On recognition of direct powers of finite simple linear groups by spectrum

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Abstract

The spectrum of a finite group is the set of its element orders. We give an affirmative answer to Problem 20.58(a) from the *Kourovka Notebook* proving that for every positive integer k, the k-th direct power of the simple linear group $L_n(2)$ is uniquely determined by its spectrum in the class of finite groups provided n is a power of 2 greater than or equal to $56k^2$.

1 Introduction

All groups considered in this paper are finite, the simple sporadic groups and simple groups of Lie type are denoted according to the notation of *Atlas of finite groups* [4], the symmetric and alternating groups of degree n are denoted by Sym_n and Alt_n, respectively.

Given a group G, denote by $\omega(G)$ the spectrum of G, that is the set of all its element orders. Groups whose spectra coincide are said to be isospectral. We refer to a group G as *recognizable* (by spectrum) if every finite group isospectral to G is isomorphic to G, as *almost recognizable* (by spectrum) if there is only a finite number of pairwise nonisomorphic groups isospectral to G, and as *unrecognizable* otherwise. It is known that if a finite group is almost recognizable, then its socle is a direct product of nonabelian simple groups [19, Lemma 1]. On the other hand, if G is a nonabelian simple group, then in most cases G is almost recognizable [9, Theorem 1.1]. More information on the recognition of simple groups and related topics can be found in the recent survey article [11].

Though the recognition problem is solved for most of the simple groups, very little is known about recognizability of (nontrivial) direct products of simple groups. The recognizability of groups $Sz(2^7) \times Sz(2^7)$ and $J_4 \times J_4$ was proved in [18] and [7], respectively. Recently, it has been proved in [27] that the direct squares of Suzuki groups $Sz(q)$, where $q \geq 8$ and $q \neq 32$, are recognizable and the group $Sz(32)\times Sz(32)$ is almost recognizable. For cubes of simple groups, it is only known that the group $L_n(2)\times L_n(2)\times L_n(2)$ is recognizable for all $n = 2^l \geq 64$ [8].

Can a recognizable group be a direct product of arbitrary many simple groups? If we do not presuppose that all simple factors are isomorphic, then the answer is affirmative as shown in [11, Theorem 19]: for every $k > 1$ there exists a set $\Delta(k)$ of k primes such that the group $\prod_{p\in\Delta(k)} Sz(2^p)$ is recognizable. If we fix a simple group L (or even any finite group) and consider its direct powers, then as easily seen (cf. [11, Section 4.3]), there is k_0 depending

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on L such that $\omega(L^k) = \omega(L^{k_0})$ for all $k \geq k_0$, in particular, L^k is unrecognizable for every such k. The remaining question is if one can, vice versa, start with an integer k and find an appropriate simple group L.

Problem. [11, Problem 4.8], [16, Problem 20.58(a)] *Is it true that for every k there is a recognizable group that is the k-th direct power of a nonabelian simple group?*

In the present paper we develop techniques from [8] and obtain an affirmative answer to this problem.

Theorem 1. *Let k and l be positive integers and $n = 2^l \geq 56k^2$. Suppose that $L = L_n(2)$ and P is the k-th direct power of L. If G is a finite group with $\omega(G) = \omega(P)$, then $G \simeq P$.*

Remark 1. Suppose that given a positive integer k, one wish to find the smallest n_0 such that $P = L_n(2)^k$ is recognizable for every $n = 2^l \geq n_0$. Theorem 1 provides a upper bound on n_0, which is quadratic. It is not hard to show (and we do this in the last section, see Proposition 5.3) that there is a linear lower bound: the group P is unrecognizable for all $n < 2k$. The exact value of n_0 as a function of k is not known for all $k > 1$ (if $k = 1$, then it follows from [28, Corollary 1]) that $n_0 = 4$).

Remark 2. One of the key tools in the proof of Theorem 1 that helps to establish a quadratic upper bound is Lemma 2.1. This lemma provides upper bounds on the number of distinct prime divisors of the order of a solvable group in terms of the maximum number k of distinct prime divisors of an element order of this group. It seems that the linear bound $6k$ from item (ii) of this lemma is the best known (see, e.g., the proof of the main result in [13], where the bound $7k$ was established).

The next theorem shows that the situation described in Theorem 1 is quite specific. Namely, for a wide range of simple groups of arbitrarily large dimension even their squares or cubes are unrecognizable by spectrum. The standard abbreviations $L_n^+(q) = L_n(q)$ and $L_n^-(q) = U_n(q)$ are used; the cyclic group of order r is denoted by $\mathbb{Z}_r$.

Theorem 2. *Let $n \geq 2$ be an integer and q a power of a prime p. The following hold.*

(i) *If $L = L_n^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$, and there exists a prime r dividing $q - \varepsilon 1$ and coprime to n, then $\omega(L^3) = \omega(L^3 \times \mathbb{Z}_r^m)$ for every positive integer m. Moreover, if $n - 1$ is not a power of p, then $\omega(L^2) = \omega(L^2 \times \mathbb{Z}_r^m)$ for every positive integer m.*

(ii) *If $L = S_{2n}(q)$ and q is odd, then $\omega(L^3) = \omega(L^3 \times \mathbb{Z}_2^m)$ for every positive integer m. Moreover, if $2n - 1$ is not a power of p, then $\omega(L^2) = \omega(L^2 \times \mathbb{Z}_2^m)$ for every positive integer m.*

The paper is organized as follows. In Section 2, we discuss arithmetic properties of spectra of simple groups alongside with some number theoretic facts. In Section 3, we list auxiliary group theoretic results that are used in our proofs. Section 4 is devoted to the proof of Theorem 1. Finally, in Section 5, we prove Theorem 2 and Proposition 5.3.

2 Preliminaries: arithmetic of spectra of simple groups

Given a nonzero integer n, we put $\pi(n)$ for the set of prime divisors of n. If G is a group and $g \in G$, then we write $\pi(G)$ for $\pi(|G|)$ and $\pi(g)$ for $\pi(|g|)$. Denote

$$\rho(k) = \max\{|\pi(G)| \mid G \text{ is solvable and } |\pi(g)| \leq k \text{ for every } g \in G\}.$$

Lemma 2.1. *The following hold:*

(i) $\rho(k) \leq \frac{k(k+3)}{2}$ *for every* $k \geq 1$;

(ii) $\rho(k) \leq 6k$ *for every* $k \geq 1$;

(iii) $\rho(k) \leq 7k - 9$ *for every* $k \geq 2$.

Proof. Item (i) is [29, Theorem 1].

To prove (ii) assume to the contrary that there exists k such that $\rho(k) > 6k$. Since $(k+3)/2 \leq 6$ for $k \leq 9$, we infer that $k \geq 10$, so $\rho(k) \geq 61$. Let $\tau(x)$ denote the number of primes less than or equal to x. It follows from [15, Theorem 7] that $\rho(k) \leq \tau(\rho(k)) + 4k$ for every positive integer k. By [5, Corollary 5.2], $\tau(x) \leq \frac{1.2551 \cdot x}{\ln(x)}$ for every $x > 1$. Therefore, $\rho(k) \leq \frac{1.2551}{\ln(61)}\rho(k) + 4k < 0.3054\rho(k) + 4k$ and hence $\rho(k) < 5.8k$; a contradiction.

Item (iii) easily follows from (i) and (ii). Indeed, if $k > 9$, then (ii) yields $\rho(k) \leq 6k \leq 7k - 9$. On the other hand, $k(k+3)/2 \leq 7k - 9$ is equivalent to $(k-2)(k-9) \leq 0$, so $\rho(k) \leq 7k - 9$ for $2 \leq k \leq 9$ in view of (i). $\square$

The prime graph (or the Gruenberg-Kegel graph) $\Gamma(G)$ of a group G is defined as follows. The vertex set is the set $\pi(G)$. Two vertices corresponding to distinct primes r and s are adjacent in $\Gamma(G)$ if and only if $rs \in \omega(G)$. Recall that an independent set of vertices or a *coclique* of a graph Γ is any subset of pairwise nonadjacent vertices of Γ. We write $t(G)$ to denote the greatest size of a coclique in $\Gamma(G)$. The following obvious observation is a key to our technique and will be repeatedly used in further considerations.

