ON COMPUTING THE CLOSURES OF SOLVABLE PERMUTATION GROUPS

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Abstract. Let $m \geq 3$ be an integer. It is proved that the m-closure of a given solvable permutation group of degree n can be constructed in time $n^{O(m)}$.

Keywords. Permutation group, closure, polynomial-time algorithm.

1. Introduction

Let m be a positive integer and let Ω be a finite set. The *m-closure* $G^{(m)}$ of $G \leq \mathrm{Sym}(\Omega)$ is the largest permutation group on Ω having the same orbits as G in its induced action on the cartesian power Ω^m. The m-closure of a permutation group was introduced by H. Wielandt in [22], where it was, in particular, proved that $G^{(m)}$ can be treated as the full automorphism group of the set of all m-ary relations invariant with respect to G. Since then the theory was developed in different directions, e.g., there were studied the closures of primitive groups [13, 17, 25], the behavior of the closure with respect to permutation group operations [7, 10, 20], totally closed abstract groups [1, 2, 8], etc.

From the computational point of view, the m-closure problem consisting in finding the m-closure of a given permutation group is of special interest; here and below, it is assumed that permutation groups are given by generating sets, see [19]. When the number m is given as a part of input, the problem seems to be very hard even if the input group is abelian. It is quite natural therefore to restrict the m-closure problem to the case when m is fixed and the input group belongs to a certain class of groups. In this setting, polynomial-time algorithms for finding the m-closure were constructed for the groups of odd order [9] and, if $m = 2$, also for nilpotent and supersolvable groups [15, 16]. Note that the case $m = 1$ is trivial, because the 1-closure of any permutation group G is equal to the direct product of symmetric groups acting on the orbits of G.

The goal of the present paper is to solve the m-closure problem for $m \geq 3$ in the class of all solvable groups (note that there is an efficient algorithm testing whether or not a given permutation group is solvable).

Theorem 1.1. *Given an integer $m \geq 3$, the m-closure of a solvable permutation group of degree n can be found in time $n^{O(m)}$.*

The proof of Theorem 1.1 is given in Section 3. A starting point in our approach to the proof is the main result in [14] stating that for $m \geq 3$ the m-closure of every solvable permutation group is solvable. To apply this result, it suffices for a given solvable group G to find a solvable overgroup and then find $G^{(m)}$ inside it with

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the help of the Babai-Luks algorithm [3]; the latter enables, in particular, to find efficiently the relative m-closure $G^{(m)} \cap H$ of an arbitrary group G with respect to a solvable group H.

To explain how to find the overgroup, we recall that a permutation group is said to be *non-basic* if it is contained in a wreath product with the product action; it is *basic* otherwise, see [4, Section 4.3]. A classification of the primitive solvable linear groups having a faithful regular orbit [24] implies that for a sufficiently large primitive basic solvable group G, we have $G = G^{(m)}$ for all $m \geq 3$. This reduces the problem to solvable groups that are not basic, that is, to those that can be embedded in a direct or wreath product of smaller groups. In fact, we only need to test whether the corresponding embedding exists and (if so) to find it explicitly. This is a subject of Section 2.

All undefined terms can be found in [4] (for permutation groups) and [19] (for permutation group algorithms).

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2. The embedding problem

Given permutation groups $K \leq \mathrm{Sym}(\Delta)$ and $L \leq \mathrm{Sym}(\Gamma)$, we denote by $K \times L$ (respectively, $K \wr L$, $K \uparrow L$) the permutation group induced by the action of direct (respectively, wreath) product of K and L on $\Delta \cup \Gamma$ (respectively, $\Delta \times \Gamma$, $\Delta^{|\Gamma|}$).

Theorem 2.1. *Let $m \geq 2$ be an integer, K, L permutation groups, and $\star \in \{\times, \wr, \uparrow\}$. Then*

$$(K \star L)^{(m)} \leq K^{(m)} \star L^{(m)}$$

unless $\star = \uparrow$, $m = 2$, and K is 2-transitive.

