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On the Almost Simple Automorphism Groups of Rank 3 Graphs

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To the 95th anniversary of the birth of Mikhail Ivanovich Kargapolov

Abstract—A permutation group G of a finite set Ω acts componentwise on the Cartesian square Ω^2. The largest subgroup of $\mathrm{Sym}(\Omega)$ that has the same orbits on Ω^2 as G is called the 2-closure of G. The rank of G is the number of its orbits on Ω^2. If the rank of G is 3 and the order is even, then an undirected graph with vertex set Ω is defined, up to taking the complement, for which one of the two non-diagonal orbits of G on Ω^2 is taken as the edge set. Such a graph is called a rank 3 graph. The full automorphism group of this graph coincides with the 2-closure of G and contains G as a subgroup. At present, with the exception of the case when G is an almost simple group, there exists an explicit description of the 2-closures of rank 3 groups G. In this paper, we fill the existing gap, thereby completing the description of the automorphism groups of rank 3 graphs.

Keywords—almost simple group, 2-closure of permutation group, rank 3 permutation group, rank 3 graph, automorphism group of a graph.

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INTRODUCTION

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group on a finite set Ω. The orbits of the componentwise action of G on the Cartesian square $\Omega \times \Omega$ are called its *orbitals* or 2-*orbits*, and their number is the *rank* of G. The largest subgroup of the symmetric group $\mathrm{Sym}(\Omega)$ with the same orbitals as G is called the 2-*closure* of the group G and is denoted by $G^{(2)}$. A graph Γ (possibly directed), whose vertex set is Ω and whose edge set is one of the orbitals of G, is called an *orbital graph* of G.

If G is a rank 3 group, then it is evidently transitive, and thus one of its orbitals is the diagonal of the square $\Omega \times \Omega$. If G is of odd order, then the two remaining irreflexive orbitals are mutual transposes; therefore, the corresponding orbital graphs are *tournaments* (i.e., complete directed graphs) that are opposite to each other. If G is of even order, then these orbitals are symmetric, and the corresponding orbital graphs are undirected graphs complementary to each other. They are called *rank* 3 *graphs* (corresponding to G). It is straightforward to see that the full automorphism group $\mathrm{Aut}\,\Gamma$ of such a graph (for a pair of graphs that are complementary to each other, the automorphism group is obviously the same) is the 2-closure of G, and G itself is a subgroup of $\mathrm{Aut}\,\Gamma = G^{(2)}$.

The rank 3 graphs represent a most important and well-studied subclass of the class of *strongly regular graphs*, i.e., graphs in which the number of common neighbors of two vertices depends only on

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whether they are equal, adjacent, or non-adjacent (complete graphs and their complements are not considered strongly regular graphs). The recently published monograph "Strongly Regular Graphs" by Brouwer and Van Maldeghem [2] provides a description of the rank 3 graphs, i.e., it specifies all pairs of the form (Γ, G), where Γ is a strongly regular graph and G is some (not necessarily full) automorphism group of this graph, whose rank is equal to 3 (the group G serves as a certificate that Γ is a rank 3 graph). It is important to note that a description of the full automorphism groups of rank 3 graphs or, equivalently, a description of the 2-closures of rank 3 groups, is not presented in this book (or in any literature available to the authors). Despite the fact that there is a large volume of information about the automorphism groups of rank 3 graphs (for example, in [2, Table 11.8] for all rank 3 graphs with no more than 1024 vertices, their full automorphism groups are indicated), completing their description seems to be a task of significant importance for both group theory and graph theory.

In recent works [17] and [10], the solution to this problem was reduced to the problem of describing the 2-closures for almost simple rank 3 groups G. The objective of this work is to provide such a description, thereby completing both the description of the 2-closures of rank 3 groups and the description of the full automorphism groups of rank 3 graphs (a rank 3 graph is considered given if some group of its automorphisms acting on the vertices of the graph as a rank 3 permutation group is specified).

Theorem 1. *If G is a finite rank 3 group, then the 2-closure $G^{(2)}$ is known.*

Corollary 1. *If Γ is a finite rank 3 graph, then the automorphism group $\operatorname{Aut}\Gamma$ is known.*

Let us recall that the *socle* $\operatorname{Soc} G$ of a group G is the subgroup generated by all its minimal normal subgroups. A group G is *almost simple* if its socle $L = \operatorname{Soc} G$ is a nonabelian simple group. Equivalently, G is almost simple if there exists a nonabelian simple group L such that $L \simeq \operatorname{Inn} L \leq G \leq \operatorname{Aut} L$, i.e., G is squeezed between L and its automorphism group $\operatorname{Aut} L$.

Now let $G \leq \operatorname{Sym}(\Omega)$ be a rank 3 permutation group. As noted, G is transitive on Ω. If G is imprimitive, then it has exactly one non-trivial imprimitive system Σ; the corresponding orbital graph Γ_G (or its complement) is the union of pairwise disjoint cliques of the same size, whose vertex sets are the blocks of the system Σ, see [2, Sect. 1.1.3]. If Δ is one of the blocks of Σ, then the automorphism group $\operatorname{Aut}\Gamma$ is permutationally isomorphic to the wreath product $\operatorname{Sym}(\Delta) \wr \operatorname{Sym}(\Sigma)$ of two symmetric groups on the sets Δ and Σ (see, for example, [17, Proposition 2.2]). Thus, we can assume that G is primitive.

From [13, Theorem 1], it follows that if $G \leq \operatorname{Sym}(\Omega)$ is a primitive almost simple group with socle L, then, except for some cases explicitly described there, the 2-closure $G^{(2)}$ is contained in the normalizer $N_{\operatorname{Sym}(\Omega)}(L)$ of the socle L in the symmetric group $\operatorname{Sym}(\Omega)$. Moreover, if G is a rank 3 group, then from [13, Propositions 1, 2] it can be easily deduced that the number of such exceptions is finite, and the group $G^{(2)}$ remains almost simple, albeit with a different socle; see Proposition 4 in Section 2 of this paper, where we indicate the 2-closures in all exceptional cases. In all other cases, $G^{(2)} \leq N_{\operatorname{Sym}(\Omega)}(L)$ and, by [7, Theorem 4.3B],

$$L \leq G \leq G^{(2)} \leq \operatorname{Aut} L,$$

i.e., the 2-closure is also an almost simple group with the same socle group as the initial group G (see also [15, Theorem 2]). Thus, the problem reduces to the problem of finding, for every faithful primitive permutation representation of rank 3 of an almost simple group G with socle L, the largest subgroup of $\operatorname{Aut} L$ to which it lifts while remaining faithful and preserving rank 3 (see Lemma 2 in Section 3). The latter problem is solved in Propositions 1–3 (see Section 2) for almost simple groups whose socles are alternating groups, sporadic groups, and groups of Lie type, respectively.

Bannai provided a description of rank 3 groups whose socles are alternating groups [1] (see also [2, Theorem 11.3.1]). In the case of sporadic groups, the same was done by Brouwer, Soicher, and R. Wilson using the “Atlas of Finite Groups” [5] (see [2, Theorem 11.3.5]). Our results on 2-closures of these groups—Propositions 1 and 2—follow from the above by simple verification and more like a reference.

The case of groups of Lie type requires more effort. The rank 3 groups with a classical socle were described by Kantor and Liebler in [11] (see [2, Theorems 11.3.2 and 11.3.3]), and those with an exceptional socle were described by Liebeck and Saxl [14] (see also [2, Theorem 11.3.4]). To obtain their 2-closures, we use information about how the maximal subgroups of such groups behave with respect to their automorphisms from [4, 6, 12].

Let us recall that, according to Steinberg's classical theorem [18], every element of the automorphism group $\mathrm{Aut}\,L$ of a group of Lie type L is the product of some inner, diagonal, field, and graph automorphisms. Let $\widetilde{L}$ denote the subgroup of $\mathrm{Aut}\,L$ generated by all inner, diagonal, and field automorphisms (see Section 1 for exact definitions). It turns out that in most cases, the 2-closure of an almost simple rank 3 permutation group G with socle L is isomorphic to $\mathrm{Aut}\,L$ or $\widetilde{L}$ (if the group, whose socle is a group of Lie type, is considered). More precisely, as follows from Propositions 1–4 in Section 2, the following theorem holds.

Theorem 2. *Let G be a primitive almost simple rank 3 permutation group on a finite set Ω, and let L be the socle of G. Then either $G^{(2)} = \mathrm{Aut}\,L$, or one of the following holds:*

(1) *$L = \mathrm{PSL}_n(q)$, $n > 4$, the action of G on Ω is defined in item 1 of Proposition 3 and $G^{(2)} = \widetilde{L}$ is a subgroup of index 2 in $\mathrm{Aut}\,L$.*

(2) *$L = \mathrm{PSp}_4(q)$, q is even, the action of G on Ω is defined in items 6–8 of Proposition 3 and $G^{(2)} = \widetilde{L}$ is a subgroup of index 2 in $\mathrm{Aut}\,L$.*

(3) *$L = \mathrm{P\Omega}^+_{10}(q)$, the action of G on Ω is defined in item 12 of Proposition 3 and $G^{(2)} = \widetilde{L}$ is a subgroup of index 2 in $\mathrm{Aut}\,L$.*

(4) *$L = \mathrm{P\Omega}^+_8(q)$, the action of G on Ω, as well as $G^{(2)}$ are defined in items 11, 15, 16 of Proposition 3.*

(5) *$L = \mathrm{P\Omega}^{\varepsilon}_{2n}(3)$, $n \geq 3$, $\varepsilon \in \{+,-\}$, $(n,\varepsilon) \neq (4,+)$, the action of G on Ω is defined in item 16 of Proposition 3 and $G^{(2)}$ is a subgroup of index 2 in $\mathrm{Aut}\,L$.*

(6) *$L = E_6(q)$, the action of G on Ω is defined in item 17 of Proposition 3, and $G^{(2)} = \widetilde{L}$ is a subgroup of index 2 in $\mathrm{Aut}\,L$.*

(7) *L is a group from items 4, 12 of Table 1, items 2, 6–9, 11 of Table 2 or a group from Table 3.*

The article is structured as follows: Section 1 provides the necessary information about simple groups and their automorphisms. Section 2 collects our results, which provide detailed descriptions of the 2-closures of rank 3 groups, equivalently, descriptions of the full automorphism groups of rank 3 graphs. Section 3 contains the proofs of these results.