Lemma 2.2. *Suppose that Ω is a coclique of size t in the prime graph of a group G. Then for every positive integer $k < t$, the k-th direct power G^k of G does not contain an element of order equal to the product of all primes from Ω.*

For a set of nonzero integers $n_1, \dots, n_k$, we denote by $(n_1, \dots, n_k)$ and $[n_1, \dots, n_k]$ their greatest common divisor and least common multiple, respectively. If n is a nonzero integer and r is an odd prime with $(r, n) = 1$, then $e(r, n)$ denotes the multiplicative order of n modulo r. Given an odd integer n, we put $e(2, n) = 1$ if $n \equiv 1 \pmod 4$, and $e(2, n) = 2$ otherwise. Fix an integer a with $|a| > 1$. A prime r is said to be a primitive prime divisor of $a^i - 1$ if $e(r, a) = i$. We write $r_i(a)$ to denote some primitive prime divisor of $a^i - 1$, if such a prime exists, and $R_i(a)$ to denote the set of all such divisors. For $i \neq 2$ the product of all primitive divisors of $a^i - 1$ taken with multiplicities is denoted by $k_i(a)$. Put $k_2(a) = k_1(-a)$. It is well known that that primitive prime divisors exist for almost all pairs (a, i).

Lemma 2.3. [1, 30] *Let a be an integer and $|a| > 1$. For every positive integer i the set $R_i(a)$ is nonempty, except for the pairs $(a, i) \in \{(2, 1), (2, 6), (-2, 2), (-2, 3), (3, 1), (-3, 2)\}$.*

Sometimes it is convenient to consider primitive divisors $r_i(-q)$ instead of $r_i(q)$ (e.g., for unitary groups). The following lemma helps to deal with the numbers $e(r_i(-q), q)$ and $e(r_i(q), -q)$.

Lemma 2.4. [22, Lemma 1.3] *Let a and i be integers with $|a| > 1$ and $i > 0$. If i is odd then $k_i(-a) = k_{2i}(a)$, and if i is a multiple of 4 then $k_i(-a) = k_i(a)$.*

For convenience, given a classical group L, we put $\mathrm{prk}(L)$ to denote its dimension if L is a linear or unitary group, and its Lie rank if L is a symplectic or orthogonal group.

Define the following function on positive integers:

$$\eta(k) = \begin{cases} k, \text{ if } k \text{ is odd,} \\ k/2, \text{ if } k \text{ is even.} \end{cases}$$

Following [22], we introduce a function φ in order to unify further arguments. Namely, given a simple classical group L over a field of order q and a prime r coprime to q, we put

$$\varphi(r, L) = \begin{cases} e(r, \varepsilon q), \text{ if } L = L_n^\varepsilon(q), \text{ where } \varepsilon \in \{+, -\}; \\ \eta(e(r, q)), \text{ if } L \text{ is symplectic or orthogonal.} \end{cases}$$

Lemma 2.5. [22, Lemma 2.4] *Let L be a simple classical group over a field of order q and characteristic p, and let $\operatorname{prk}(L) = n \geq 4$.*

(i) *If $r \in \pi(L) \setminus \{p\}$, then $\varphi(r, L) \leq n$.*

(ii) *If r and s are distinct primes from $\pi(L) \setminus \{p\}$ with $\varphi(r, L) \leq n/2$ and $\varphi(s, L) \leq n/2$, then r and s are adjacent in $\Gamma(L)$.*

(iii) *If r and s are distinct primes from $\pi(L) \setminus \{p\}$ with $n/2 < \varphi(r, L) \leq n$ and $n/2 < \varphi(s, L) \leq n$, then r and s are adjacent in $\Gamma(L)$ if and only if $e(r, q) = e(s, q)$.*

(iv) *If r and s are distinct primes from $\pi(L) \setminus \{p\}$ and $e(r, q) = e(s, q)$, then r and s are adjacent in $\Gamma(L)$.*

Item (iv) of the previous lemma can be generalized as follows (cf. [22, Lemma 2.13]).

Lemma 2.6. *Let m be a positive integer and L a simple classical group over a field of order q and characteristic p. For $j = 1, \ldots, m$, suppose that pairwise distinct primes r_j lie in $\pi(L) \setminus \{p\}$ and put $i_j = e(r_j, q)$. If $i_1, i_2, \ldots, i_m$ are greater than 2 and pairwise distinct, then $r_1 r_2 \cdots r_m \in \omega(L)$ if and only if $k_{i_1}(q) k_{i_2}(q) \cdots k_{i_m}(q) \in \omega(L)$.*

We need the descriptions of the spectra for linear, unitary and symplectic simple groups obtained in [2, 3].

Lemma 2.7. [2, Corollary 3] *Let $G = L_n^\varepsilon(q)$, where $n \geq 2$, $\varepsilon \in \{+, -\}$, and q is a power of a prime p. Put $d = (n, q - \varepsilon 1)$. Then $\omega(G)$ consists of all divisors of the following numbers:*

(i) $\frac{q^n - (\varepsilon 1)^n}{d(q - \varepsilon 1)}$;

(ii) $\frac{[q^{n_1} - (\varepsilon 1)^{n_1}, q^{n_2} - (\varepsilon 1)^{n_2}]}{(n/(n_1, n_2), q - \varepsilon 1)}$ *for $n_1, n_2 > 0$ such that $n_1 + n_2 = n$;*

(iii) $[q^{n_1} - (\varepsilon 1)^{n_1}, q^{n_2} - (\varepsilon 1)^{n_2}, \ldots, q^{n_s} - (\varepsilon 1)^{n_s}]$ *for $s \geq 3$ and for $n_1, n_2, \ldots, n_s > 0$ such that $n_1 + n_2 + \ldots + n_s = n$;*

(iv) $p^k \cdot \frac{q^{n_1} - (\varepsilon 1)^{n_1}}{d}$ *for $k, n_1 > 0$ such that $p^{k-1} + 1 + n_1 = n$;*

(v) $p^k \cdot [q^{n_1} - (\varepsilon 1)^{n_1}, q^{n_2} - (\varepsilon 1)^{n_2}, \ldots, q^{n_s} - (\varepsilon 1)^{n_s}]$ *for $s \geq 2$ and $k, n_1, n_2, \ldots, n_s > 0$ such that $p^{k-1} + 1 + n_1 + n_2 + \ldots + n_s = n$;*

(vi) p^k *if $p^{k-1} + 1 = n$ for $k > 0$.*

Lemma 2.8. [3, Corollary 2] *Let $G = S_{2n}(q)$, where $n \geq 2$ and q is a power of an odd prime number p. Then $\omega(G)$ consists of all divisors of the following numbers:*

(i) $\frac{q^n \pm 1}{2}$;

(ii) $[q^{n_1}+\varepsilon_1 1, q^{n_2}+\varepsilon_2 1, \ldots, q^{n_s}+\varepsilon_s 1]$ *for all* $s \geq 2$, $\varepsilon_i \in \{+,-\}$, $1 \leq i \leq s$, *and positive* $\{n_j\}$, *with* $n_1+n_2+\ldots+n_s=n$;

(iii) $p^k \cdot [q^{n_1}+\varepsilon_1 1, q^{n_2}+\varepsilon_2 1, \ldots, q^{n_s}+\varepsilon_s 1]$ *for all* $s \geq 1$, $\varepsilon_i \in \{+,-\}$, $1 \leq i \leq s$, *and positive* k *and* $\{n_j\}$, *with* $p^{k-1}+1+2n_1+2n_2+\ldots+2n_s=2n$;

(iv) p^k *if* $p^{k-1}+1=2n$ *for some* $k>1$.

The following two lemmas are almost direct corollaries of the above descriptions.

Lemma 2.9. *Suppose that L is a simple classical group over a field of odd characteristic. If* $\mathrm{prk}(L) \geq 2^k+2$ *for an integer k, then* $2^{k+2} \in \omega(L)$.

Proof. Suppose that u is the order of the underlining field of L. It follows from Lemmas 2.7, 2.8 and [3, Corollaries 6, 8, 9] that $u^{2^k}-1 \in \omega(L)$. Note that u^2-1 is divisible by 8. Since $u^{2^k}-1=(u^2-1)\prod_{i=1}^{k-1}(u^{2^i}+1)$, we infer that 2^{k+2} divides $u^{2^k}-1$, so $2^{k+2} \in \omega(L)$. □

For a real number x, denote by $[x]$ the integral part of x that is the largest integer less than or equal to x.

Lemma 2.10. *Suppose that* $L=L_n^{\varepsilon}(q)$, *where* $\varepsilon \in \{+,-\}$. *Then the following hold:*

(i) *if* $n \geq 12$, *then* $\Omega=\{r_i(\varepsilon q) \mid n/2<i\leq n\}$ *is a coclique of size* $[(n+1)/2]$ *in* $\Gamma(L)$;

(ii) *if* $n=2^l$ *and q is even, then* $2^l \in \omega(L)$ *and* $2^{l+1} \notin \omega(L)$;

(iii) *if* $a \in \omega(L)$, *then* $a \leq q^n/(q-1)$.

Proof. The first assertion is a consequence of [24, Prop. 6.9 and Table 8]. The second and the third ones follow from Lemma 2.7(vi) and [21, Lemma 1.3], respectively. □

In three following lemmas, we concentrate on properties of the spectra of linear groups over field of order 2 and their direct products.