Proof. See [14, Theorems 3.1, 3.2] and [20, Theorem 1.2]. □

Theorem 2.1 is used to reduce the study of the m-closure of a group $G \leq \mathrm{Sym}(\Omega)$ to permutation groups on smaller sets. From the algorithmic point of view, we need to solve the $\star$-*embedding problem*: test whether there exists an embedding of G to $K \star L$ for some sections $K \leq \mathrm{Sym}(\Delta)$ and $L \leq \mathrm{Sym}(\Gamma)$ of the group G, such that $|\Delta| < |\Omega|$ and $|\Gamma| < |\Omega|$, and if so, then to find the embedding explicitly. By this, we mean finding a bijection f from Ω to the underlying set of $K \star L$, such that

$$f^{-1} G f \leq K \star L. \quad (1)$$

The $\star$-embedding problem is easy if G is intransitive and $\star = \times$, or imprimitive and $\star = \wr$. In the rest of the section, we focus on the $\star$-embedding problem for primitive G and $\star = \uparrow$.

A *cartesian decomposition* of Ω is defined in [18] as a finite set $\mathcal{P} = \{P_1, \ldots, P_k\}$ of partitions of Ω such that $|P_i| \geq 2$ for each i and $|\Delta_1 \cap \cdots \cap \Delta_k| = 1$ for each $\Delta_1 \in P_1, \ldots, \Delta_k \in P_k$. A cartesian decomposition $\mathcal{P}$ is said to be *trivial* if $\mathcal{P}$ contains only one partition, namely, the partition into singletons, and $\mathcal{P}$ is said to be *homogeneous* if the number $|P_i|$ does not depend on $i = 1, \ldots, k$. Every partition π of $\mathcal{P}$ defines a cartesian decomposition $\mathcal{P}_\pi$ consisting of the meets $P_{i_1} \wedge \cdots \wedge P_{i_\ell}$, where $\{P_{i_1}, \ldots, P_{i_\ell}\}$ is a class of π.

A group $G \leq \mathrm{Sym}(\Omega)$ *preserves* (respectively, *stabilizes*) the cartesian decomposition $\mathcal{P}$ if any element of G permutes the P_i (respectively, leaves each P_i fixed). In this case, we say that $\mathcal{P}$ is *maximal* for G if $\mathcal{P} = \mathcal{Q}_\pi$ for no cartesian decomposition $\mathcal{Q}$ preserved (respectively, stabilized) by G and no nontrivial partition π

of $\mathcal{Q}$. Note that if G preserves $\mathcal{P}$ and the action of G on $\mathcal{P}$ is transitive, then $\mathcal{P}$ is homogeneous. Furthermore, if G stabilizes a nontrivial $\mathcal{P}$, then G cannot be primitive.

A natural example of cartesian decomposition comes from the wreath product $G = K \uparrow L$, where as before $K \leq \mathrm{Sym}(\Delta)$ and $L \leq \mathrm{Sym}(\Gamma)$. The underlying set of G is equal to Δ^k, where $k = |\Gamma|$, and one can define a partition P_i $(i = 1, \ldots, k)$ with $|\Delta|$ classes of the form

$$\{(\delta_1, \ldots, \delta_k) \in \Delta^k : \ \delta_i \text{ is a fixed element of } \Delta\}.$$

The partitions $P_1, \ldots, P_k$ form a cartesian decomposition $\mathcal{P}$ of Ω, which is preserved by G and stabilized by K^k; we say that $\mathcal{P}$ is a *standard* cartesian decomposition for G. Clearly, it can be found efficiently for any given K and L. Well-known properties of a wreath product with the product action [11] imply that if G is primitive, then the standard cartesian decomposition (a) is homogeneous, and (b) is maximal (among those that are preserved by G) if and only if K is basic.

Lemma 2.2. *Let $G \leq \mathrm{Sym}(\Omega)$ be a primitive group. Then G is non-basic if and only if G preserves a nontrivial homogeneous cartesian decomposition of Ω. Moreover, given such a decomposition, an embedding of G to a wreath product with product action can be found efficiently.*

Proof. Let G be non-basic. Then there is an embedding of G to a group $K \uparrow L$ for some $K \leq \mathrm{Sym}(\Delta)$ and $L \leq \mathrm{Sym}(\Gamma)$, such that $|\Delta| < |\Omega|$ and $|\Gamma| < |\Omega|$. Denote by f the corresponding bijection from Ω to $\Delta^{|\Gamma|}$. Then G preserves a homogeneous nontrivial cartesian decomposition $f^{-1}(\mathcal{P})$, where $\mathcal{P}$ is the standard cartesian decomposition for $K \uparrow L$.