1. PRELIMINARIES ON SIMPLE GROUPS AND THEIR AUTOMORPHISMS

In the notation and terminology for graphs, we follow [2]. The notation related to groups is based on [4, 5, 12]. The following information on automorphism groups of nonabelian simple groups is essential to grasping the subject.

It is known that 14 of the 26 sporadic groups, namely,

$$M_{11}, M_{23}, M_{24}, J_1, J_4, \mathrm{Ru}, \mathrm{Ly}, \mathrm{Co}_1, \mathrm{Co}_2, \mathrm{Co}_3, \mathrm{Fi}_{23}, \mathrm{Th}, B, M$$

coincide with their automorphism group, and the remaining 12,

$$M_{12}, M_{22}, J_2, J_3, \mathrm{HS}, \mathrm{McL}, \mathrm{O'N}, \mathrm{He}, \mathrm{HN}, \mathrm{Suz}, \mathrm{Fi}_{22}, \mathrm{Fi}_{24}{}',$$

have outer automorphism groups of order 2.

It should be noted that, of the four Janko groups J_1–J_4, only the group J_2 plays a role in the description of rank 3 permutation groups. In accordance with the notation and terminology of [2], which is also used in [5], this group is referred to as the Hall–Janko group and denoted by the symbol HJ.

The automorphism group of the alternating group $\mathrm{Alt}(n)$ coincides with $\mathrm{Sym}(n)$, except when $n = 6$. The outer automorphism group of the group $\mathrm{Alt}(6) \simeq \mathrm{PSL}_2(9) \simeq \mathrm{PSp}_4(2)'$ is the elementary abelian group of order 4.

For a finite classical simple group L and for $L = E_6(q)$, we define a subgroup $\widetilde{L} \le \mathrm{Aut}\, L$ containing L. It is sufficient to specify the image of the group $\widetilde{L}$ with respect to the canonical epimorphism

$$\mathrm{Aut}\, L \to \mathrm{Out}\, L = \mathrm{Aut}\, L/L.$$

For this purpose, following the notation from [4], we define automorphisms γ, δ, δ', ϕ, φ, and τ. The same symbols will denote both the automorphisms themselves and their images in $\mathrm{Out}\, L$.

Hereafter, L is one of the following groups: $\mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q)$, $\mathrm{PSp}_{2n}(q)$, $\mathrm{P\Omega}_{2n+1}(q)$, $\mathrm{P\Omega}^{\pm}_{2n}(q)$, $E_6(q)$. We assume that $q = p^e$ for some prime number p, and F is the field of order q^2 in the case $L = \mathrm{PSU}_n(q)$ and of order q in the other cases. We denote by ω a primitive element of the multiplicative group $F^\times$ of the field F.

We simultaneously consider each classical group as the image in the factor group modulo the scalar matrices of some matrix group, as well as the factor group modulo multiplication by nonzero scalars of some group of linear transformations of a vector space V, on which a bilinear, Hermitian or quadratic form is defined (see definitions in [4, Sects. 1.5, 1.6]).

Case $L = \mathrm{PSL}_n(q)$. The automorphism δ is the conjugation by the projective image of the matrix $\mathrm{diag}(\omega, 1, \dots, 1)$. The automorphism ϕ of L is induced by the action of the Frobenius automorphism

$$(\alpha_{ij}) \mapsto (\alpha_{ij}^p) \tag{1.1}$$

on the matrices (α_{ij}) from $\mathrm{GL}_n(q)$. The automorphism γ is defined for $n \ge 3$ and induced by the automorphism

$$g \mapsto (g^{-1})^\intercal \tag{1.2}$$

of $\mathrm{GL}_n(q)$, where $g^\intercal$ denotes the transpose of matrix g. In the group $\mathrm{Out}\, L$, the following relations hold

$$\delta^{(n,q-1)} = \gamma^2 = \phi^e = [\gamma, \phi] = 1, \quad \delta^\gamma = \delta^{-1}, \quad \delta^\phi = \delta^p,$$

whereas $\mathrm{Out}\, L = \langle \delta, \phi \rangle$ when $n = 2$ and $\mathrm{Out}\, L = \langle \delta, \phi, \gamma \rangle$ when $n \ge 3$. The image of $\widetilde{L}$ in $\mathrm{Out}\, L$ is equal to $\langle \delta, \phi \rangle$. According to the notation of [4, Table 1.2], $\widetilde{L} = \mathrm{P\Gamma L}_n(q)$.

Case $L = \mathrm{PSU}_n(q)$, $n \ge 3$. Let us identify $\mathrm{GU}_n(q)$ as a matrix group with the subgroup of $\mathrm{GL}_n(q^2)$ of matrices (α_{ij}) such that

$$(\alpha_{ij}^q)^\intercal = (\alpha_{ij})^{-1}.$$

The automorphism δ is the conjugation by the projective image of the matrix $\mathrm{diag}(\omega^{q-1}, 1, \dots, 1)$ from $\mathrm{GU}_n(q)$. As above, the automorphism ϕ of L is induced by the action of the Frobenius automorphism (1.1) on the matrices (α_{ij}) from $\mathrm{GU}_n(q)$. The automorphism of $\mathrm{GL}_n(q^2)$ defined in (1.2) leaves the group $\mathrm{GU}_n(q)$ invariant and induces the automorphism γ on $\mathrm{PSU}_n(q)$. In the group $\mathrm{Out}\, L$, the following relations hold

$$\delta^{(n,q+1)} = \gamma^2 = [\gamma, \phi] = 1, \quad \phi^e = \gamma, \quad \delta^\gamma = \delta^{-1}, \quad \delta^\phi = \delta^p$$

and $\mathrm{Out}\, L = \langle \delta, \phi, \gamma \rangle = \langle \delta, \phi \rangle$. According to the notation of [4, Table 1.2], $\mathrm{Aut}\, L = \widetilde{L} = \mathrm{P\Gamma U}_n(q)$.

Case $L = \mathrm{PSp}_{2n}(q)$, $n \geq 2$. Let us identify $\mathrm{Sp}_{2n}(q)$ as a matrix group with the subgroup of $\mathrm{GL}_{2n}(q)$ of matrices g such that $gg^{\intercal}$ is the block-diagonal matrix with n blocks of the form

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The automorphism δ is the conjugation by the projective image of the matrix $\mathrm{diag}(\omega,\dots,\omega,1,\dots,1)$ from $\mathrm{GL}_{2n}(q)$, where the number of symbols ω is equal to n. The automorphism ϕ of L is induced by the action of the Frobenius automorphism (1.1) on the matrices (α_{ij}) from $\mathrm{Sp}_{2n}(q)$. In the group $\mathrm{Out}\,L$, the following relations hold

$$\delta^{(2,q-1)} = \phi^e = [\delta,\phi] = 1,$$

whereas $\mathrm{Out}\,L = \langle\delta,\phi\rangle$ when $(n,p) \neq (4,2)$ and $|\,\mathrm{Out}\,L : \langle\delta,\phi\rangle| = 2$ when $(n,p) = (4,2)$. The image of $\widetilde{L}$ in $\mathrm{Out}\,L$ is equal to $\langle\delta,\phi\rangle$. According to the notation of [4, Table 1.2], $\widetilde{L} = \mathrm{PC\Gamma Sp}_{2n}(q)$.

Next, we consider orthogonal groups. Geometrically, the groups $\mathrm{GO}^{\varepsilon}_d(q)$, where ε is the empty symbol for odd d and $\varepsilon = \pm$ for even d, can be considered as the groups of invertible transformations of a finite-dimensional vector space V over the field F of order q, preserving a given non-degenerate quadratic form $Q\colon V \to F$ on V. In all cases, up to equivalence, there are two quadratic forms, but the groups that preserve them are isomorphic if d is odd, and not isomorphic and indexed by the sign "+" or "−" if d is even. The group $\mathrm{GO}^{\varepsilon}_d(q)$ is embedded as a normal subgroup into the group $\mathrm{CGO}^{\varepsilon}_d(q)$ of similarities of Q, i.e., such transformations under which the value of the form on any vector is multiplied by a fixed nonzero element of the field depending only on the transformation. The form Q is associated with a non-degenerate symmetric bilinear form defined by the equality

$$(u,v) = \frac{1}{(2,q-1)}\bigl(Q(u+v) - Q(u) - Q(v)\bigr). \tag{1.3}$$

The determinant of each transformation from $\mathrm{GO}^{\varepsilon}_d(q)$ is equal to ±1, and $\mathrm{SO}^{\varepsilon}_d(q)$ is the subgroup of $\mathrm{GO}^{\varepsilon}_d(q)$, consisting of transformations with determinant 1. In turn, $\mathrm{SO}^{\varepsilon}_d(q)$ contains a (normal) subgroup of index 2, denoted by $\Omega^{\varepsilon}_d(q)$. A vector v is called *non-singular* if $Q(v) \neq 0$. Let q be odd. The value of a quadratic form on a non-singular vector may or may not be a square in F. For a non-singular vector $v \in V$, the reflection

$$r_v\colon u \mapsto u - 2\frac{(u,v)}{(v,v)}v$$

is defined. The reflections generate the group $\mathrm{GO}^{\varepsilon}_d(q)$, and the group $\Omega^{\varepsilon}_d(q)$ consists of those transformations in $\mathrm{SO}^{\varepsilon}_d(q)$ for which, in any decomposition into a product of reflections, there is only an even number of reflections corresponding to non-singular vectors with non-square form values.