Lemma 2.11. *If* $G=\mathrm{Aut}(L)$, *where* $L=L_n(2)$, $n=2^l \geq 4$, *then* $2^{l+1} \in \omega(G)$.

Proof. Observe that $L=GL_n(2)$ is the full general linear group and $G=L \rtimes \langle g \rangle$, where g is the invert-transpose automorphism of L. Let x be a unipotent matrix from L whose Jordan normal form consists of $2^{l-1}-1$ Jordan blocks of size one and one block of size $2^{l-1}+1$. Clearly, $|x|=2^l$. It follows from [26, Theorem 2.3.1] that there exists $h \in L$ such that $x=hh^g=(hg)^2$. Then $|hg|=2^{l+1}$, as required. □

Lemma 2.12. *Suppose that* $i,k \geq 2$ *are integers and n is a power of* 2. *If* $n \geq i$ *and* $n>18(k+1)$, *then there exists a set* $\Psi(i)$ *consisting of* $2k$ *distinct integers* $n/2<i_1,i_2,\ldots,i_k \leq n$ *and* $n/3<j_1,j_2,\ldots,j_k<n/2$ *such that* $i_m+j_m=n$ *for* $1 \leq m \leq k$ *and if* $a,b \in \Psi(i)$, $a \neq b$, *then a, b, and i do not divide each other.*

Proof. We will find a set $\Psi(i)$ of required numbers among $2(k+1)$ numbers of the form

$$i_m=n/2+\sum_{l=1}^{m} t_l \text{ and } j_m=n/2-\sum_{l=1}^{m} t_l, \text{ where } t_l \in \{1,2,3\} \text{ and } m=1,\ldots,k+1.$$

Clearly, $i_m + j_m = n$ for all $m = 1, \ldots, k+1$. Since $n > 18(k+1)$, it follows that $i_m \leq n/2 + 3m \leq n/2 + 3(k+1) < n$ and $j_m \geq n/2 - 3m \geq n/2 - 3(k+1) > n/3$ for every $m = 1, \ldots, k+1$.

Let us now check the desired divisibility conditions. If one of the elements of $\Psi(i)$ divides the other, say, a divides b, then it is clear that $a = n/2 - \sum_{l=1}^{x} t_l$, $b = n/2 + \sum_{l=1}^{y} t_l$ for some $x, y \in \{1, \ldots, k+1\}$, and $b = 2a$. Since $\sum_{l=1}^{k+1} t_l \leq 3(k+1)$, we arrive to a contradiction, because $a = 2b$ yields the impossible inequality $n/2 \leq 9(k+1)$.

There is $t_1 \in \{1, 2, 3\}$ such that $i_1 = n/2 + t_1$ and $j_1 = n/2 - t_1$ are both not divisible by i. Indeed, if not, there exists $\varepsilon \in \{+, -\}$ such that i divides $n/2 + \varepsilon 1$ and $n/2 + \varepsilon 3$ simultaneously. The latter is possible only if $i = 2$, but $n/2+1$ and $n/2-1$ are odd. Assume by induction on m that we have already proved the same for all i_s and j_s with $s < m$. Then again there is $t_m \in \{1, 2, 3\}$ such that $i_m = i_{m-1} + t_m$ and $j_m = j_{m-1} - t_m$ are both not divisible by i. Otherwise, there exists $\varepsilon \in \{+, -\}$ such that i divides $i_{m-1} + \varepsilon 1$ and $i_{m-1} + \varepsilon 3$ simultaneously. Again this implies that $i = 2$, but either $i_{m-1} \pm 1$ or $i_{m-1} \pm 2$ are both odd.

Now we claim that there exists at most one index $m \in \{1, \ldots, k+1\}$ such that i_m or j_m divides i. By construction, we know that $i_m > n/2 > j_m > n/3$, so the only possibility is $i = 2j_m$. Excluding, if necessary, the pair i_m and j_m with this property, we obtain $2k$ required numbers. ☐

Lemma 2.13. *Suppose that numbers i, k, n, and the set $\Psi(i)$ are as in Lemma 2.12. If $\Pi(i) = \prod_{j \in \Psi(i)} r_j(2)$, then the following hold:*

(i) $\Pi(i) \in \omega(L_n(2)^k)$;

(ii) *if $\Pi(i) \in \omega(L_{n_1}(2) \times L_{n_2}(2) \ldots \times L_{n_k}(2))$, where $4 \leq n_m \leq n$ for $m \in \{1, \ldots, k\}$, then $n_m = n$ for every m;*

(iii) $2 \cdot \Pi(i) \notin \omega(L_n(2)^k)$ *and if $i \neq 6$, then $r_i(2) \cdot \Pi(i) \notin \omega(L_n(2)^k)$.*

Proof. Recall that $\Psi(i) = \{i_1, \ldots, i_k, j_1, \ldots, j_k\}$, where $i_1 < i_2 < \ldots < i_k$ and $n = i_1 + j_1 = \ldots = i_k + j_k$. It follows from Lemma 2.7(ii) that $r_{i_m}(2) \cdot r_{j_m}(2) \in \omega(L_n(2))$ for $m = 1, \ldots, k$. Hence $\Pi(i) \in \omega(L_n(2)^k)$, which proves (i).

To prove (ii), put $L_m = L_{n_m}(2)$ for $m = 1, \ldots, k$, and $P = L_1 \times \ldots \times L_k$. Let $g = g_1 \cdot \ldots \cdot g_k$ be an element of order $\Pi(i)$ in P and $g_m \in L_m$ for $m = 1, \ldots, k$. Since $i_1, \ldots, i_k$ are greater than $n/2$ and do not divide each other, Lemma 2.5 implies that the primes $r_{i_1}(2), \ldots, r_{i_k}(2)$ divide the orders of pairwise distinct elements among $g_1, \ldots, g_k$. Hence up to reordering, we may assume that $r_{i_m}(2)$ divides $|g_m|$ for $m = 1, \ldots, k$.

If $r_{j_1}(2)$ divides $|g_m|$ for $m > 1$, then $r_{j_1}(2) r_{i_m}(2) \in \omega(L_m)$, which is impossible, because $i_m + j_1 > n$ and i_m and j_1 do not divide each other. Therefore, $r_{j_1}(2) r_{i_1}(2)$ divides $|g_1|$. In particular, we have $n_1 = n$, because $i_1 + j_1 = n$. Furthermore, as easy to deduce from Lemma 2.7, $|g_1| = r_{j_1}(2) r_{i_1}(2)$. It follows now by induction on k that $|g_m| = r_{j_m}(2) r_{i_m}(2)$ for each $m = 2, \ldots, k$. Thus, $L_m = L_n(2)$ for all m, as required.

Proving (iii), we apply very similar arguments. Let $P = L_1 \times \ldots \times L_k$, where $L_m = L_n(2)$ for all m, and let $g = g_1 \cdot \ldots \cdot g_k$ be an element of order $r \cdot \Pi(i)$ in P, where $g_m \in L_m$ for $m = 1, \ldots, k$ and either $r = 2$ or $r = r_i(2)$. Arguing as in the previous paragraph, we obtain that $|g_m| = r_{j_m}(2) r_{i_m}(2)$ for each $m = 1, \ldots, k$, so the order of g must be exactly $\Pi(i)$, which proves (iii). ☐

We complete the section with two simple number-theoretic observations.

Lemma 2.14. *If $n \geq 59$ is an integer, then there are at least $[2n/3]+3$ composite numbers among $n, n+1, \dots, 2n$.*

Proof. Denote $I = \{n, n+1, \dots, 2n\}$. If x and y are positive integers, then y divides exactly $[x/y]+1$ numbers among $0, \dots, x$. Therefore, at least $[n/2]$ numbers in I are divisible by 2, at least $[n/3]$ are divisible by 3, and at most $[n/6]+1$ are divisible by 6. Hence 2 or 3 divide at least $[n/2]+[n/3]-[n/6]-1$ numbers in I. Note that I contains 60 consecutive integers, so there exist $a, b, c, d \in I$ such that $a \equiv 5 \pmod{60}$, $b \equiv 25 \pmod{60}$, $c \equiv 35 \pmod{60}$, and $d \equiv 55 \pmod{60}$. These four numbers are composite and coprime to 6. Suppose that $n \equiv r \pmod 3$, where $0 \leq r \leq 2$. Then we have at least $n/2-1/2+n/3-r/3-n/6-1+4 = 2n/3 - r/3 + 5/2 \geq [2n/3] - 1/3 + 5/2 > [2n/3] + 2$ composite numbers in I. Since this number is an integer, the result follows. □

Lemma 2.15. *Suppose that $x \geq 0$ and $\Theta = \{i \in \mathbb{Z}, i \geq 1 \mid \eta(i) \leq x\}$. Then $\frac{3x}{2} - \frac{3}{2} \leq |\Theta| \leq \frac{3x}{2} + \frac{1}{2}$.*

Proof. There are two cases depending on the parity of $[x]$. Suppose that $[x] = 2k$. Then $\Theta = \{1, 2, \dots, 2k, 2k+2, \dots, 4k\}$, so $|\Theta| = 3k$. Since $x - 1 < 2k \leq x$, it follows that $(3x)/2 - 3/2 < 3k \leq (3x)/2 < (3x)/2 + 1/2$.