Let G preserve a nontrivial homogeneous cartesian decomposition $P_1, \ldots, P_k$ of Ω. Denote by L (respectively, K) the permutation group induced by the action of G (respectively, the stabilizer of P_1 in G) on the set $\Gamma = \{P_1, \ldots, P_k\}$ (respectively, $\Delta = P_1$). Following the proof of [18, Theorem 5.13], one can efficiently identify each P_i with Δ. Then the bijection f from $\Delta^k = P_1 \times \ldots \times P_k$ onto Ω taking the cartesian product $\Delta_1 \times \ldots \times \Delta_k \in P_1 \times \ldots \times P_k$ to the unique point in $\Delta_1 \cap \ldots \cap \Delta_k$ can be found efficiently. Now the bijection f^{-1} moves G to a subgroup of $K \uparrow L$. □

Theorem 2.3. *Let G be a permutation group of degree n, and $\star \in \{\times, \wr, \uparrow\}$. Assume that G is imprimitive if $\star = \wr$, and primitive if $\star = \uparrow$. Then the $\star$-embedding problem for G can be solved in time* $\mathrm{poly}(n)$*.*

Proof. Using standard permutation group algorithms [19, Section 3.1], one can solve the $\star$-embedding problem for $G \leq \mathrm{Sym}(\Omega)$ if $\star = \times$ or $\wr$. Assume that $\star = \uparrow$ and G is primitive. Then (again by standard permutation group algorithms), one can find in time $\mathrm{poly}(n)$ the socle $S = \mathrm{Soc}(G)$ of G and test whether or not S is abelian.

In the abelian case, the required statement can be proved in almost the same way as was done in [9, Section 5.1] for solvable groups. Indeed, in this case, the group S is elementary abelian of order $n = p^k$ and can naturally be identified with Ω, which therefore can be treated as a linear space over the field of order p. The procedure **BLOCK** described in the cited paper, efficiently finds a minimal subspace $\Delta \subseteq \Omega$ so that Ω is the direct sum of the subspaces belonging to the set $\Gamma = \{\Delta^g : \ g \in G\}$. Now the required embedding of G exists only if $\Delta \neq \Omega$, and then as L and K one can take the group G^Γ and the restriction of its stabilizer of Δ (as a point) to Δ (as a set).

Let S be nonabelian. Then S is a direct product of pairwise isomorphic nonabelian simple groups. We need two auxiliary statements.

Claim 1. *There is at most one maximal nontrivial cartesian decomposition $\mathcal{P}$ stabilized by S. Moreover, one can test in time $\mathrm{poly}(n)$ whether $\mathcal{P}$ does exist, and if so, then find it within the same time.*

Proof. We will show that up to the language (in fact, the language of coherent configurations, see [5]) this claim is an almost direct consequence of results in [6, 12]. We start by noting that the cartesian decompositions stabilized by S are exactly the tensor decompositions of the coherent configuration $\mathcal{X}$ associated with S (see [6] for details). Thus, in view of [6, Theorem 1], the cartesian decompositions stabilized by S are in a 1-1 (efficiently computable) correspondence with the cartesian decompositions of the coherent configuration $\mathcal{X}$ itself. Moreover, if every subdegree of S is at least 2, i.e., $\mathcal{X}$ is thick in terms of [6], then there is at most one maximal nontrivial cartesian decomposition $\mathcal{P}$ of $\mathcal{X}$ [6, Theorem 2]. The polynomial-time algorithm in [6, Lemma 13] enables us to find a certificate that $\mathcal{X}$ has only the trivial cartesian decomposition, or to construct $\mathcal{P}$.

Assume that at least one (nontrivial) subdegree of S is equal to 1. Since the union of singleton orbits of a one point stabilizer of S is a block of the primitive group G, this union is the whole set Ω and the group S is regular. In this case, the coherent configuration $\mathcal{X}$ is also regular, $S = \mathrm{Aut}(\mathcal{X})$, and from the above mentioned [6, Theorem 1], it follows that the cartesian decompositions of $\mathcal{X}$ are in a 1-1 correspondence with the direct decompositions of the group S itself. If this group is simple, then S stabilizes only the trivial cartesian decomposition. Otherwise, the decomposition of S into the direct product of pairwise isomorphic (nonabelian) simple groups gives the maximal nontrivial cartesian decomposition $\mathcal{P}$ stabilized by S. It remains to note that $\mathcal{P}$ can be found efficiently by the main algorithm in [12]. □