Case $L = \mathrm{P\Omega}_{2n+1}(q)$, $n \geq 3$, q odd. Up to equivalence, two non-degenerate quadratic forms are defined on V, which nevertheless correspond to isomorphic orthogonal groups. By choosing a basis in V according to [12, Proposition 2.5.3(iii)], we can identify $\mathrm{GO}_{2n+1}(q)$ as a matrix group with the subgroup of $\mathrm{GL}_{2n+1}(q)$ of all such matrices g such that $gg^{\intercal}$ is the block-diagonal matrix with n blocks of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.4}$$

and one block (1). The automorphism ϕ of L is induced by the action of the Frobenius automorphism (1.1) on the matrices (α_{ij}) from $\mathrm{GO}_{2n+1}(q)$. The automorphism δ of L is the conjugation by the projective image of the matrix from $\mathrm{SO}_{2n+1}(q) \setminus \Omega_{2n+1}(q)$ (for example, by the product $r_v r_w$, where $Q(v)$ is a square and $Q(w)$ is not a square in $F^\times$ and $(v,w)=0$). Note that $\mathrm{PSO}_{2n+1}(q)=\mathrm{PGO}_{2n+1}(q)$. In the group $\operatorname{Out} L$, the following relations hold

$$\delta^2=\phi^e=[\delta,\phi]=1,$$

where $\operatorname{Out} L=\langle\delta,\phi\rangle$. In the notation of [4, Table 1.2], $\operatorname{Aut} L=\widetilde{L}=\mathrm{PC\Gamma O}_{2n+1}(q)$.

Case $L=\mathrm{P\Omega}^{\varepsilon}_{2n}(q)$, $n\geq 3$, $\varepsilon\in\{+,-\}$. By choosing a suitable basis in V, we can identify $\mathrm{GO}^{\varepsilon}_{2n}(q)$ as a matrix group with the subgroup of $\mathrm{GL}_{2n}(q)$ of all such matrices g such that $gg^{\intercal}$ is the matrix B defined as follows:

- B is the block-diagonal matrix consisting of n blocks of the form (1.4), if $\varepsilon=+$;
- B is the identity matrix if $\varepsilon=-$ and both q and $n(q-1)/2$ are odd;
- in other cases, B is the block-diagonal matrix consisting of $n-1$ blocks of the form (1.4) and one block

$$\begin{pmatrix}1&0\\0&\mu\end{pmatrix},$$

  where $\mu\in F$ is chosen such that the polynomial $t^2+t+\mu\in F[t]$ is irreducible over F.

As follows from [12, Proposition 2.5.13], for odd q, the discriminant of the form Q, i.e., the determinant of B, will be a square in the field F if and only if $(-1)^{n(q-1)/2}=\varepsilon$.

The automorphism γ is induced by an element of order 2 from $\mathrm{GO}^{\varepsilon}_{2n}(q)\setminus\mathrm{SO}^{\varepsilon}_{2n}(q)$ when q is odd and from $\mathrm{SO}^{\varepsilon}_{2n}(q)\setminus\Omega^{\varepsilon}_{2n}(q)$ when q is even.

The mapping (1.1) leaves the matrix group $\mathrm{GO}^{\varepsilon}_{2n}(q)$ invariant if $\varepsilon=+$ or if $\varepsilon=-$ and the numbers q and $n(q-1)$ are both odd. In these cases, ϕ is the automorphism induced on $L=\mathrm{P\Omega}^{\varepsilon}_{2n}(q)$ by the mapping (1.1). The order of the image of ϕ in $\operatorname{Out} L$ in this case is equal to e. In the remaining cases, consider the automorphism φ induced on $\mathrm{P\Omega}^{-}_{2n}(q)$ by the composition of the Frobenius mapping and the conjugation by some matrix c. This matrix is chosen such that the given composition leaves invariant the matrix B and, therefore, the group $\mathrm{GO}^{-}_{2n}(q)$ defined above. The exact form of the matrix B is given in [12, Sect. 2.8] and [4, Sect. 1.7.1]. Furthermore, the order of the image φ in $\operatorname{Out} L$ is equal to $2e$ and $\varphi^e=\gamma$.

In the case where q is odd, the diagonal automorphisms δ and δ' of the group $L=\mathrm{P\Omega}^{\varepsilon}_{2n}(q)$ are defined. Let us first define δ'. In the case when $(-1)^{n(q-1)/2}=\varepsilon$, the automorphism δ' has order 2 and is induced by the conjugation of some element from $\mathrm{SO}^{\varepsilon}_{2n}(q)\setminus\Omega^{\varepsilon}_{2n}(q)$ (for example, $r_v r_w$ where v,w are non-singular orthogonal vectors such that $Q(v)$ is a square and $Q(w)$ is not a square in F). In other cases, we consider δ' to be equal to 1.

The automorphism δ is induced by the conjugation of some matrix from $\mathrm{CGO}^{\varepsilon}_{2n}(q)\setminus\mathrm{GO}^{\varepsilon}_{2n}(q)$. For the exact form of the matrix, see [4, Sect. 1.7.1]. The order of δ in the group $\operatorname{Out} L$ is as follows:

$$|\delta|=\begin{cases}2 & \text{if } (-1)^{n(q-1)/2}=(\varepsilon 1)^n,\\ 4 & \text{if } (-1)^{n(q-1)/2}\neq(\varepsilon 1)^n.\end{cases}$$

In this case, if $|\delta|=4$, then $\delta^2=\delta'$ in $\operatorname{Out} L$.

When q is odd, the images of the groups $\mathrm{PSO}^{\varepsilon}_{2n}(q)$, $\mathrm{PGO}^{\varepsilon}_{2n}(q)$ and $\mathrm{PCGO}^{\varepsilon}_{2n}(q)$ in $\operatorname{Out} L$ coincide with $\langle\delta'\rangle$, $\langle\delta',\gamma\rangle$, and $\langle\delta,\delta',\gamma\rangle$, respectively.

For convenience, we assume that $\delta=\delta'=1$ if q is even. The image of the group $\operatorname{Inndiag} L$ of inner-diagonal automorphisms is equal to $\langle\delta,\delta'\rangle$ (see [8, Definition 2.5.10]). When q is odd, we denote the group $\operatorname{Inndiag} L$, following [9], by $\mathrm{PCSO}^{\varepsilon}_{2n}(q)$. The groups $\mathrm{PSO}^{\varepsilon}_{2n}(q)$, $\mathrm{PCSO}^{\varepsilon}_{2n}(q)$, $\mathrm{PGO}^{\varepsilon}_{2n}(q)$, and $\mathrm{PCGO}^{\varepsilon}_{2n}(q)$ are represented in the diagram in Fig. 1.

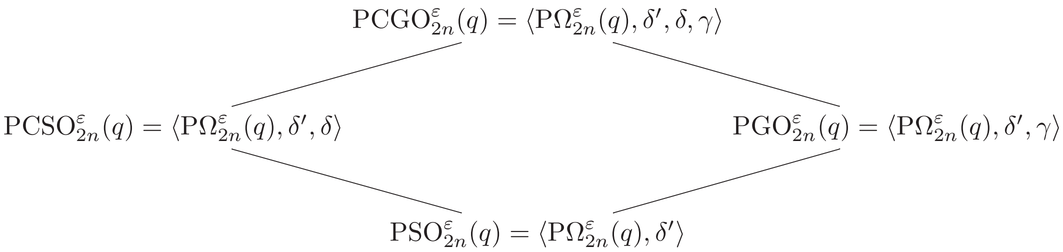

Fig. 1.

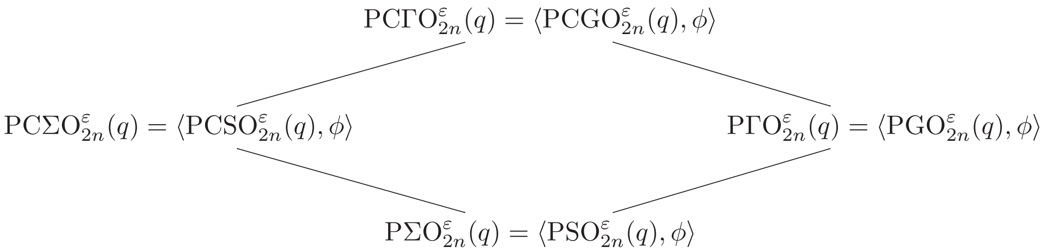

Fig. 2.

We denote[1] the preimages of subgroups $\langle \delta', \phi \rangle$, $\langle \delta', \phi, \gamma \rangle$, $\langle \delta, \delta', \phi \rangle$, and $\langle \delta, \delta', \gamma, \phi \rangle$ in $\operatorname{Aut} L$ by the symbols $\mathrm{P\Sigma O}^{\varepsilon}_{2n}(q)$, $\mathrm{P\Gamma O}^{\varepsilon}_{2n}(q)$, $\mathrm{PC\Sigma O}^{\varepsilon}_{2n}(q)$, and $\mathrm{PC\Gamma O}^{\varepsilon}_{2n}(q)$, respectively. If necessary, ϕ should be replaced by φ. The groups $\mathrm{P\Sigma O}^{\varepsilon}_{2n}(q)$, $\mathrm{PC\Sigma O}^{\varepsilon}_{2n}(q)$, $\mathrm{P\Gamma O}^{\varepsilon}_{2n}(q)$, and $\mathrm{PC\Gamma O}^{\varepsilon}_{2n}(q)$ are represented in the diagram in Fig. 2.

