p(G) \cup \omega_{p'}(G)).$$

In 1995, Testerman [21] described the set $\omega_p(G)$ for each finite group G of Lie type in terms of the root system of G (note that by the root system of a twisted group we understand the system of the corresponding split group). Recall that there exists a unique root α_0 of maximum height in each indecomposable root system. The height $ht(\alpha_0)$ is equal to n for type A_n, to $2n-1$ for types B_n and C_n, to $2n-3$ for type D_n, and to 11, 17, 29, 11, 5 for types E_6, E_7, E_8, F_4, G_2, respectively (see Ref. [1]). The following result is a direct consequence of Proposition 0.5 in Ref. [21].

Theorem 8 *Let G be a finite group of Lie type over field of characteristic p. Suppose that α_0 is the highest root in the root system of G. Then $p^k \in \omega(G)$ if and only if $p^{k-1} \leqslant ht(\alpha_0)$.*

Since each semisimple element of G arises as an element of some maximal tori of G (see Chapter 3 in Ref. [5]), to describe $\omega_{p'}(G)$ it is sufficient to determine periods of maximal tori of G. Each maximal torus T being a finite abelian group can be represented as a direct product of cyclic groups. We refer to determining terms of such a decomposition as describing cyclic structure of T. The cyclic structure of maximal tori of the exceptional groups of Lie type is known (see Ref. [11] for details). For the classical groups the problem was solved by Buturlakin and Grechkoseeva [4].

Quite recently, Buturlakin [3] described the set $\omega_{mix}(G)$ for all finite linear and unitary groups. Thus, he provided an arithmetic criterion for a natural number to lie in the spectrum of a given simple linear or unitary group.

Theorem 9 *Let $G = L_n^\varepsilon(q)$ where $n \geqslant 2$ and q be a power of a prime p, and let $d = (n, q - \varepsilon 1)$. The set $\omega(G)$ is exactly the set of all divisors of the following numbers:*

(1) $\frac{q^n - (\varepsilon 1)^n}{d(q-\varepsilon 1)}$;

(2) $\frac{[q^{n_1} - (\varepsilon 1)^{n_1}, q^{n_2} - (\varepsilon 1)^{n_2}]}{(n/(n_1,n_2), q-\varepsilon 1)}$ *for all $n_1, n_2 > 0$ such that $n_1 + n_2 = n$;*

(3) $[q^{n_1} - (\varepsilon 1)^{n_1}, q^{n_2} - (\varepsilon 1)^{n_2}, \dots, q^{n_s} - (\varepsilon 1)^{n_s}]$ *for each $s \geqslant 3$ and all $n_1, n_2, \dots, n_s > 0$ such that $n_1 + n_2 + \dots + n_s = n$;*

(4) $p^k \frac{q^{n_1} - (\varepsilon 1)^{n_1}}{d}$ *for all $k, n_1 > 0$ such that $p^{k-1} + 1 + n_1 = n$;*

(5) $p^k [q^{n_1} - (\varepsilon 1)^{n_1}, q^{n_2} - (\varepsilon 1)^{n_2}, \dots, q^{n_s} - (\varepsilon 1)^{n_s}]$ *for each $s \geqslant 2$ and all $k, n_1, n_2, \dots, n_s > 0$ such that $p^{k-1} + 1 + n_1 + n_2 + \dots + n_s = n$;*

(6) p^k *if $k > 0$ and $p^{k-1} + 1 = n$.*

For simple symplectic and orthogonal groups, such criteria are not obtained yet. Nevertheless, one of the most interesting questions on spectra of these groups has been recently clarified. It is well known that for every n and every q orders of groups $B_n(q)$ and $C_n(q)$ are equal. Furthermore, graphs $GK(B_n(q))$ and $GK(C_n(q))$ were proved to coincide by Vasilev and Vdovin (Proposition 7.5 in Ref. [26]). The result on maximal tori by Buturlakin and Grechkoseeva together with Theorem 8 imply that the spectra of $B_n(q)$ and $C_n(q)$ are very close; namely, if q is a power of a prime p, then

$$\omega_p(B_n(q)) = \omega_p(C_n(q)), \quad \omega_{p'}(B_n(q)) = \omega_{p'}(C_n(q)).$$

Thus, the question about whether the spectra of non-isomorphic groups $B_n(q)$ and $C_n(q)$ differ arises. The question was answered independently by Shi [20] and Grechkoseeva [7].

Theorem 10 *Let $n \geqslant 2$ and q be a power of an odd prime p. Then*

$$p(q^{n-1} + 1) \in \omega(C_n(q)) \setminus \omega(B_n(q)).$$

We conclude the section with the following natural problem.