If $[x] = 2k+1$, then $|\Theta| = 3k+2$. We have $x - 1 < 2k+1 \leq x$, so $(3x)/2 - 3/2 < 3k + 3/2 \leq (3x)/2$. Then $|\Theta| > 3k+3/2 > (3x)/2 - 3/2$ and, on the other hand, $|\Theta| - 1/2 = 3k + 3/2 \leq (3x)/2$, as required. □

3 Preliminaries: group theory

The first well-known lemma helps to deal with groups having trivial solvable radical.

Lemma 3.1. [20, 3.3.20] *Let $R \simeq R_1 \times \cdots \times R_k$, where R_i is the n_i-th direct power of a simple group S_i, and $S_i \not\simeq S_j$ for $i \neq j$. Then $\operatorname{Aut} R \simeq \operatorname{Aut} S_1 \times \cdots \times \operatorname{Aut} S_k$ and $\operatorname{Aut} R_i \simeq (AutS_i) \wr Sym_{n_i}$, where in this wreath product $\operatorname{Aut} S_i$ appears in its right regular representation and the symmetric group Sym_{n_i} in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms $\operatorname{Out} R \simeq \operatorname{Out} R_1 \times \cdots \times \operatorname{Out} R_k$ and $\operatorname{Out} R_i \simeq (\operatorname{Out} S_i) \wr Sym_{n_i}$.*

The next two lemmas provide an existence of appropriate large solvable subgroups.

Lemma 3.2. *Let G be a group and $(A_1, \dots, A_m)$ a tuple of all chief factors of G. Suppose that there exist k distinct integers $i_1, i_2, \dots, i_k$ and distinct primes $p_1, p_2, \dots, p_k$ such that $1 \leq i_j \leq m$ and p_j divides $|A_{i_j}|$ for every $j \in \{1, \dots, k\}$. Then G includes a solvable subgroup K such that $\pi(K) = \{p_1, \dots, p_k\}$.*

Proof. We proceed by induction on $|G|$. If G is simple, then $k = 1$ and a Sylow p_1-subgroup of G fits as K. Thus, G includes a proper nontrivial minimal normal subgroup M. If $|M|$ is coprime to $\pi = \{p_1, p_2, \dots, p_k\}$, then, by induction, G/M includes a required π-subgroup and, by the Schur–Zassenhaus theorem, so does G.

Let now $\pi \cap \pi(M) \neq \varnothing$. Without loss of generality, we may assume that $M = A_1$. If R is a Sylow p_1-subgroup of M, then $G = N_G(R)M$ due to the Frattini argument. Since $N_G(R)$ satisfies the hypothesis of the lemma, $G = N_G(R)$ by inductive arguments. In view of the Jordan–Hölder theorem, the groups $A_2, \dots, A_k$ are among the chief factors of $\overline{G} = G/R$. Therefore, $\overline{G}$ includes a solvable subgroup $\overline{K}$ with $\pi(\overline{K}) = \{p_2, \dots, p_k\}$. The preimage K of $\overline{K}$ in G is a required solvable subgroup. □

Lemma 3.3. *Suppose that K is a nontrivial normal subgroup of a group G and $r \in \pi(K)$. Then there exists a subgroup H in G with a normal solvable subgroup M such that $H/M \simeq G/K$, $\pi(M) \subseteq \pi(K)$, and M is a product of its Sylow subgroups $T_1, \ldots, T_k$, where T_1 is an r-group, and $T_j \subseteq N_N(T_i)$ for every $j > i$.*

Proof. By induction on $|G|$ and the Frattini argument, we may assume that $G = N_G(T_1)$. Put $\overline{G} = G/T_1$ and $\overline{K} = K/T_1$. If $K = T_1$, then we are done, so $\overline{K}$ and $\overline{G}$ satisfies the lemma hypothesis for some prime divisor s of $\overline{K}$. Therefore, the conclusion of lemma holds for some subgroups $\overline{H}$ and $\overline{M}$ of $\overline{G}$. The preimages H and M of these subgroups in G are as required. □

Lemma 3.4. *Let G be a group, $A \leq \mathrm{Aut}(G)$, and let N be an A-invariant normal subgroup of G with $(|A|, |N|) = 1$. Then $C_{G/N}(A)$ is the image of $C_G(A)$ in G/N.*

Proof. Apply [14, Corollary 3.28] and the Feit–Thompson theorem. □

The following lemma shows that the simple linear groups $L_n(2)$ are saturated with various Frobenius subgroups.

Lemma 3.5. *The group $L_n(2)$, $n \geq 2$, includes a Frobenius subgroups with kernel of order $2^n - 1$ and cyclic complement of order n, as well as a Frobenius subgroup with kernel of order 2^k and cyclic complement of order $2^k - 1$ for every $2 \leq k \leq n - 1$.*

Proof. Both assertions are well known, see, e.g., [23, Lemma 5] and [10, Lemma 2.5]. □

The last two lemmas show how one can use Frobenius subgroups to deal with spectra of group extensions.

Lemma 3.6. [17, Lemma 1] *Let P be a normal p-subgroup of a finite group G and let G/P be a Frobenius group with kernel F and cyclic complement C. If p does not divide F and $F \not\subseteq PC_G(P)/P$, then G contains an element of order $p|C|$.*

Lemma 3.7. *Suppose that K is a normal subgroup of G and $G/K \simeq S_1 \times S_2 \times \ldots \times S_m$, where S_i are nonabelian simple groups. Suppose that each S_i includes a Frobenius subgroup X_i which kernel is a p_i-group for some prime p_i and complement is of prime order $s_i \notin \pi(K)$, where s_i is not a Fermat prime. If $r \in \pi(K)$ and $r \notin \{p_1, \ldots, p_m\}$, then $rs_1 \cdots s_m \in \omega(G)$.*

Proof. By Lemma 3.3, we may assume that K is a product of its Sylow subgroups $T_1, \ldots, T_k$ such that T_1 is a r-group and $T_j \leq N_K(T_i)$ for every i, j with $1 \leq i < j \leq k$. Factoring G and K by the Frattini subgroup $\Phi(T_1)$, we arrive at a situation where T_1 is elementary abelian. If $T_1 \cap Z(G) \neq 1$, then there is nothing to prove, so T_1 acts faithfully by conjugation on G and can be considered as a subgroup of $\mathrm{Aut}(G)$.

We proceed by induction on m and begin with $m = 1$. If $C_G(T_1)K/K \neq 1$, then $C_G(T_1)K/K = S_1$ and, clearly, there is an element in G whose order equals rs_1. Therefore, $C_G(T_1) \leq K$ and there exists a Hall r'-subgroup J of $C_G(T_1)$. Then $C_G(T_1) = T_1 \times J$ and J is normal in G, because $C_G(T_1)$ is a normal in G. By Lemma 3.4, the images of $C_G(T_1)$ and T_1 in G/J coincide. Hence $C_G(T_1) = T_1$.

Recall that $s_1 \notin \pi(K)$ and at most one of T_i is a p_1-group. Therefore, applying consequently the Schur-Zassenhaus theorem to the preimages of X_1 in factors $G/(T_1 \cdots T_i)$ for $i = k, k-1, \ldots, 1$ we obtain that there exists a subgroup $X = Y \rtimes \langle g \rangle$ of G/T_1, where Y is a p_1-group, $|g| = s_1$ and X_1 is an image of X in G/K. The subgroup $[Y, g]$ is the preimage of a Frobenius kernel of X_1, so $[Y, g] \neq 1$. The action of $[Y, g]$ on T_1 is faithful, because

$C_G(T_1) = T_1$. It follows now from the cross-characteristic analogue of the Hall–Higman theorem (see, e.g., [22, Lemma 3.6]) that $C_{T_1}(g) \neq 1$, so $rs_1 \in \omega(G)$.