When q is even, the group $\mathrm{PSO}^{\varepsilon}_{2n}(q) = \mathrm{PGO}^{\varepsilon}_{2n}(q) = \mathrm{PCSO}^{\varepsilon}_{2n}(q) = \mathrm{PCGO}^{\varepsilon}_{2n}(q)$ has the image in $\operatorname{Out} L$ equal to $\langle \gamma \rangle$. The group $\operatorname{Inndiag} L$ coincides with L.

Below, the relations in $\operatorname{Out} L$ are presented, as well as the group $\widetilde{L}$ defined by its image in $\operatorname{Out} L$.

If q is even and $\varepsilon = +$, then

$$\gamma^2 = \phi^e = 1, \quad [\phi, \gamma] = 1,$$

where $\operatorname{Out} L = \langle \gamma, \phi \rangle$, except when $n = 4$. The image of $\widetilde{L}$ coincides with $\langle \phi \rangle$.

If q is even and $\varepsilon = -$, then

$$\gamma^2 = 1, \quad \varphi^e = \gamma$$

and $\operatorname{Out} L = \langle \varphi \rangle$. The image of $\widetilde{L}$ coincides with $\langle \varphi \rangle$.

If q is odd, $\varepsilon = +$ and $n \geq 4$ is even, then

$${\delta'}^2 = \delta^2 = \gamma^2 = \phi^e = [\phi, \gamma] = [\delta, \phi] = 1, \quad (\delta\gamma)^2 = \delta',$$

where $\operatorname{Out} L = \langle \delta, \delta', \gamma, \phi \rangle = \langle \delta, \gamma, \phi \rangle$, except when $n = 4$. The image of $\widetilde{L}$ in $\operatorname{Out} L$ coincides with $\langle \delta, \delta', \phi \rangle$, i.e., $\widetilde{L}$ coincides with $\mathrm{PC\Sigma O}^{\varepsilon}_{2n}(q)$.

If $q \equiv 1 \pmod 4$, $\varepsilon = +$ and $n \geq 3$ is odd, then

$${\delta'}^2 = \gamma^2 = \phi^e = [\phi, \gamma] = 1, \quad \delta^2 = \delta', \quad \delta^\gamma = \delta^{-1}, \quad \delta^\phi = \delta^p$$

and $\operatorname{Out} L = \langle \delta, \delta', \gamma, \phi \rangle = \langle \delta, \gamma, \phi \rangle$. The image of $\widetilde{L}$ in $\operatorname{Out} L$ coincides with $\langle \delta, \delta', \phi \rangle = \langle \delta, \phi \rangle$, i.e., $\widetilde{L}$ coincides with $\mathrm{PC\Sigma O}^{\varepsilon}_{2n}(q)$.

[1] Note that in the literature, including [2, 3], the symbol $\mathrm{P\Gamma O}^{\varepsilon}_{d}(q)$ sometimes denotes the group which we, following [4], denote by the symbol $\mathrm{PC\Gamma O}^{\varepsilon}_{d}(q)$. From our point of view, the symbol $\mathrm{P\Gamma O}^{\varepsilon}_{d}(q)$ for the extension of the group $\mathrm{PGO}^{\varepsilon}_{d}(q)$ by the group of field automorphisms is well consistent with the notation from [4], and we use it here exactly in this sense.

If $q \equiv 3 \pmod 4$, $\varepsilon = +$ and $n \geq 3$ is odd, then

$$\delta^2 = \gamma^2 = \phi^e = [\phi, \gamma] = [\phi, \delta] = [\gamma, \delta] = 1$$

and $\operatorname{Out} L = \langle \delta, \gamma, \phi \rangle$. The image of $\widetilde{L}$ in $\operatorname{Out} L$ coincides with $\langle \delta, \phi \rangle$, i.e., $\widetilde{L}$ coincides with $\mathrm{PC\Sigma O}^{\varepsilon}_{2n}(q)$.

If q is odd, $\varepsilon = -$, $n \geq 3$ and either n is even or $q \equiv 1 \pmod 4$, then

$$\delta^2 = \gamma^2 = [\varphi, \delta] = [\gamma, \delta] = 1, \quad \varphi^e = \gamma,$$

where $\operatorname{Out} L = \langle \delta, \gamma, \varphi \rangle = \langle \delta, \varphi \rangle$. The image of $\widetilde{L}$ in $\operatorname{Out} L$ coincides with $\langle \delta, \varphi \rangle$, i.e., $\widetilde{L}$ coincides with $\mathrm{PC\Sigma O}^{\varepsilon}_{2n}(q)$.

If $q \equiv 3 \pmod 4$, $\varepsilon = -$ and $n \geq 3$ is odd, then

$$\delta'^2 = \gamma^2 = \phi^e = [\phi, \gamma] = [\phi, \delta] = 1, \quad \delta^2 = \delta', \quad \delta^\gamma = \delta^{-1}$$

and $\operatorname{Out} L = \langle \delta, \delta', \gamma, \phi \rangle = \langle \delta, \gamma, \phi \rangle$. The image of $\widetilde{L}$ in $\operatorname{Out} L$ coincides with $\langle \delta, \delta', \phi \rangle = \langle \delta, \phi \rangle$, i.e., $\widetilde{L}$ coincides with $\mathrm{PC\Sigma O}^{\varepsilon}_{2n}(q)$ and coincides with $\operatorname{Aut} L$.

In the case when $(\varepsilon, n) = (+, 4)$, the group $L = \mathrm{P\Omega}^+_8(q)$ also admits an automorphism τ of order 3 such that $\langle \gamma, \tau \rangle \simeq \mathrm{Sym}(3)$. The following relations hold in the group $\operatorname{Out} L$.

If q is even and $(\varepsilon, n) = (+, 4)$, then

$$\gamma^2 = \tau^3 = (\gamma\tau)^2 = \phi^e = 1, \quad [\phi, \gamma] = [\phi, \tau] = 1$$

and $\operatorname{Out} L = \langle \gamma, \phi, \tau \rangle$.

If q is odd and $(\varepsilon, n) = (+, 4)$, then

$$\delta'^2 = \delta^2 = \gamma^2 = \tau^3 = (\gamma\tau)^2 = \phi^e = [\phi, \gamma] = [\delta, \phi] = [\tau, \phi] = 1, \quad (\delta\gamma)^2 = \delta', \quad \delta^\tau = \delta', \quad \delta'^\tau = \delta\delta'$$

and $\operatorname{Out} L = \langle \delta, \delta', \gamma, \phi, \tau \rangle = \langle \delta, \gamma, \phi, \tau \rangle$.

Case $L = E_6(q)$. It is known [8, Theorem 2.5.12] that the group $\operatorname{Out} L$ is generated by automorphisms δ, ϕ, and γ, where δ is a diagonal automorphism, ϕ is a field automorphism, γ is a graph automorphism, and the following relations are satisfied

$$\delta^{(3,q-1)} = \phi^e = \gamma^2 = [\phi, \gamma] = 1, \quad \delta^\gamma = \delta^{-1}, \quad \delta^\phi = \delta^p.$$

We assume that in this case the group $\widetilde{L}$ is such that its image in $\operatorname{Out} L$ is equal to $\langle \delta, \phi \rangle$.

Let us note the important difference in terminology and notation between [2] and [4, 12]. A vector space V of dimension $2n+1$ with a quadratic form Q over the field $\mathbb{F}_q$ of characteristic 2 always has the non-trivial radical $V^\perp$ with respect to the bilinear form (1.3) associated with the quadratic form Q. In [4, 12], the form Q is considered degenerate. However, in [2, Sect. 3.1.1], the form Q is deemed non-degenerate on V if $V^\perp$ has dimension 1 and is spanned by some non-singular vector. On the factor space $V/V^\perp$, the form (1.3) induces a non-degenerate symplectic form, whose isometry group is isomorphic to the group $\mathrm{Sp}_{2n}(q) = \mathrm{PSp}_{2n}(q)$, as well as to the isometry group of the quadratic form Q of V. Therefore, when q is even, where [2] refers to groups $\Omega_{2n+1}(q) = \mathrm{P\Omega}_{2n+1}(q)$, we write $\mathrm{Sp}_{2n}(q) = \mathrm{PSp}_{2n}(q)$, and instead of $\mathrm{P\Gamma O}_{2n+1}(q)$, we write $\mathrm{P\Gamma Sp}_{2n}(q)$ and so on. In this instance, V will be designated as a *parabolic* space of rank n. In addition, it is convenient to call V parabolic when V is a space of odd dimension with a non-degenerate quadratic form and the characteristic of the field is odd.

Additionally, when q is even, the graph $\Gamma(Q_{2n}(q))$ associated with V is isomorphic to the graph $\Gamma(W_{2n-1}(q))$ (see [2, Sect. 2.6.4]), and we consider it in this case only as $\Gamma(W_{2n-1}(q))$.

2. RANK 3 GROUPS AND GRAPHS

Let us recall that a rank 3 permutation group G of a set Ω defines two rank 3 graphs on Ω that are complementary to each other. The graph Γ_G, which appears in Propositions 1–4 below, is defined up to taking the complement.