Claim 2. *Assume that G is non-basic. Then G preserves a nontrivial homogeneous cartesian decomposition of the form $\mathcal{P}_\pi$ for some partition π of the cartesian decomposition $\mathcal{P}$ from Claim 1.*

Proof. Since G is non-basic, we may assume that $G \leq K \uparrow L$, where K is basic primitive and L is transitive. Denote by Q the corresponding standard cartesian decomposition (Lemma 2.2). We may also assume that K is the permutation group induced by the action on $P \in Q$ of the stabilizer of P in G. Then in virtue of [4, Theorem 4.7] (and the remark after it), the socle S of G is a subgroup of the base group of the wreath product $K \uparrow L$. It follows that S stabilizes Q. Thus, by Claim 1, there exists the unique maximal nontrivial cartesian decomposition $\mathcal{P}$ stabilized by S and $Q = \mathcal{P}_\pi$ for some partition π of $\mathcal{P}$. Since G acts transitively on $\mathcal{P}$, the decomposition Q is homogeneous. □

Let us complete the proof. By Lemma 2.2, it suffices to test whether G preserves a nontrivial cartesian decomposition and, if so, find it efficiently. Applying the algorithm of Claim 1, we test efficiently whether S stabilizes a nontrivial cartesian decomposition. If not, then G cannot preserve a nontrivial cartesian decomposition (see Claim 2). Otherwise, we efficiently find the cartesian decomposition $\mathcal{P}$ from

Claim 1. By Claim 2, all we need is to test whether there exist a partition π of $\mathcal{P}$, such that $\mathcal{P}_\pi$ is a nontrivial homogeneous cartesian decomposition preserved by G and, if so, find it efficiently. Since $\mathcal{P}$ is nontrivial, we have $|\mathcal{P}| \le \log n$ and the power set $2^{\mathcal{P}}$ has cardinality at most $2^{\log n} = n$. Furthermore, the cartesian decompositions $\mathcal{P}_\pi$ preserved by G are in one-to-one correspondence with those subsets $Q \subseteq \mathcal{P}$ for which

$$\{Q^g : \ g \in G\} \text{ is a homogeneous partition of } \mathcal{P}.$$

Since this condition can be tested only for the generators g of G, we are done. □

3. Proof of Theorem 1.1

We deduce Theorem 1.1 from a more general statement valid for any complete class of groups. A class of (abstract) groups is said to be *complete* if it is closed with respect to taking subgroups, quotients, and extensions [21, Definition 11.3]. Any complete class is obviously closed with respect to direct and wreath product and taking sections. The class of all permutation groups of degree at most n that belong to $\mathfrak{K}$ is denoted by $\mathfrak{K}_n$.

Theorem 3.1. *Let $m, n \in \mathbb{N}$, $m \ge 3$, and $\mathfrak{K}$ a complete class of groups. Then*

- (i) *$\mathfrak{K}_n$ is closed with respect to taking the m-closure if and only if $\mathfrak{K}_n$ contains the m-closure of every primitive basic group in $\mathfrak{K}_n$,*
- (ii) *the m-closure of any group in $\mathfrak{K}_n$ can be found in time* $\mathrm{poly}(n)$ *by accessing oracles for finding the m-closure of every primitive basic group in $\mathfrak{K}_n$ and the relative m-closure of every group in $\mathfrak{K}_n$ with respect to any group in $\mathfrak{K}_n$.*

Proof. The "only if" part of statement (i) is obvious. To prove the "if" part and statement (ii), we present a more or less standard recursive algorithm for finding the m-closure $G^{(m)}$ of a group $G \in \mathfrak{K}_n$. At each step we will verify that $G^{(m)} \in \mathfrak{K}_n$.

Depending on whether $G \le \mathrm{Sym}(\Omega)$ is intransitive, imprimitive, or primitive, we set $\star = \times$, $\wr$, or $\uparrow$, respectively. Solving the $\star$-embedding problem for G by Theorem 2.3, one can can test in time $\mathrm{poly}(n)$ whether there exists an embedding of G to $K \star L$ for some sections

$$K \le \mathrm{Sym}(\Delta) \quad \text{and} \quad L \le \mathrm{Sym}(\Gamma)$$

of G, such that the numbers $n_K = |\Delta|$ and $n_L = |\Gamma|$ are less than $n = |\Omega|$, and if so, then find the embedding explicitly. If there is no such embedding, then G is primitive basic, belongs to $\mathfrak{K}_n$, and the m-closure of G can be found for the cost of one call of the corresponding oracle.