Let $m \geq 2$. By the above arguments, there exists an element $g \in G$ of order s_1 such that $C_{T_1}(g) \neq 1$ and the image g_1 of g in G/K lies in S_1. Lemma 3.4 implies that $C_G(g)K/K = C_{G/K}(g_1)$. Hence $C_G(g)/(K \cap C_G(g))$ being isomorphic to $C_G(g)K/K$ includes a subgroup S isomorphic to $S_2 \times \cdots \times S_m$. Applying the inductive hypothesis to the preimage of S in $C_G(g)$, we obtain an element of order $rs_2 \cdots s_m$ in $C_G(g)$. Thus, $rs_1s_2 \cdots s_m \in \omega(G)$, as required. □

4 Proof of Theorem 1

According to the hypothesis of Theorem 1, $L = L_n(2)$, where $n = 2^l \geq 56k^2$, and G is a finite group isospectral to $P = L^k$. By [23], we may assume that $k \geq 2$ and thereby $n \geq 2^8$. We fix a chief series $G_0 = 1 \lhd G_1 \lhd \ldots \lhd G_s = G$ of G, where $A_i = G_i/G_{i-1}$ is a nontrivial minimal normal subgroup of G/G_{i-1} for $1 \leq i \leq s$.

Primitive prime divisors $r_i(2)$ for $2 \leq i \leq n$ and $i \neq 6$ are denoted by r_i. Put $\Omega = \{r_{n/2+1}, r_{n/2+2}, \ldots, r_n\}$ and observe that Ω is a coclique of size $n/2$ in the prime graph $\Gamma(L)$ due to Lemma 2.10. It is also clear from the definition of r_i and Fermat's little theorem that $r > n/2 \geq 2^7$ for every $r \in \Omega$.

Lemma 4.1. *There exists a normal subgroup N of G such that $\overline{G} = G/N$ includes a normal subgroup K satisfying the following conditions:*

(i) $C_{\overline{G}}(K) = 1$;

(ii) *$K = S_1 \times S_2 \times \ldots \times S_m$, where $m \leq k$ and each S_i is a nonabelain simple group with $|\pi(S_i) \cap \Omega| > k$;*

(iii) *there exists $\Delta \subseteq \Omega$ such that $|\Delta| \geq 3n/8+8k$ and each $p \in \Delta$ is coprime to $|N| \cdot |\overline{G}/K|$.*

Proof. Suppose that for every chief factor A_i of G the set $\pi(A_i) \cap \Omega$ has at most k elements. By Lemma 2.1(ii), $|\Omega| = n/2 \geq 28k^2 > k \cdot \rho(k)$. Therefore, one can choose at least $|\Omega|/k > \rho(k)$ distinct primes in Ω dividing orders of different chief factors of G. Lemma 3.2 implies that G includes a solvable subgroup H of order divisible by each of these primes. It follows from Lemma 2.2 that $|\pi(h)| \leq k$ for every $h \in H$. Since $|\pi(H)| > \rho(k)$, we arrive at a contradiction with the definition of $\rho(k)$.

Let N be a normal subgroup of G of the largest possible order such that $|\pi(A) \cap \Omega| \leq k$ for every chief factor A of N. By above, $\overline{G} = G/N \neq 1$. We claim that N and $K = \mathrm{Soc}(\overline{G})$ satisfy the conclusion of the theorem.

Firstly, $K \simeq M_1 \times \ldots \times M_t$, where M_j are the minimal normal subgroups of $\overline{G}$. By the choice of N, we have $|\pi(M_j) \cap \Omega| > k \geq 2$ for each $j \in \{1, \ldots, t\}$. Since every subgroup M_j is characteristically simple in G, it must be a direct product of groups isomorphic to some nonabelian simple group R_j. In particular, $K \cap C_{\overline{G}}(K) = 1$. On the other hand, K includes all minimal normal subgroups of $\overline{G}$ and hence $C_{\overline{G}}(K) = 1$, which proves (i).

Since M_j is a direct power of R_j, it follows that $\pi(R_j) = \pi(M_j)$ for each $j \in \{1, \ldots, t\}$. Therefore, $|\pi(S_i) \cap \Omega| > k$ for every composition factor S_i of $K = S_1 \times \ldots \times S_m$. Lemma 2.2 yields $m \leq k$, so (ii) follows.

We claim that $|\pi(N) \cap \Omega| \leq k \cdot \rho(k)$. Assume the opposite and consider chief factors $T_1, \ldots, T_y$ of N such that $\pi(N) \cap \Omega \subseteq \pi(T_1) \cup \ldots \cup \pi(T_y)$ and y is minimal. Since $|\pi(T_i) \cap \Omega| \leq k$

for each factor T_i, $i = 1, \ldots, y$, it follows that there are more than $\rho(k)$ factors T_i having pairwise distinct primes from Ω. Lemma 3.2 implies that N includes a solvable subgroup M with $|\pi(M) \cap \Omega| > \rho(k)$. Therefore, there exists $g \in M$ such that $|\pi(g) \cap \Omega| > k$, which contradicts Lemma 2.2.

Now we prove that at most k primes from Ω divide $|\overline{G}/K|$. Since $C_{\overline{G}}(K) = 1$, Lemma 3.1 implies that $\overline{G}/K$ is isomorphic to a subgroup of $(\mathrm{Out}(S_1) \times \ldots \times \mathrm{Out}(S_m)) \cdot Sym_m$. Since $r > n/2 > k \geq m$ for every $r \in \Omega$, it follows that

$$\pi(\overline{G}/K) \cap \Omega \subseteq \bigcup_{i=1}^{m} \pi(\mathrm{Out}(S_i)).$$

Thus, it suffices to prove that $|\pi(S_i) \cap \Omega| \leq 1$ for all factors S_i. If not, then without loss of generality we may suppose that r and r' are two distinct primes from $\pi(\mathrm{Out}(S_1)) \cap \Omega$. The simple group S_1 must be a group of Lie type, because $2 \notin \Omega$. Let the underlying field of S_1 be of order $u = v^d$, where v is a prime.

If rr' divides d, then S_1 contains an element of order greater or equal to

$$(u-1)/2 \geq (2^{rr'} - 1)/2 > (2^{n^2/4} - 1)/2 > 2^n,$$

which contradicts Lemma 2.10(iii). Therefore, only one of the primes r and r' can divide the order of the field automorphism of S_1. It follows that S_1 is a linear or unitary group and at least one of r and r' divides the dimension n_1 of S_1. By Lemma 2.7, the group S_1 includes an element of order at least $u^{n_1-2} - 1$. If rr' divides n_1, then $u^{n_1-2} - 1 \geq (2^{rr'-2} - 1) > 2^n$. If r' divides n_1 and r does not, then r divides d and $u^{n_1-2} - 1 \geq (2^{r(r'-2)} - 1) > 2^n$. In the both cases, we again arrive at a contradiction to Lemma 2.10(iii).

Put $\Delta = \Omega \setminus (\pi(N) \cup \pi(\overline{G}/K))$. Then $|\Delta| \geq |\Omega| - k\rho(k) - k = n/2 - k(\rho(k) + 1)$. Lemma 2.1(iii) yields $7k \geq \rho(k) + 9$, so $n/8 \geq 7k^2 \geq k(\rho(k) + 9)$. As easily seen, the latter inequality is equivalent to $n/2 - k(\rho(k)+1) \geq 3n/8 + 8k$. This implies that $|\Delta| \geq 3n/8 + 8k$, as required. □

Now we fix the subgroup N of G, subgroups $K, S_1, \ldots, S_m$ of $\overline{G} = G/N$, and subset Δ of Ω as in Lemma 4.1, and denote $\Delta_i = \pi(S_i) \cap \Delta$ for $i = 1, \ldots, m$.

Lemma 4.2. *For any k distinct primes $p_1, \ldots, p_k$ from Δ, there is an element $g \in K$ of order $p_1 \cdots p_k$. For every such element g and every $i = 1, \ldots, m$, $\pi(g) \cap \pi(S_i) \neq \varnothing$.*

Proof. The first statement of the lemma holds, because $p_1 \cdots p_k \in \omega(G) = \omega(P)$ and each p_i does not divide $|N| \cdot |\overline{G}/K|$ in view of Lemma 4.1(iii).

If $\pi(g) \cap \pi(S_i) = \varnothing$ for some $i \in \{1, \ldots, m\}$, then g centralizes S_i and one can take an element $h \in S_i$ of order $r \in \pi(S_i) \cap \Omega \setminus \{p_1, \ldots, p_k\}$. It follows that $rp_1 \cdots p_k \in \omega(K) \setminus \omega(G)$; a contradiction. □

Lemma 4.3. *For every $i = 1, \ldots, m$, $|\Delta_i| > 3n/8 + 7k$.*

Proof. If there exist k distinct primes $p_1, \ldots, p_k$ in $\Delta \setminus \Delta_i$, then one can take an element $g \in K$ of order $p_1 \cdots p_k$ according to the first statement of Lemma 4.2. However, this contradicts the second statement of the same lemma. Thus, $|\Delta \setminus \Delta_i| < k$, so $|\Delta_i| > |\Delta| - k \geq 3n/8 + 7k$. □

Recall that $t(G)$ denotes the maximal size of a coclique in the prime graph of a group G.