Proposition 1. *Let $G \le \mathrm{Sym}(\Omega)$ be a primitive almost simple rank 3 permutation group on a finite set Ω, such that its socle L is an alternating group of degree $n \ge 5$. Let M be the stabilizer of a point in this action and let Γ_G be the corresponding rank 3 graph. If $\mathrm{Soc}\, G^{(2)} = L$, then up to permutation isomorphism, one of the following holds*:

(1) *Ω is the set of 2-element subsets of an n-element set, $\Gamma_G = T(n)$ is the triangle graph. Then $G^{(2)} = \mathrm{Sym}(n)$, in particular, $G^{(2)} = \mathrm{Aut}\, L$ when $n \ne 6$.*

(2) *$n \in \{8, 10\}$, Ω is the set of partitions of an n-element set into a union of disjoint subsets of size $n/2$, Γ_G is the S_n-graph. Then $G^{(2)} = \mathrm{Sym}(n) = \mathrm{Aut}\, L$.*

Proposition 2. *Let $G \le \mathrm{Sym}(\Omega)$ be a primitive almost simple rank 3 permutation group on a finite set Ω of size v, such that its socle L is a sporadic group. Let M be the stabilizer of a point in this action and let Γ_G be the corresponding rank 3 graph. If $\mathrm{Soc}\, G^{(2)} = L$, then up to permutation isomorphism, $v = |\Omega|$, L, $M \cap L$, Γ_G and the 2-closure $G^{(2)}$ of G are listed in Table 1.*

Proposition 3. *Let $G \le \mathrm{Sym}(\Omega)$ be a primitive almost simple rank 3 permutation group on a finite set Ω, such that its socle L is a simple group of Lie type over the finite field of order q and characteristic p. Let M be the stabilizer of a point in this action and let Γ_G be the corresponding rank 3 graph. If $\mathrm{Soc}\, G^{(2)} = L$, then up to permutation isomorphism, either $v = |\Omega|$, L, $M \cap L$, Γ_G and the 2-closure $G^{(2)}$ of G are listed in Table 2, or one of the following holds*:

(1) *$L = \mathrm{PSL}_n(q)$, $n \ge 4$, Ω is the set of lines for L in the associated projective space, $\Gamma_G = J_q(n, 2)$ is the Grassmann graph. Then $G^{(2)} = \widetilde{L} = \mathrm{P\Gamma L}_n(q)$ when $n > 4$ and $G^{(2)} = \mathrm{Aut}\, L = \langle \mathrm{P\Gamma L}_n(q), \gamma \rangle$ when $n = 4$.*

(2) *$L = \mathrm{PSU}_n(q)$, $n \ge 3$, Ω is the set of isotropic points in the associated projective space, $\Gamma_G = \Gamma(H_{n-1}(q^2))$ is the Hermitian polar graph. Then $G^{(2)} = \mathrm{Aut}\, L = \widetilde{L} = \mathrm{P\Gamma U}_n(q)$.*

(3) *$L = \mathrm{PSU}_4(q)$, Ω is the set of maximal isotropic subspaces in the associated projective space, $\Gamma_G = \mathrm{GQ}(q, q^2)$ is the generalized quadrangle. Then $G^{(2)} = \mathrm{Aut}\, L = \widetilde{L} = \mathrm{P\Gamma U}_4(q)$.*

Table 1. Almost simple rank 3 permutation groups with sporadic socle

No.	$v = \|\Omega\|$	$L = \mathrm{Soc}\, G$	$M \cap L$	Γ_G	$G^{(2)}$	Reference
1	77	M_{22}	$2^4.\, \mathrm{Alt}(6)$	$S(3,6,22)$	$\mathrm{Aut}\, L$	[2, Sect. 10.27]
2	100	HS	M_{22}	Higman–Sims	$\mathrm{Aut}\, L$	[2, Sect. 10.31]
3	100	HJ	$\mathrm{PSU}_3(3)$	Hall–Janko	$\mathrm{Aut}\, L$	[2, Sect. 10.32]
4	176	M_{22}	$\mathrm{Alt}(7)$	$S(4,7,23) \setminus S(3,6,22)$	L	[2, Sect. 10.51]
5	253	M_{23}	$2^4.\, \mathrm{Alt}(7)$	$S(4,7,23)$	$L = \mathrm{Aut}\, L$	[2, Sect. 10.56]
6	275	McL	$\mathrm{PSU}_4(3)$	McLaughlin	$\mathrm{Aut}\, L$	[2, Sect. 10.61]
7	1288	M_{24}	$M_{12}.2$	Dodecad graph	$L = \mathrm{Aut}\, L$	[2, Sect. 10.80]
8	1782	Suz	$G_2(4)$	Suzuki	$\mathrm{Aut}\, L$	[2, Sect. 10.83]
9	2300	Co_2	$\mathrm{PSU}_6(2).2$	Conway	$L = \mathrm{Aut}\, L$	[2, Sect. 10.88]
10	3510	Fi_{22}	$2.\, \mathrm{PSU}_6(2)$	Fi_{22}-graph	$\mathrm{Aut}\, L$	[2, Sect. 10.90]
11	4060	Ru	${}^2F_4(2)$	Rudvalis	$L = \mathrm{Aut}\, L$	[2, Sect. 10.91]
12	14 080	Fi_{22}	$\Omega_7(3)$	Fi_{22}-graph	L	[2, Sect. 10.94]
13	31 671	Fi_{23}	$2.\mathrm{Fi}_{22}$	Fi_{23}-graph	$L = \mathrm{Aut}\, L$	[2, Sect. 10.96]
14	137 632	Fi_{23}	$\mathrm{P}\Omega_8^+(3).\, \mathrm{Sym}(3)$	Fi_{23}-graph	$L = \mathrm{Aut}\, L$	[2, Sect. 10.97]
15	306 936	${\mathrm{Fi}_{24}}'$	Fi_{23}	Fi_{24}-graph	$\mathrm{Aut}\, L$	[2, Sect. 10.99]

Table 2. Almost simple rank 3 permutation groups with a socle of Lie type

No.	$v=\|\Omega\|$	$L=\operatorname{Soc} G$	$M\cap L$	Γ_G	$G^{(2)}$	Reference
1	10	$\mathrm{PSL}_2(5)\simeq\mathrm{Alt}(5)$	$\mathrm{Sym}(3)$	$T(5)$	$\operatorname{Aut} L\simeq\mathrm{Sym}(5)$	[2, Sect. 10.3]
2	15	$\mathrm{PSL}_2(9)\simeq\mathrm{Alt}(6)$	$\mathrm{Sym}(4)$	$T(6)$	$\mathrm{Sym}(6)$	[2, Sect. 10.5]
3	28	$\mathrm{PSL}_4(2)\simeq\mathrm{Alt}(8)$	$\mathrm{Sym}(6)$	$T(8)$	$\operatorname{Aut} L\simeq\mathrm{Sym}(7)$	[2, Sect. 10.11]
4	35	$\mathrm{PSL}_4(2)\simeq\mathrm{Alt}(8)$	$2^4:(\mathrm{Sym}(3)\times\mathrm{Sym}(3))$	S_8-graph	$\operatorname{Aut} L\simeq\mathrm{Sym}(8)$	[2, Sect. 10.13]
5	36	$\mathrm{PSU}_3(3)$	$\mathrm{PSL}_3(2)$	$G_2(2)$-graph	$\operatorname{Aut} L$	[2, Sect. 10.14]
6*	50	$\mathrm{PSU}_3(5)$	$\mathrm{Alt}(7)$	Hoffman–Singleton	$L.2$	[2, Sect. 10.19]
7*	56	$\mathrm{PSL}_3(4)$	$\mathrm{Alt}(6)$	Gewirtz	$L:2^2$	[2, Sect. 10.20]
8	117	$\mathrm{PSL}_4(3)\simeq\mathrm{P}\Omega_6^+(3)$	$\mathrm{PSp}_4(3)\simeq\mathrm{P}\Omega_5(3)$	$NO_6^+(3)$	$\mathrm{PGO}_6^+(3)$	[2, Sect. 10.35]
9*	162	$\mathrm{PSU}_4(3)$	$\mathrm{PSL}_3(4)$	$U_4(3)$-graph	$L.(2^2)_{133}$	[2, Sect. 10.48]
10	416	$G_2(4)$	HJ	$G_2(4)$-graph	$\operatorname{Aut} L$	[2, Sect. 10.68]
11*	1408	$\mathrm{PSU}_6(2)$	$\mathrm{PSU}_4(3).2$	Conway	$L.2$	[2, Sect. 10.81]

Note: The group $L=\mathrm{PSU}_4(3)$ has the outer automorphism group that is isomorphic to D_8, which contains three classes of involutions, denoted in [5] by the symbols 2_1 (the central element in D_8), 2_2 and 2_3, and two elementary abelian subgroups of order 4, denoted by $(2^2)_{122}$ and $(2^2)_{133}$; the extension of L by the latter is the group $G^{(2)}$ in item 9 of Table 2. For items 6, 7, and 11, the structure of $G^{(2)}$ is uniquely restored from its description in Table 2 according to [5].

(4) $L=\mathrm{PSU}_5(q)$, Ω *is the set of maximal isotropic subspaces in the associated projective space,* $\Gamma_G=\mathrm{GQ}(q^3,q^2)$ *is the generalized quadrangle. Then* $G^{(2)}=\operatorname{Aut} L=\widetilde{L}=\mathrm{P\Gamma U}_5(q)$.