Problem 6 Suppose that G is a finite simple orthogonal, or symplectic, or exceptional group of Lie type. Give an arithmetic criterion for a natural number to lie in the spectrum of G.

5 Recent results on recognition of groups with connected prime graph

The last attempt to compile a full list of finite groups with solved recognition problem was made by Mazurov [13] in 2004 (his survey [14] is available only in Russian). We do not intend to give the contemporary version of this list in the present paper; we rather concentrate on the finite groups for which methods of solving the recognition problem have recently appeared, i.e., on groups with connected prime graph (see Section 2).

In 2005, Vasilev [22] proved that the simple orthogonal groups $O^-_{4n}(2^m)$ are quasirecognizable for $n \geqslant 8$, thus providing the first example of an infinite series of quasirecognizable groups with connected prime graph.

The same year Vasilev and Grechkoseeva [25] established that the simple linear groups $L_n(2^m)$ are recognizable for $n = 2^k \geqslant 32$. Thus, an infinite series of recognizable groups with connected prime graph was obtained for the first time. Groups $L_{16}(2^m)$ were proved to be recognizable by the same authors and Shi [10]. Quite recently, Chen and Mazurov [15] showed that groups $L_4(2^m)$ and $U_4(2^m)$ are recognizable for $m > 1$. Shen, Shi and Zinov'eva [17] showed that groups $B_p(3)$ are recognizable for $p > 3$.

We make a special emphasis on the following result for its generality.

Theorem 11 *For every natural $n > 2$ the simple linear group $L_n(2)$ is recognizable by spectrum.*

When $n < 9$ the prime graph of $L_n(2)$ is disconnected, and the assertion of Theorem 8 for the corresponding groups was proved in 2003 in a number of papers (see Ref. [9] for details). In 2005, Grechkoseeva, Mazurov, Moghaddamfar, Lucido, and Vasilev [9] proved that $L_n(2)$ are quasirecognizable for all $n \geqslant 9$. In 2006, after having investigated covers of simple linear and unitary groups, Mazurov and Zavarnitsine [30] completed the proof of Theorem 11.

The latest and most general result on recognition of linear groups over arbitrary finite fields of characteristic 2 was obtained by Grechkoseeva [8].

Theorem 12 *Let $L = L_n(q)$ where $q = 2^m$ and $11 \leqslant n \leqslant 17$ or $n > 24$. If either $n = 2^k + 1$ for some natural m or $(n, \frac{q-1}{(n,q-1)}, m) = 1$, then L is recognizable by spectrum. Otherwise, $h(L)$ is equal to the number of divisors of $(n, \frac{q-1}{(n,q-1)}, m)$.*

In view of the listed results, to give the ultimate solution of the recognition problem for simple linear groups over fields of characteristic 2, it remains to solve the following problem.

Problem 7 Suppose that $L = L_n(2^m)$ where $m > 1$ and $5 \leqslant n \leqslant 10$ or $18 \leqslant n \leqslant 24$. Determine the value of $h(L)$ in terms of n and m.

As mentioned above, the most part of simple classical groups of Lie type has connected prime graph. For simple exceptional groups of Lie type a rule is quite the opposite. Namely, groups $E_7(q)$ where $q > 3$ have connected prime graph, while the prime graphs of the remaining exceptional groups are disconnected. Recently, Vasilev [23] proved that groups $E_7(q)$ with $q > 3$ are quasirecognizable by spectrum. This result, together with the previous works on recognition of exceptional groups (see Ref. [23] for details), implies the following general theorem.

Theorem 13 *Let L be a finite simple exceptional group of Lie type. If G is a finite group with $\omega(G) = \omega(L)$, then*

$$L \leqslant G/K \leqslant \mathrm{Aut}L,$$

where K is the maximal normal soluble subgroup of G. In particular, L is quasirecognizable by spectrum.

Note that the problem of the existence of a finite simple exceptional group of Lie type which is not recognizable by spectrum (see Question 16.24 in Ref. [16]) is still open.

We observe that the above results by Vasilev and Vdovin [22,26] can be employed to solve the recognition problem for groups with disconnected prime graph as well. By using these results together with Gruenberg–Kegel Theorem, Shen, Shi and Zinovyeva [17] showed that groups $B_p(3)$ are recognizable for $p > 3$.