Lemma 4.4. *For every $i = 1, \ldots, m$, either $t(S_i) > 3n/8 + 6k$ or $|\pi(g) \cap \pi(S_i)| \geq 2$ for every $g \in K$ with $|\pi(g)| = k$ and $\pi(g) \subseteq \Delta$.*

Proof. Suppose that for some $i \in \{1, \ldots, m\}$ there exists $g \in K$ with $|\pi(g)| = k$, $\pi(g) \subseteq \Delta$, and $|\pi(g) \cap \pi(S_i)| \leq 1$. We claim that every two distinct primes in $\Delta_i \setminus \pi(g)$ are nonadjacent in $\Gamma(S_i)$. If this is not the case, then there is an element h of order $rs \in \pi(S_i)$, where r and s are distinct primes in Δ_i and at most one of them divides the order of g. If $g = g_1 \cdot \ldots \cdot g_m$, where $g_j \in S_j$ for $1 \leq j \leq m$, and $g' = g_i^{-1}g$, then $g' \in C_K(h)$, so $|\pi(g'h) \cap \Delta| > k$, which is impossible by Lemma 2.2. Therefore, $t(S_i) \geq |\Delta_i| - k > 3n/8 + 6k$ by Lemma 4.3. □

Lemma 4.5. *Suppose that $g, h \in K$ such that $\pi(g) \cap \pi(h) = \varnothing$, $|\pi(g)| = |\pi(h)| = k$, and $\pi(g), \pi(h) \subseteq \Delta$. Then $|\pi(g) \cap \pi(S_i)| = |\pi(h) \cap \pi(S_i)|$ for every $i = 1, \ldots, m$.*

Proof. Write $h = h_1 \cdots h_m$ and $g = g_1 \cdots g_m$, where $h_i, g_i \in S_i$ for $i = 1, \ldots, m$. Assume to the contrary that there exists $i \in \{1, \ldots, m\}$ such that $|\pi(h_i)| > |\pi(g_i)|$. Since $\pi(g) \cap \pi(h) = \varnothing$, it follows that $|\pi(g')| > |\pi(g)| = k$ for $g' = g_1 \cdots g_{i-1}h_ig_{i+1} \cdots g_m$ and $\pi(g') \subseteq \Delta$, which contradicts Lemma 2.2. □

Lemma 4.6. *For every $i = 1, \ldots, m$, the factor S_i is not a sporadic group.*

Proof. By Lemma 4.3, $|\Delta_i| \geq 1$ for each $i = 1, \ldots, m$. If $r \in \Delta_i$, then $r > n/2 \geq 128$. However, it is well known, see, e.g., [4], that the prime divisors of the orders of the sporadic groups are less than 100, a contradiction. □

Lemma 4.7. *For every $i = 1, \ldots, m$, the factor S_i is not an alternating group.*

Proof. Suppose that one of S_i is an alternating group of degree d, for definiteness, $S_1 \simeq Alt_d$. To arrive at a contradiction it suffices to show that $d \geq 2n + 2$, because in this case $2^{l+1} = 2n \in \omega(S_i) \subseteq \omega(G)$, which is impossible by Lemma 2.10(ii).

First, we suppose that $t(S_1) > 3n/8$. Then, see, e.g., [25, Proposition 1.1],

$$3n/8 + 1 \leq t(S_1) \leq 1 + |\{x \leq p \leq 2x \mid p \text{ is prime}\}|,$$

where $x = [(d + 1)/2]$. Since $n \geq 256$, it follows that $x > 59$ and Lemma 2.14 yields $|\{x \leq p \leq 2x \mid p \text{ is prime}\}| \leq x + 1 - ([(2x)/3] + 3) \leq x/3 - 4/3$. Hence $3n/8 \leq x/3 - 4/3$. Then $x \geq n + 4$ and, consequently, $d \geq 2n + 2$, a contradiction.

Suppose now that $t(S_1) \leq 3n/8$. There exists an element $g \in K$ such that $|\pi(g)| = k$, $\pi(g) \subseteq \Delta$ and, by Lemma 4.4, $|\pi(g) \cap \pi(S_1)| \geq 2$. Denote $\Omega' = \{r_i \in \Omega \mid i+1 \text{ is composite}\}$. By little Fermat's theorem i divides $r_i - 1$ and, if $i + 1$ is composite, then $i < r_i - 1$, so $r_i - 1 \geq 2i$. It follows that $r > 2n + 1$ for every $r \in \Omega'$.

By Lemma 2.14, the set $\{n/2 + 2, \ldots, 2 \cdot (n/2 + 2)\}$ contains at least $[n/3 + 4/3] + 3$ composite numbers, so $|\Omega'| \geq n/3$. Therefore,

$$|\Omega' \cap \Delta_1| = |\Omega'| + |\Delta_1| - |\Omega' \cup \Delta_1| \geq |\Omega'| + |\Delta_1| - |\Omega| \geq n/3 + 3n/8 + 7k + 1 - n/2 \geq 2k.$$

It follows that there are at least k primes in $(\Omega' \cap \Delta_1) \setminus \pi(g)$. Take an element $h = h_1 \ldots h_m$ with $h_i \in S_i$, $i = 1, \ldots, m$, such that $\pi(h)$ consists of these k primes. Lemma 4.5 yields $|\pi(h_1)| \geq 2$, so there are at least two primes greater than $n + 1$ and adjacent in $\Gamma(S_1)$. It is possible only if $d \geq 2n + 2$, which leads to a final contradiction. □

Lemma 4.8. *For every $i = 1, \ldots, m$, the factor S_i is not an exceptional group of Lie type.*

Proof. In this lemma, we use well-known information on the orders of simple exceptional groups of Lie type and their maximal tori, see, e.g., [4, Table 6], [24, Lemma 1.3], and [25, Lemma 2.6]. Assume that S_1 is an exceptional group of Lie type over a field of order u

and characteristic v. If $S_1 \in \{E_8(u), E_7(u), E_6(u), {}^2E_6(u), F_4(u)\}$, then each prime $r \in \pi(S)$ either is equal to v or belongs to the set $R_j(u)$ of primitive prime divisors of $u^j - 1$ for some integer j, where in each of these cases, the number of possible indices j is at most 18. For each $R_j(u)$, there exists a maximal torus T_j of S_i such that $R_j(u) \subseteq \pi(T_j)$. If $S_1 \notin \{E_8(u), E_7(u), E_6(u), {}^2E_6(u), F_4(u)\}$, then S_1 includes (up to conjugation) at most 13 maximal tori. It is clear that the order of each of these maximal tori is divisible by at most k primes from Δ. Therefore, $|\Delta_1| \leq 18k + 1$ in all cases. Lemma 4.3 implies that $3n/8 + 7k < 18k + 1$, which is impossible since $n \geq 56k^2$. □

Lemma 4.9. *The equality $m = k$ holds.*

Proof. Lemma 4.1(ii) yields $m \leq k$. Assume that $m < k$. Take an element $g \in K$ such that $|\pi(g)| = k$ and $\pi(g) \subseteq \Delta$. Then without loss of generality, we may assume that $|\pi(g) \cap \pi(S_1)| \geq 2$ and put $n_1 = \operatorname{prk} S_1$.

By Lemmas 4.6-4.8, each group S_i, $i = 1, \ldots, m$, is a classical group of Lie type. Let S_1 be a classical group over a field of order u and characteristic v. First we consider the case when v is odd. If $n_1 \leq 3$, then S_1 includes at most 13 maximal tori (up to conjugation) (see, e.g., [24, Lemma 1.2]), and we arrive at a contradiction as in Lemma 4.8. Hence $n_1 \geq 4$.