(5) $L=\mathrm{PSU}_n(2)$, $n\geq 4$, Ω *is the set of anisotropic points in the associated projective space,* $\Gamma_G=\overline{NU}_m(2)$ *is the graph whose adjacency relation is orthogonality. Then* $G^{(2)}=\operatorname{Aut} L=\widetilde{L}=\mathrm{P\Gamma U}_n(2)$.

(6) $L=\mathrm{PSp}_{2n}(q)$, $n\geq 2$, Ω *is the set of isotropic points in the associated projective space,* $\Gamma_G=\Gamma(W_{2n-1}(q))$ *is the symplectic polar graph of rank* n. *Then* $G^{(2)}=\widetilde{L}=\mathrm{PC\Gamma Sp}_{2n}(q)$, *and* $G^{(2)}=\operatorname{Aut} L$ *if* $(n,p)\neq(2,2)$.

(7) $L=\mathrm{PSp}_4(q)$, Ω *is the set of maximal isotropic subspaces in the associated projective space,* $\Gamma_G=\mathrm{GQ}(q,q)$ *is the generalized quadrangle. Then* $G^{(2)}=\widetilde{L}=\mathrm{PC\Gamma Sp}_4(q)$, *where* $G^{(2)}=\operatorname{Aut} L$, *if* q *is odd.*

(8) $L=\mathrm{PSp}_{2n}(q)$, $n\geq 2$, $q\in\{4,8\}$, Ω *is the set of anisotropic hyperplanes of the same type (either hyperbolic or elliptic) in the parabolic space of rank* n *over the field of* q *elements,* $\Gamma_G=NO^{\pm}_{2n+1}(q)$. *Then* $G^{(2)}=\widetilde{L}=\mathrm{PC\Gamma Sp}_{2n}(q)$. *In this case,* $G=G^{(2)}=\mathrm{PC\Gamma Sp}_{2n}(q)$, *if* $q=8$. *Furthermore, if* $n>2$, *then the group* $G^{(2)}$ *coincides with* $\operatorname{Aut} L$.

(9) $L=\mathrm{P}\Omega_{2n+1}(3)$, $n\geq 2$, Ω *is the set of anisotropic hyperplanes of the same type (either hyperbolic or elliptic) in the parabolic space of rank* n *over the field of* 3 *elements,* $\Gamma_G=NO^{\pm}_{2n+1}(3)$. *Then* $G^{(2)}=\operatorname{Aut} L=\widetilde{L}=\mathrm{PSO}_{2n+1}(3)$.

(10) $L=\mathrm{P}\Omega_{2n+1}(q)$, $n\geq 2$, q *is odd,* Ω *is the set of isotropic points in the associated projective space,* $\Gamma_G=\Gamma(Q_{2n}(q))$ *is the parabolic orthogonal polar graph of rank* n. *Then* $G^{(2)}=\operatorname{Aut} L=\widetilde{L}=\mathrm{PC\Gamma O}_{2n+1}(q)$.

(11) $L=\mathrm{P}\Omega^+_{2n}(q)$, $n\geq 3$, Ω *is the set of isotropic points in the associated projective space,* $\Gamma_G=\Gamma(Q_{2n-1}(q))$ *is the hyperbolic orthogonal polar graph of rank* n. *Then* $G^{(2)}=\mathrm{PC\Gamma O}^+_{2n}(q)$, *whereas* $G^{(2)}=\operatorname{Aut} L$ *if* $n\neq 4$, *and* $G^{(2)}$ *has index* 3 *in* $\operatorname{Aut} L$ *if* $n=4$.

(12) $L=\mathrm{P}\Omega^+_{10}(q)$, Ω *is the set of maximal isotropic subspaces in the associated projective space,* $\Gamma_G=\Delta_{1/2}$ *is the halved graph of the bipartite distance-regular graph* $\Delta(X,\Omega)$ *of the orthogonal polar space of rank* 5. *Then* $G^{(2)}=\widetilde{L}=\mathrm{PC\Sigma O}^+_{10}(q)$.

Table 3. Almost simple permutation groups G of rank 3, for which $\operatorname{Soc} G^{(2)} \neq \operatorname{Soc} G$

No.	$v = \lvert\Omega\rvert$	$L = \operatorname{Soc} G$	$M \cap L$	Γ_G	$G^{(2)}$	Reference
1*	36	$\mathrm{PSL}_2(8)$	D_{14}	$T(9)$	$\mathrm{Sym}(9)$	[2, Sect. 10.15]
2	55	M_{11}	$M_9.2$	$T(11)$	$\mathrm{Sym}(11)$	[2, Sect. 11.3.5]
3	66	M_{12}	$M_{10}.2$	$T(12)$	$\mathrm{Sym}(12)$	[2, Sect. 11.3.5]
4	120	$\mathrm{PSp}_6(2)$	$G_2(2)$	$NO_8^+(2)$	$\mathrm{PSO}_8^+(2)$	[2, Sect. 10.39]
5	120	$\mathrm{Alt}(9)$	$\mathrm{P\Gamma L}_2(8)$	$NO_8^+(2)$	$\mathrm{PSO}_8^+(2)$	[2, Sect. 11.3.1]
6	253	M_{23}	$M_{21}.2$	$T(23)$	$\mathrm{Sym}(23)$	[2, Sect. 11.3.5]
7	276	M_{24}	$M_{22}.2$	$T(24)$	$\mathrm{Sym}(24)$	[2, Sect. 11.3.5]
8	351	$G_2(3)$	$\mathrm{PSU}_3(3).2$	$NO_7^{-\perp}(3)$	$\mathrm{PSO}_7(3)$	[2, Sect. 10.66]
9	1080	$\mathrm{P\Omega}_7(3)$	$G_2(3)$	$NO_8^+(3)$	$\mathrm{PGO}_8^+(3)$	[2, Sect. 10.78]
10	2016	$G_2(4)$	$\mathrm{PSU}_3(4).2$	$NO_7^-(4)$	$\mathrm{P\Gamma Sp}_6(4)$	[2, Sect. 3.1.4]
11*	130 816	$G_2(8)$	$\mathrm{PSU}_3(8).2$	$NO_7^-(8)$	$\mathrm{P\Gamma Sp}_6(8)$	[2, Sect. 3.1.4]

Note: Groups G with socles $L = \mathrm{PSL}_2(8)$ and $L = G_2(8)$ from items 1 and 11 of Table 3, respectively, have rank 3 only if $G = \operatorname{Aut} L$.

(13) $L = \mathrm{P\Omega}^-_{2n+2}(q)$, $n \geq 2$, Ω *is the set of isotropic points in the associated projective space,* $\Gamma_G = \Gamma(Q_{2n+1}(q))$ *is the elliptic orthogonal polar graph of rank* n. *Then* $G^{(2)} = \operatorname{Aut} L = \mathrm{PC\Gamma O}^-_{2n+2}(q)$.

(14) $L = \mathrm{P\Omega}^-_6(q)$, Ω *is the set of maximal isotropic subspaces in the associated projective space,* $\Gamma_G = \mathrm{GQ}(q^2, q)$ *is the generalized quadrangle. Then* $G^{(2)} = \operatorname{Aut} L = \mathrm{PC\Gamma O}^-_6(q)$.

(15) $L = \mathrm{P\Omega}^\varepsilon_{2n}(2)$, $n \geq 3$, $\varepsilon \in \{+, -\}$, Ω *is the set of anisotropic points in the associated projective space,* $\Gamma_G = NO^\varepsilon_{2n}(2)$ *is the graph whose adjacency relation is orthogonality. Then* $G^{(2)} = \mathrm{PSO}^\varepsilon_{2n}(2)$, *and* $G^{(2)} = \operatorname{Aut} L$ *if* $(n, \varepsilon) \neq (4, +)$, *and* $G^{(2)}$ *has index* 3 *in* $\operatorname{Aut} L$ *if* $(n, \varepsilon) = (4, +)$.

(16) $L = \mathrm{P\Omega}^\varepsilon_{2n}(3)$, $n \geq 3$, $\varepsilon \in \{+, -\}$, Ω *is the set of anisotropic points with fixed value of quadratic form in the associated projective space,* $\Gamma_G = NO^\varepsilon_{2n}(3)$ *is the graph whose adjacency relation is orthogonality. Then* $G^{(2)} = \mathrm{PGO}^\varepsilon_{2n}(3)$ *is the subgroup whose image in* $\operatorname{Out} L$ *is equal to* $\langle \delta', \phi, \gamma \rangle$; *the subgroup* $G^{(2)}$ *has index* 2 *in* $\operatorname{Aut} L$ *if* $(n, \varepsilon) \neq (4, +)$, *and index* 6 *if* $(n, \varepsilon) = (4, +)$.

(17) $L = E_6(q)$, Ω *is the set of cosets of a subgroup* M, *where* $M \cap L$ *is a parabolic* $D_5(q)$*-subgroup of* L, Γ_G *is the collinearity graph of the geometry* $E_{6,1}(q)$. *Then* $G^{(2)} = \widetilde{L} = \langle \operatorname{Inndiag} L, \phi \rangle$ *is the subgroup of* $\operatorname{Aut} L$ *of index* 2, *whose image in* $\operatorname{Out} L$ *is equal to* $\langle \delta, \phi \rangle$.

Proposition 4. *Let* $G \leq \mathrm{Sym}(\Omega)$ *be a primitive almost simple rank* 3 *permutation group on a finite set* Ω, *let* L *be its socle, let* M *be the stabilizer of a point in this action, and let* Γ_G *be the corresponding rank* 3 *graph. If* $\operatorname{Soc} G^{(2)} \neq L$, *then up to permutation isomorphism* $v = |\Omega|$, L, $M \cap L$, Γ_G *and the* 2*-closure* $G^{(2)}$ *of* G *are listed in Table* 3.