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References

1. Bourbaki N. Groupes et Algèbres de Lie: Chapitres 4–6 (Éléments de Mathématique). Paris: Hermann, 1968; Translation: Lie Groups and Lie Algebras: Chapters 4–6 (Elements of Mathematics). Berlin: Springer, 1989
2. Brandl R, Shi W J. Finite groups whose element orders are consecutive integers. J Algebra, 1991, 143: 388–400
3. Buturlakin A A. Spectra of finite linear and unitary groups. Algebra and Logic (in press)
4. Buturlakin A A, Grechkoseeva M A. The cyclic structure of maximal tori of the finite classical groups. Algebra and Logic, 2007, 46(2): 73–89
5. Carter R W. Finite Groups of Lie Type: Conjugacy Classes and Complex Characters. New York: John Wiley & Sons, 1985
6. Conway J H, Curtis R T, Norton S P, et al. Atlas of Finite Groups. Oxford: Clarendon Press, 1985
7. Grechkoseeva M A. On difference between the spectra of the simple groups $B_n(q)$ and $C_n(q)$. Sib Math J, 2007, 48(1): 73–75
8. Grechkoseeva M A. Recognition by spectrum of finite simple linear groups over fields of characteristic 2. Algebra and Logic (in press)
9. Grechkoseeva M A, Lucido M S, Mazurov V D, et al. On recognition of the projective special linear groups over the binary field. Sib Electron Math Rep, 2005, 2: 253–263 (http://semr.math.nsc.ru)
10. Grechkoseeva M A, Shi W J, Vasilev A V. Recognition by spectrum of $L_{16}(2^m)$. Algebra Colloq, 2007, 14(4): 585–591
11. Kantor W M, Seress A. Prime power graphs for groups of Lie type. J Algebra, 2002, 247: 370–434
12. Kondratiev A S. On prime graph components for the finite simple groups. Mat Sbornik, 1989, 180(6): 787–797 (in Russian)
13. Mazurov V D. Characterization of groups by arithmetic properties. Algebra Colloq, 2004, 11(1): 129–140
14. Mazurov V D. Groups with a prescribed spectrum. Izv Ural Gos Univ Mat Mekh, 2005, 36(7): 119–138 (in Russian)

15. Mazurov V D, Chen G Y. Recognizability of finite simple groups $L_4(2^m)$ and $U_4(2^m)$ by spectrum. Algebra and Logic (in press)
16. Mazurov V D, Khukhro E I. The Kourovka Notebook: Unsolved Problems in Group Theory. 16th ed. Novosibirsk: Sobolev Inst Mat, 2006
17. Shen R L, Shi W J, Zinovyeva M R. Characterization of simple groups $B_p(3)$. (in press)
18. Shi W J. A characteristic property of A_5. J Southwest-China Normal Univ, 1986, 11: 11–14 (in Chinese)
19. Shi W J. A characterization of the sporadic simple groups by their element orders. Algebra Colloq, 1994, 1(2): 159–166
20. Shi Wujie. Pure quantitative characterization of finite simple groups. Frontiers of Mathematics in China, 2007, 2(1): 123–125
21. Testerman D M. A_1-Type overgroups of order p in semisimple algebraic groups and the associated finite groups. J Algebra, 1995, 177(1): 34–76
22. Vasil'ev A V. On connection between the structure of finite group and properties of its prime graph. Sib Math J, 2005, 46(3): 396–404
23. Vasilev A V. On the recognition by spectrum of finite simple exceptional groups of Lie type. Algebra and Logic (in press).
24. Vasilev A V, Gorshkov I B. On recognition of finite simple groups with connected prime graph. Sib Math J (in press)
25. Vasil'ev A V, Grechkoseeva M A. On recognition of finite simple linear groups by spectrum. Sib Math J, 2005, 46(4): 593–600
26. Vasil'ev A V, Vdovin E P. An adjacency criterion for the prime graph of a finite simple group. Algebra and Logic, 2005, 44(6): 381–406
27. Williams J S. Prime graph components of finite groups. J Algebra, 1981, 69: 487–513
28. Zavarnitsin A V. Recognition of the simple groups $U_3(q)$ by element orders. Algebra and Logic, 2006, 45(2): 106–116
29. Zavarnitsin A V, Mazurov V D. Element orders in coverings of symmetric and alternating groups. Algebra and Logic, 1999, 38(3): 159–170
30. Zavarnitsin A V, Mazurov V D. On orders of elements in coverings of simple groups $L_n(q)$ and $U_n(q)$. Proceedings of the Steklov Institute of Mathematics, 2007, Suppl 1: 145–154
31. Zavarnitsine A V. Element orders in coverings of the groups $L_n(q)$ and recognition of the alternating group A_{16}. NII Diskret Mat Inform, Novosibirsk, Preprint No. 48, 2000 (in Russian)
32. Zavarnitsine A V. Properties of element orders in cover of $L_n(q)$ and $U_n(q)$. Sib Math J (in press)