If $S_1 \simeq L^-_{n_1}(u)$ or $S_1 \simeq O^-_{n_1}(u)$, then put $\varepsilon = -$, otherwise put $\varepsilon = +$. Each $r \in \Delta_i$ either is equal to v or belongs to $R_j(\varepsilon u)$ for some integer $j \in \{1, \ldots, n_1\}$. Put $\Theta = \{e(r, \varepsilon u) \mid r \in \Delta_i, r \neq v\}$. By [22, Lemma 2.13] (see also Remark 2 after it), S_1 has an element of order $k_j(\varepsilon u)$ for each $j \in \Theta$. Therefore, $|R_j(\varepsilon u)| \leq k$. If $|\Delta_1| - |\Theta| \geq 2k + 1$, then one can find k primes $p_1, p_2, \ldots, p_k$ in $\Delta_1 \setminus \{v\}$ with $e(p_j, \varepsilon u) > 2$. If, additionally, $|\Delta_i| - |\Theta| \geq 3k + 1$, then for every p_j, $j = 1, \ldots, k$, there is $p'_j \in \Delta_i \setminus \{p_1, \ldots, p_k\}$ with $e(p_j, \varepsilon u) = e(p'_j, \varepsilon u)$. Take $h \in K$ such that $|h| = p_1 p_2 \cdots p_k$ and put $h = h_1 h_2$, where $\pi(h_1) = \pi(h) \cap \pi(S_1)$ and $\pi(h_2) \cap \pi(S_1) = \varnothing$. Note that $h_1 \neq 1$ by Lemma 4.2, so we may assume that $\pi(h_1) = \{p_1, \ldots, p_s\}$, where $1 \leq s \leq k$. By the choice of p_j, $e(p_j, \varepsilon u) > 2$ for every $1 \leq j \leq k$, so Lemma 2.6 implies that there is $h'_1 \in S_1$ of order $p'_1 p_1 \cdots p_s$. Since h_2 centralizes S_1, we obtain the element $h'_1 h_2 \in K$ of order $p'_1 p_1 \ldots p_k$; a contradiction. Therefore, $|\Delta_1| - |\Theta| \leq 3k$ and hence $|\Theta| \geq |\Delta_1| - 3k > 3n/8 + 4k$ due to Lemma 4.3.

Suppose that there are at least $2k$ elements in Θ greater than $n_1/2$. Then there exists $h \in K$ such that $|h| = p_1 \cdots p_k$, where $p_1, \ldots, p_k$ do not not belong to $\pi(g)$, $e(p_i, \varepsilon u) > n_1/2$ for $i = 1, \ldots, k$, and $e(p_i, \varepsilon u) \neq e(p_j, \varepsilon u)$ for $i \neq j$. Lemma 4.5 implies that $|\pi(h) \cap \pi(S_1)| \geq 2$, which contradicts Lemma 2.5(iii). If S_1 is a linear or unitary group, then $\Theta \subseteq \{1, \ldots, n_1\}$ and hence $|\Theta| \leq n_1/2 + 2k - 1$. Assume that S_1 is a symplectic or orthogonal group. As above, we see that the set $\{j \in \Theta \mid n_1/2 < \eta(j)\}|$ has at most $2k - 1$ elements. By Lemma 2.15, the set $\{j \in \Theta \mid \eta(j) \leq n_1/2\}$ has at least $(3n_1)/4 - 3/2$ elements and the set $\{j \in \Theta \mid \eta(j) \leq n_1\}$ has at most $(3n_1)/2 + 1/2$ elements. Hence $|\Theta| \leq 2k - 1 + (3n_1)/2 + 1/2 - (3n_1)/4 + 3/2 = 2k + (3n_1)/4 + 1$.

Therefore, $(3n_1)/4 + 2k + 1 \geq |\Theta| \geq 3n/8 + 4k + 1$. It follows that $n_1 \geq n/2 + 8k/3 > n/2 + 2$. Lemma 2.9 yields $2^{l+1} \in \omega(S_1)$; a contradiction with Lemma 2.10(ii).

Thus, S_1 is a group of even characteristic, i.e., $u = 2^f$ for some integer f. Then $R_{n_1 f}(2) \subseteq \pi(S_1)$ or $R_{2n_1 f}(2) \subseteq \pi(S_1)$. It follows that $n_1 f \leq n$. Consider the set $\Omega' = \{r_i \in \Omega \mid i \text{ is odd}\}$ of size $n/4$. Since $|\Omega' \cap \Delta| \geq n/4 + |\Delta| - n/2 = |\Delta| - n/4$ and $|\Delta| \geq 3n/8 + 8k + 1$, we have $|\Omega' \cap \Delta| > 2k$. Hence, there exist k distinct primes from $(\Omega' \cap \Delta) \setminus \pi(g)$ and an element h in K whose order equal to their product. It follows from Lemma 4.5 that $\pi(h) \cap \pi(S_1) \geq 2$.

Therefore, there are two distinct primes $p_1, p_2 \in \Omega' \cap \Delta \cap \pi(S_1)$ such that $p_1 p_2 \in \omega(S_1)$. By the choice of these primes, $j_1 = e(p_1, 2) \neq j_2 = e(p_2, 2)$. Put also $e_1 = \varphi(p_1, S_1)$ and $e_2 = \varphi(p_2, S_1)$. Since $p_1, p_2 \in \Omega'$, the numbers j_1 and j_2 are odd. Note that p_1 divides

$2^{2e_1f}-1$. Hence j_1 divides $2e_1f$, so j_1 divides e_1f. If $e_1f \geq 2j_1 > n$, then $n_1f > n$; a contradiction. It follows that $e_1f = j_1$, so $e_1f = j_1 > n/2 \geq n_1f/2$ and hence $e_1 > n_1/2$. Similarly, $j_2 = e_2f$ and $e_2 > n_1/2$. Since $p_1p_2 \in \omega(S_1)$, Lemma 2.5 yields $e_1 = e_2$. Then $j_1 = j_2$, which contradicts the choice of p_1 and p_2, thus completing the proof. □

Lemma 4.10. *For every $i = 1, \dots, m$, the factor S_i is a linear group over a field of order 2 and $7n/8 + 7k \leq \operatorname{prk}(S_i) \leq n$.*

Proof. By Lemma 4.2, there is an element $g \in K$ such that $|\pi(g)| = k$, $\pi(g) \subseteq \Delta$. For every $i = 1, \dots, m$ and any such element g, the same lemma yields $|\pi(g) \cap \pi(S_i)| \neq \varnothing$. Since $m = k$ in view of Lemma 4.9, it follows that $|\pi(g) \cap \pi(S_i)| = 1$ and, by Lemma 4.4, the inequality $t(S_i) \geq 3n/8 + 6k$ holds.

Consider one of the groups S_i, say S_1. Denote by u the order of the underlying field of S_1 and put $n_1 = \operatorname{prk} S_1$. It follows from [25, Tables 2, 3] that $t(S_1) \leq (3n_1 + 5)/4$. Therefore, $(3n_1 + 5)/4 \geq 3n/8 + 6k$ and hence $n_1 > n/2 + 2$.

If u is odd, then we immediately arrive at a contradiction in view of Lemma 2.9(ii). Assume now that $u = 2^f$. If S_1 is symplectic or orthogonal group, then $R_{2t}(u) \subseteq \pi(S_1)$ for every $t < n_1$. Since $n_1 > n/2 + 2$, it follows that $R_{n+2}(u) \subseteq \pi(S_1)$, so $R_{f(n+2)}(2) \subseteq \pi(S_1)$, a contradiction. If S_1 is an unitary group, then $\pi(S_1)$ includes either $R_{2(n_1-1)}(u)$ or $R_{2n_1}(u)$. Hence $R_{2jf}(2) \subseteq \pi(S_1)$ for some $j \in \{n_1 - 1, n_1\}$. Since $2j > n$, we arrive at a contradiction. If S_1 is a linear group, then $R_{n_1f}(2) \subseteq R_{n_1}(u) \subseteq \pi(S_1)$. Since $2n_1 > n$, it follows that $u = 2$, as required.

Clearly, $\operatorname{prk}(S_1) \leq n$. Since $|\Omega| = n/2$ and $|\Delta_1| > 3n/8 + 7k$ in view of Lemma 4.3, there are at most $n/8 - 7k - 1$ integers a between $n/2 + 1$ and n such that $e(s, 2) \neq a$ for some $s \in \pi(S_1)$. Therefore, $\operatorname{prk}(S_i) \geq 7n/8 + 7k$, and we are done. □

Lemma 4.11. *If $r \in \pi(N) \cup \pi(\overline{G}/K)$, then $e(r, 2) < n/3$. In particular, $\Delta = \Omega$.*

Proof. Suppose that $r \in \pi(\overline{G}/K)$. Since $C_{\overline{G}}(K) = 1$, we infer that $\overline{G}$ embeds into $\operatorname{Aut}(K)$. In view of Lemma 3.1, this implies that $r \in \pi(\operatorname{Out}(L_a(2)))$ or $r \in \pi(Sym_b)$, where $b \leq k$. Since $|\operatorname{Out}(L_a(2))| = 2$, it follows that $e(r, 2) < r \leq b \leq k < n/3$.

Suppose now that $r \in \pi(N)$ and $e = e(r, 2) > n/3$. We arrive at a contradiction by showing that there is an element in G whose order is a product of $k + 1$ primes pairwise nonadjacent in $\Gamma(L)$. Clearly, e divides at most one integer between $2n/3$ and n. Observe also that every $s \in \Omega \setminus \{r_n\}$ is not a Fermat's prime, because $e(s, 2)$ is not a power of 2. For $i = 1, \dots, m$, set $n_i = \operatorname{prk}(S_i)$ and $\Delta_i' = \Delta_i \setminus \{r\}$.