3. PROOFS OF THE MAIN RESULTS

We need several auxiliary lemmas. Henceforth, let Ω be a finite set with $|\Omega| > 1$.

Lemma 1. *Let* $G \leq G^* \leq \mathrm{Sym}(\Omega)$ *and let* G *be transitive. Let* $\alpha \in \Omega$, *let* $H = G_\alpha$ *and* $H^* = G^*_\alpha$ *be the stabilizers of* α *in* G *and* G^*, *respectively. The following hold:*

(1) $H = G \cap H^*$.

(2) $G^* = H^* G$.

(3) *If both groups* G *and* G^* *have the same rank, then the orbits* H *and* H^* *on* Ω *coincide.*

(4) *If* $G \trianglelefteq G^*$, *then* $H^G = H^{G^*}$.

(5) *If* $G \trianglelefteq G^*$ *and* H *has no orbits of length* 1 *other than* $\{\alpha\}$, *then* $H^* = N_{G^*}(H)$; *in particular,* $H = N_G(H)$.

Proof. Statement 1 is obvious. Since G is a transitive group, for any $g^* \in G^*$ there exists an element $g \in G$ such that $\alpha g^* = \alpha g$ and, therefore, $g^*g^{-1} \in H^*$ and $g^* \in H^*G$. Thus, $G^* \subseteq H^*G$ and item 2 is proven. Since the orbits of H^* on Ω are unions of orbits of H, it follows from the coincidence of the number of the orbits of H and H^* (i.e., from the coincidence of the ranks of G and G^*) that each orbit of H will also be an orbit of H^*. Thus, the orbits of H and H^* are the same, as stated in item 3. Item 4 follows from items 1 and 2: if $G \trianglelefteq G^*$, then $H = H^* \cap G \trianglelefteq H^*$ and

$$H^{G^*} = H^{H^*G} = H^G.$$

Let us prove item 5. It is clear that $H^* \leq N_{G^*}(H)$ if $G \trianglelefteq G^*$. Suppose that $g \in N_{G^*}(H) \setminus H^*$. Then $\beta := \alpha g \neq \alpha$. On the other hand,

$$\beta H = \beta H^g = \alpha g H^g = \alpha H g = \alpha g = \beta;$$

which contradicts the fact that $\{\alpha\}$ is the only orbit of length 1 of the group H. □

Lemma 2. *Let* $G \trianglelefteq X$ *and* $C_X(G) = 1$. *Let* $\phi\colon G \to \mathrm{Sym}(\Omega)$ *be a faithful transitive permutation representation of rank* 3 *of the group* G *and let* $H = G_\alpha$ *be the stabilizer of a point* $\alpha \in \Omega$. *Suppose that the orbits on* Ω *of the group* H *have pairwise distinct lengths. Then* $GN_X(H)$ *is the largest subgroup of* X *onto which the mapping* ϕ *extends, remaining a faithful transitive permutation representation of rank* 3 *on the set* Ω.

Proof. To keep things concise, let $H^0 := N_X(H)$ and $G^0 := H^0G$.

First, we show that the mapping $\phi\colon G \to \mathrm{Sym}(\Omega)$ can be extended to a homomorphism $G^0 \to \mathrm{Sym}(\Omega)$, which defines the action of G^0 on Ω. By the hypothesis, it follows that $\{\alpha\}$ is the only orbit of length 1 of the group H. Now, by item 5 of Lemma 1, we have $H = N_G(H) = H^0 \cap G$. Consider

$$\Delta^0 = \{H^0g \mid g \in G^0\} \quad \text{and} \quad \Delta = \{Hg \mid g \in G\}.$$

From the equalities $G^0 = H^0G$ and $H = H^0 \cap G$, it follows that $\Delta^0 = \{H^0g \mid g \in G\}$ and that the rule $Hg \leftrightarrow H^0g$ for all $g \in G$ correctly defines a natural one-to-one correspondence between the elements of Δ^0 and Δ. This correspondence shows that the actions of G by right multiplication on Δ and Δ^0 are equivalent. The first of these actions is canonically equivalent to the action of G on Ω, and the action of G on Δ^0 is the restriction to G of the canonical action of G^0 on Δ^0. Consequently, the mapping ϕ by equivalence extends to the action of G^0 on Ω.

We will show below that the action of G^0 on Ω defined in this way still has rank 3. The stabilizer of α under this action will be the subgroup H^0, so we need to ensure that H^0 has exactly three orbits. Every orbit of H^0 is a union of orbits of H, since $H \leq H^0$, and one of the orbits of H^0 coincides with the orbit $\{\alpha\}$ of H. Thus, if we assume that the number of orbits of H^0 is not equal to 3, it is equal to two, and $\Omega \setminus \{\alpha\}$ is an orbit of H^0. However, since $H \trianglelefteq H^0$, the orbits of H on $\Omega \setminus \{\alpha\}$, i.e., the two orbits of H other than $\{\alpha\}$, form an imprimitivity system for the action of H^0 on $\Omega \setminus \{\alpha\}$. In particular, they have equal lengths, contrary to the hypothesis.

Finally, we show that the action of G^0 on Ω is faithful. Since G acts faithfully, the kernel K of this action intersects the normal subgroup G trivially. Therefore, $[G, K] \leq G \cap K = 1$ and $K \leq C_X(G) = 1$.

If now G^* is another subgroup of X, containing G and such that the action of G on Ω can be extended to a faithful transitive permutation representation of the rank 3 group G^*, then Lemma 1 yields $G^* = H^*G$, where H^* is the stabilizer of α in G^* and $H^* = N_{G^*}(H) \leq N_X(H) = H^0$. Thus,

$G^* = H^*G \leq H^0G = G^0$, i.e., G^0 is the largest subgroup of X with the required properties, as claimed in the lemma. □

Remark 1. If, under the hypothesis of Lemma 2, we consider the action of X on the set of classes of conjugate subgroups of G induced by conjugations, then $GN_X(H)$ coincides with the stabilizer in X of the class H^G.

The following statement is well known and is included here only for completeness. It allows us to directly apply [13, Theorem 1] in the proof of Propositions 1–4. Alternatively, one can use [15, Theorem 2].

Lemma 3. *Let $G \leq \mathrm{Sym}(\Omega)$ be a primitive almost simple rank 3 permutation group and $L = \mathrm{Soc}\, G$. Then $N_{\mathrm{Sym}(\Omega)}(L)$ is also an almost simple group with socle L. In particular, the socle of $G^{(2)}$ is equal to L if and only if $G^{(2)} \leq N_{\mathrm{Sym}(\Omega)}(L)$.*

Proof. Since L is the socle and the unique minimal normal subgroup of the primitive group G, the group $N_{\mathrm{Sym}(\Omega)}(L)$ is also primitive and, according to [7, Theorem 4.3B], either L is regular on Ω or $C_{\mathrm{Sym}(\Omega)}(L) = 1$. Suppose that L is regular on Ω. Then $|\Omega| = |L|$. Since G is a rank 3 group, the length of one of the orbits of G on Ω^2 is at least $(|L|^2 - |L|)/2$. Let the pair (α, β) belong to this orbit. Then

$$\frac{|L|^2 - |L|}{2} \leq |G : G_{\alpha\beta}| \leq |G| \leq |\operatorname{Aut} L|,$$

hence $(|L| - 1)/2 \leq |\operatorname{Out} L|$. However, according to [16, Lemma 2.2], for every nonabelian simple group L, the inequality $|\operatorname{Out} L| \leq |L|/30$ holds, which clearly contradicts the previous inequality. Therefore, $C_{\mathrm{Sym}(\Omega)}(L) = 1$ and the normalizer $N_{\mathrm{Sym}(\Omega)}(L)$ is isomorphic to $\operatorname{Aut} L$. □

Proof of Propositions 1–4. Let us recall that G is an almost simple group with a nonabelian simple socle L, acting as a rank 3 permutation group on a finite set Ω. Thus, the strongly regular graph Γ_G of degree v and valence k is defined on Ω. Let M also be the stabilizer of a point $\alpha \in \Omega$. As follows from the hypotheses of Propositions 1–4 (see also Introduction), we assume that the group G is primitive, i.e., M is a maximal subgroup of G. □

Suppose that $\mathrm{Soc}\, G^{(2)} \neq L$, as in the hypothesis of Proposition 4. Then $G^{(2)} \not\leq N_{\mathrm{Sym}(\Omega)}(L)$ by Lemma 3 and from [13, Theorem 1] it follows that one of the following Cases A.I–A.III occurs.

Case A.I. The triple $(G, v, G^{(2)})$ is present in the list

$$(\mathrm{P\Gamma L}_2(8), 36, \mathrm{Sym}(9)), \quad (M_{11}, 55, \mathrm{Sym}(11)), \quad (M_{12}, 66, \mathrm{Sym}(12)),$$
$$(M_{23}, 253, \mathrm{Sym}(23)), \quad (M_{24}, 276, \mathrm{Sym}(24)), \quad (\mathrm{Alt}(9), 120, \mathrm{PSO}_8^+(2)).$$

As shown in [2, Theorems 11.3.1–11.3.5], the group G indeed has rank 3, and each of the listed triples corresponds to items 1–3 and 5–7 in Table 3. Using the reference in the last column of Table 3, we determine $M \cap L$ and the graph Γ_G. Thus, in Case A.I, the statement of Proposition 4 holds.

Case A.II. $L = G_2(q)$, $v = |\Omega| = q^3(q^3 - 1)/2$, and if q is odd, then $\mathrm{Soc}\, G^{(2)} = \mathrm{P\Omega}_7(q)$, and if q is even, then $\mathrm{Soc}\, G^{(2)} = \mathrm{PSp}_6(q)$. From [2, Theorem 11.3.4] and [13, Proposition 1] (and the remark following it), we conclude that a group G with socle L can be a rank 3 group only if $q = 3, 4, 8$, which corresponds to items 8, 10, and 11 in Table 3, and when $q = 8$, the group G with socle $L = G_2(q)$ can be a rank 3 group only if $G = G_2(8).3 = \operatorname{Aut} L$. Using the reference in the last column of Table 3, we determine that $M \cap L = \mathrm{PSU}_3(q).2$, and $\Gamma_G = NO_7^-(q)$ when q is even, and $\Gamma_G = NO_7^{-\perp}(3)$ when $q = 3$. We will get information on the group $G^{(2)}$ later, inside the proof of items 8 and 9 of Proposition 3 (see the analysis of Case B.IV below).

Case A.III. $L = \mathrm{P\Omega}_7(q)$ when q is odd and $L = \mathrm{PSp}_6(q)$ when q is even, $v = q^3(q^4 - 1)/(2, q - 1)$ and $\mathrm{Soc}\, G^{(2)} = \mathrm{P\Omega}_8^+(q)$. From [13, Proposition 2], taking into account the previously accepted agreements, we conclude that the group G with socle L can be a rank 3 group only when $q = 2, 3$, which corresponds to items 4 and 9 in Table 3. From the reference indicated in the last column of Table 3, we determine that $M \cap L = G_2(q)$ and $\Gamma_G = NO_8^+(q)$. Complete information about the group $G^{(2)}$ will be obtained later, inside the proof of items 15 and 16 of Proposition 3 (see the analysis of Case B.IV below).

As in Propositions 1–3, let us further assume that the socle of $G^{(2)}$ coincides with L. Since a group is contained in its 2-closure, we have

$$L \trianglelefteq G \leq G^{(2)} \leq X, \tag{3.1}$$

where $X = \operatorname{Aut} L$. Let $H = L_\alpha = M \cap L$ be the stabilizer in L of $\alpha \in \Omega$. Consider Cases B.I–B.IV, when L is respectively alternating, sporadic, exceptional or classical.

Case B.I. L is an alternating group of degree at least 5 and $\operatorname{Soc} G^{(2)} = L$. All possibilities for the action of a group G with an alternating socle are described in Bannai's theorem [1] (see also [2, Theorem 11.3.1]). In particular, it follows from this theorem that the group L itself acts on Ω as a rank 3 group and, except when $L = \operatorname{Alt}(7)$ and $\Gamma_G = \Gamma_L = T(7)$, the lengths k and $v - k - 1$ of the orbits of H on Ω, other than $\{\alpha\}$, are different. If $\Gamma_G = \Gamma_L = T(7)$, then $G^{(2)} = \operatorname{Aut}\Gamma_G = \operatorname{Sym}(7)$ according to [3, Theorem 9.1.2]. In other cases, by virtue of (3.1) and Lemma 2, the group $G^{(2)}$ coincides with the stabilizer in $\operatorname{Aut} L$ of the conjugacy class H^L of H. Again, from Bannai's theorem and the structure of the automorphism group of an alternating group, we conclude that either $L \neq \operatorname{Alt}(6)$, the conjugacy class H^L is invariant under $\operatorname{Aut} L = \operatorname{Sym}(6)$, and one of items 1, 2 of Proposition 1 holds, or $L = \operatorname{Alt}(6)$ and, up to permutation equivalence, $\Gamma_L = T(6)$. In the latter case, $G^{(2)} = \operatorname{Aut}\Gamma_G = \operatorname{Sym}(6)$ is a subgroup of index 2 in $\operatorname{Aut} L$ (see [3, Theorem 9.1.2]). Thus, in the case where L is a simple alternating group, Proposition 1 holds.

Case B.II. L is a sporadic group and $\operatorname{Soc} G^{(2)} = L$. All possibilities (19 cases) when a group G with a sporadic socle acts as a rank 3 group are listed in [2, Theorem 11.3.5 and Table 11.3]. If

$$(L, v) \in \{(M_{11}, 55), (M_{12}, 66), (M_{23}, 253), (M_{24}, 276)\},$$

then, as we observed in the analysis of Case A.I, $\operatorname{Soc} G^{(2)} \neq L$. The remaining 15 cases correspond to items 1–15 in Table 1. From the corresponding subsection of [2] cited in the last column of the table, we find information about the subgroup $H = M \cap L$, the graph Γ_G, and the group $\operatorname{Aut}\Gamma_G = G^{(2)}$. Thus, we verify the validity of Proposition 2.

Case B.III. L is an exceptional group of Lie type and $\operatorname{Soc} G^{(2)} = L$. The rank 3 groups with an exceptional simple socle are described in [2, Theorem 11.3.4]. If

$$(L, v) \in \{(G_2(3), 351), (G_2(4), 2016), (G_2(8), 130\,816)\},$$

then, as we observed when discussing Case A.II, $\operatorname{Soc} G^{(2)} \neq L$. In the remaining cases, according to [2, Theorem 11.3.4], either

$$L = G_2(4), \quad v = 416, \quad \text{and} \quad H = \mathrm{HJ} = J_2,$$

or

$$L = E_6(q), \quad v = \frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)}, \quad \text{and} \quad H \text{ is a parabolic subgroup of type } D_5(q).$$

The situation $L = G_2(4)$ corresponds to item 10 in Table 2, and from [2, Sect. 10.68] we find information about Γ_G and verify that $G^{(2)} = \operatorname{Aut}\Gamma_G = \operatorname{Aut} L$, as claimed in Proposition 3. If

$L = E_6(q)$, then by [2, Theorem 11.3.4], the group L itself acts on Ω as a rank 3 group. Therefore, $G^{(2)}$ coincides with the stabilizer in $\operatorname{Aut} L$ of the conjugacy class H^L of H. It is known (see, for example, [6, Table 9]) that in the group L there are two conjugacy classes of parabolic subgroups H of the specified structure. These classes are permuted by a graph automorphism, and the stabilizer of the class coincides with the group $\widetilde{L}$, as indicated in item 17 of Proposition 3.

Case B.IV. L is a classical simple group of Lie type and $\operatorname{Soc} G^{(2)} = L$. All possibilities for the action of G are listed in [2, Theorems 11.3.2 and 11.3.3]. Some of these possibilities correspond to the cases listed in Table 2. If L and v are indicated in this table, then $H = M \cap L$; the graph Γ_G and the automorphism group of Γ_G are found in the corresponding subsection of [2] indicated in the last column of Table 3. Items 1–4 of this table correspond to cases where L is an alternating group and the group $G^{(2)}$ is known (see Case B.I) due to known isomorphisms. In other cases, the group $\operatorname{Aut}\Gamma_G$ is explicitly specified in the corresponding subsection of [2]. Furthermore, some of the possibilities in [2, Theorems 11.3.2 and 11.3.3] correspond to the case when

$$(L, v) \in \{(\mathrm{PSL}_2(8), 36), (\mathrm{PSp}_6(2), 120), (\mathrm{P\Omega}_7(3), 1080)\},$$

and then see Cases A.I and A.II.

Finally, all remaining possibilities in [2, Theorems 11.3.2 and 11.3.3] correspond to cases where the group L and the action of the group G are listed in items 1–16 of Proposition 3. Let us begin their analysis with the case when $L = \mathrm{PSp}_{2n}(q)$, where $q = 8$, and the group G with socle L acts on the set Ω of hyperbolic or elliptic hyperplanes of $(2n+1)$-dimensional vector space with a non-degenerate quadratic form; in this case, as clearly indicated in [2, Theorem 11.3.2(iv)], the group G as a subgroup of $X = \operatorname{Aut} L$ is uniquely determined, namely $G = \mathrm{PSp}_{2n}(8).3 = \mathrm{P C\Gamma Sp}_{2n}(8)$. In particular, $G^{(2)} = G = \mathrm{P C\Gamma Sp}_{2n}(8) = \widetilde{L}$. Thus, item 8 in Proposition 3 holds.

In all other cases, according to [2, Theorems 11.3.2 and 11.3.3], the socle L of G itself acts on Ω as a rank 3 group. This action, up to permutation equivalence, and the resulting graph $\Gamma_G = \Gamma_L$ coincide with the action and graph indicated in items 1–16 of Proposition 3 (with the exception of the case when $q = 8$ in item 8 considered above). It follows that H is a maximal subgroup that is a stabilizer of a certain subspace, completely isotropic or non-degenerate, in the corresponding projective module V. Therefore, H is a memeber of the Ashbacher class $\mathcal{C}_1$. The stabilizers of the conjugacy classes of memebers of Ashbacher classes in $\operatorname{Aut} L$ are known and given for the case when the projective dimension of V is at least 12 in [12, Table 3.5G] taking into account the information in column V from [12, Table 3.5A–3.5F], and for the case when the projective dimension of V is at most 11, in [4, Table 8.1–8.85]. In all cases, we verify that the stabilizer in $X = \operatorname{Aut} L$ of the conjugacy class of H coincides with the group declared as $G^{(2)}$ in the corresponding item of Proposition 3. Direct calculations using the parameters of graph Γ_L show that the lengths k and $v - k - 1$ of orbits of H, other than $\{\alpha\}$, are different (if p is the characteristic of the field over which the space V is considered, then we can compare the maximum powers of p dividing k and $v - k - 1$). By Lemma 2, each statement in items 1–16 of Proposition 3 holds.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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