Define $\Theta = \{x \in \mathbb{N} \mid 2n/3 < x < 7n/8 + 7k\}$. Direct calculation shows that for every $i = 1, \dots, m$, there are at least $n/12 + 14k$ primes s in Δ_i' such that $e(s, 2) \in \Theta$. It follows that one can choose m distinct primes $s_1, \dots, s_m$ such that $s_i \in \Delta_i'$ and $e(s_i, 2) \in \Theta$ is not a multiple of e for every $i = 1, \dots, m$. Lemma 4.10 yields $e(s_i, 2) < 7n/9 + 7k \leq n_i$ for every $i = 1, \dots, m$, so each S_i includes a Frobenius subgroup whose kernel is a 2-group and complement has order s_i by Lemma 3.5. It follows from Lemma 3.7 that there is an element $x \in G$ of order $rs_1 \cdots s_m$. All the s_i in this product are nonadjacent in $\Gamma(L)$ with each other because they are different primes from Ω, and with r because $e + e(s_i, 2) > n$ and e does not divide $e(s_i, 2)$. It remains to note that $|\pi(x)| = k + 1$ since $m = k$ in view of Lemma 4.9. □

Lemma 4.12. *For every $i = 1, \dots, m$, the factor S_i is isomorphic to L.*

Proof. Since $n \geq 56k^2 > 18(k + 1)$, we may consider a set $\Psi(2)$ as in Lemma 2.12. Lemma 2.13(i) yields

$$\Pi(2) = \prod_{j \in \Psi(i)} r_j(2) \in \omega(P) = \omega(G).$$

Moreover, $\Pi(2) \in \omega(K)$ by Lemma 4.11. In view of Lemma 4.10, we have $S_i \simeq L_{n_i}(2)$, where $4 \leq n_i \leq n$ for every $i = 1, \dots, m$. It follows from Lemma 2.13(ii) that $n_i = n$ for each i, as required. $\square$

Lemma 4.13. *$G/N = K$.*

Proof. Suppose that a prime r divides $|\overline{G}/K|$. Take an element $g \in \overline{G}/K$ or order r. Suppose that $S_i^g = S_i$ for every $i \in \{1, \dots, k\}$. Since $C_{\overline{G}}(K) = 1$, we may assume that $[S_1, g] \neq 1$. Then g acts on S_1 as a graph automorphism, so $r = 2$ and $S_1\langle g \rangle \simeq \mathrm{Aut}(L)$. It follows from Lemma 2.11 that $2^{l+1} \in \omega(\mathrm{Aut}(L)) \subseteq \omega(G)$, which contradicts Lemma 2.10(ii).

Thus, $S_i^g \neq S_i$ for some i. Then one can assume, up to reordering, that g permutes factors in the product $S_1 \times S_2 \times \ldots \times S_r$. Denote by x an element of L whose order is r^k, where k is maximal possible. If $h = (x, 1, \dots, 1) \in S_1 \times S_2 \times \ldots \times S_r$, then $|hg| = r^{k+1}$. Since $r^{k+1} \notin \omega(P)$, we get a contradiction. $\square$

Lemma 4.14. *$N = 1$.*

Proof. Suppose that $N \neq 1$. By Lemma 3.3, we may assume that N is solvable. Hence, arguing by induction on $|G|$, we may also assume that N is an elementary abelian p-group for some $p \in \pi(G)$.

Suppose that $C_G(N) = G$. If $p = r_i$, then $i \geq 2$ and we take $x = \Pi(i)$, where $\Pi(i)$ is as in Lemma 2.13. If $p = 2$, then take $x = \Pi(2)$. It follows from Lemma 2.13(iii) that $px \in \omega(G) \setminus \omega(L^k)$; a contradiction. Therefore, without loss of generality, we may assume that $S_1 \cap C_G(N)/N = 1$.

Assume that p is odd. Let t be an integer such that $p^t \in \omega(L)$ and $p^{t+1} \notin \omega(L)$. Lemma 2.7 implies that p^t divides $2^j - 1$ for some $j \in \{1, \dots, n\}$. Then $i = e(p, 2)$ divides j. Suppose that $j = n$. Lemma 4.11 yields $i < n/3$. Since n/i is a power of 2 and coprime to p, it follows that $2^i - 1$ is divisible by p^t (see, e.g. [22, Lemma 1.7]). So we can assume that $1 \leq j \leq n-1$. Lemma 3.5 implies that S_1 includes a Frobenius subgroup with kernel of order 2^j and cyclic complement of order $2^j - 1$. It follows from Lemma 3.6 that $p^{t+1} \in \omega(G)$; a contradiction.

If $p = 2$, then Lemma 3.5 implies that S_1 includes a Frobenius group with kernel of order $2^n - 1$ and complement of order $n = 2^l$. It follows from Lemma 3.6 that $2^{l+1} \in \omega(G)$, which contradicts Lemma 2.10. $\square$

5 Proof of Theorem 2

In this short section we prove Theorem 2 and Proposition 5.3. If Δ is a nonempty set of integers, then $\mu(\Delta)$ stands for the set of all maximal elements of Δ with respect to divisibility. Given a finite group G, put $\mu(G) = \mu(\omega(G))$. Note that $\omega(G)$ consists of all the divisors of $\mu(G)$ and so is completely determined by it. It is easy to see that if G and H are finite groups, then

$$\mu(G \times H) = \mu(\{[a, b] \mid a \in \mu(G), b \in \mu(H)\}). \tag{1}$$

Suppose that G is a finite group and $r \in \pi(G)$. If a is the unique element of $\mu(G)$ coprime to r, then it follows from (1) that r divides all the elements of $\omega(G \times G)$, so $\omega(G \times G) = \omega(G \times G \times \mathbb{Z}_r^m)$ for every positive integer m. Similarly, if there are only two distinct integers a and b from $\mu(G)$ both coprime to r, then $[a, b]$ is the unique element of $\mu(G \times G)$ coprime to r. Therefore, r divides all the elements of $\mu(G^3)$ and, consequently, $\omega(G^3) = \omega(G^3 \times \mathbb{Z}_r^m)$ for every m. Thus, Theorem 2 follows from the next two lemmas.

Lemma 5.1. *Let $L = L_n^\varepsilon(q)$, $\varepsilon \in \{+,-\}$, and q a power of a prime p. Suppose that there exists $r \in \pi(q-\varepsilon 1) \setminus \pi(n)$. Then there are at most two elements of $\mu(L)$ coprime to r. Moreover, if $n-1$ is not a power of p, then there is only one such element.*

Proof. Since r does not divide $d = (n, q-\varepsilon 1)$, the only integers from Lemma 2.7 coprime to r are $\frac{q^n-(\varepsilon 1)^n}{d(q-\varepsilon 1)}$ and p^k. Furthermore, $p^k \in \mu(L)$ only if $n = p^{k-1}+1$, and we are done. □

Lemma 5.2. *Suppose that $L = S_{2n}(q)$, where q is a power of an odd prime p. Then there are at most two elements of $\mu(L)$ coprime to r. Moreover, if $2n-1$ is not a power of p, then there is only one such element.*

Proof. We argue as in the previous lemma applying Lemma 2.8 instead of Lemma 2.7. □

If $n = 2^l \geq 2$, then n is even, so $k \geq [(n+3)/2] = n/2+1$ yields $n < 2k$. Therefore, the following proposition establishes the lower bound on n_0 from Remark 1 after Theorem 1 in Introduction.

Proposition 5.3. *If $L = L_n(2)$ and $k_0 = [\frac{n+3}{2}]$, then $\omega(L^k) = \omega(L^{k_0})$ for every $k \geq k_0$.*

Proof. Since $L_n(2) = 1$ for $n = 1$, we may assume that $n \geq 2$. Denote by 2^e the 2-exponent of L, put $t = [(n+2)/2]$ and denote by N the least common multiple of $k_0 = [(n+3)/2]$ integers $2^e, 2^t-1, 2^{t+1}-1, \dots, 2^n-1$. Applying Lemma 2.7, we conclude that N is the exponent of L. It follows from (1) by induction on k that $\mu(L^k) = \mu\{[a_1, \dots, a_k] \mid a_i \in \mu(L)\}$. Thus, $\mu(L^{k_0}) = \{N\}$ is a singleton, so the proposition follows. □

As observed in Introduction, the exact value of n_0 is known only for $k = 1$. For k equal to 2 and 3, we have $n_0(2) \geq 8$ and $n_0(3) \geq 16$, because $\omega(L_4(2)^2) = \omega(L_4(2) \times L_3(2))$ and $\omega(L_8(2)^3) = \omega(L_8(2)^2 \times L_7(2))$, respectively. It would be also interesting to know the asymptotic behavior of n_0 as k tends to infinity.

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