Assume that G is not primitive basic and we are given a bijection f from Ω to the underlying set of $K \star L$, such that equality (1) holds. Since $f^{-1}G^{(m)}f = (f^{-1}Gf)^{(m)}$, we may also assume that

$$G \le K \star L.$$

Note that $K \in \mathfrak{K}_{n_K}$ and $L \in \mathfrak{K}_{n_L}$, because the class $\mathfrak{K}$ is complete. Applying the algorithm recursively to K and L, we find the groups $K^{(m)}$ and $L^{(m)}$ in time $\mathrm{poly}(n_K)$ and $\mathrm{poly}(n_L)$, respectively, and then the group $K^{(m)} \star L^{(m)}$ in time $\mathrm{poly}(n)$. By induction, $K^{(m)} \in \mathfrak{K}_{n_K}$ and $L^{(m)} \in \mathfrak{K}_{n_L}$, whence $K^{(m)} \star L^{(m)} \in \mathfrak{K}_n$. On the other hand, by Theorem 2.1, we have

$$G^{(m)} \le (K \star L)^{(m)} \le K^{(m)} \star L^{(m)}.$$

Thus, $G^{(m)} \in \mathfrak{K}_n$. Accessing (one time) the oracle for finding the relative m-closure of G with respect to $K^{(m)} \star L^{(m)}$, we finally get the group $G^{(m)}$.

It remains to estimate the number of the oracles calls. Each recursive call divides the problem for a group of degree n to the same problem for a group K of degree n_K and for a group L of degree n_L. Moreover,

$$n = \begin{cases} n_K + n_L & \text{if } \star = \times, \\ n_K \cdot n_L & \text{if } \star = \wr, \\ {n_K}^{n_L} & \text{if } \star = \uparrow. \end{cases}$$

Thus the total number of recursive calls and hence the number of accessing oracles is at most n. □

An obstacle in proving Theorem 3.1 for $m = 2$ lies in the exceptional case of Theorem 2.1. Indeed, assume that the class $\mathfrak{K}$ does not contain all groups. Then it cannot contain symmetric groups of arbitrarily large degree. However the 2-closure of any two-transitive group of degree n coincides with $\mathrm{Sym}(n)$. Therefore $\mathfrak{K}$ cannot also contain two-transitive groups of sufficiently large degree. It seems that this restricts the class $\mathfrak{K}$ essentially.

Remark 3.2. *In fact, the proof of Theorem 3.1 shows that the following weakened version of this theorem holds true: both statements of Theorem 3.1 remain valid for* $m = 2$ *if "primitive basic groups" are replaced with "primitive groups".*

Proof of Theorem 1.1. Denote by $\mathfrak{K}$ the class of all solvable groups. This class is obviously complete. Moreover, the relative m-closure of any group of $\mathfrak{K}_n$ with respect to any other group from $\mathfrak{K}_n$ can be found in time $\mathrm{poly}(n)$ in view of [3, Corollary 3.6] (see also [16, Section 6.2]). By Theorem 3.1, it suffices to verify that the 3-closure of a primitive basic group $G \in \mathfrak{K}_n$ can be found in time $\mathrm{poly}(n)$; indeed, if $m > 3$, then $G^{(m)} \le G^{(3)}$ can be found as the relative m-closure of G with respect to $G^{(3)}$.

First, suppose that a point stabilizer H of G has a regular orbit. Then G is 3-closed by [14, Corollary 2.5], and there is nothing to do, because $G = G^{(3)}$. Now, if the group H has no regular orbits and n is sufficiently large, then the number $n = q$ is a prime power and $H \le \Gamma\mathrm{L}(1, q)$, see [24, Corollary 3.3]. In this case, $H = H^{(2)}$ by [23, Proposition 3.1.1] and again $G = G^{(3)}$. In the remaining case, the degree of G is bounded by an absolute constant, say N, and the group $G^{(3)}$ can be found by inspecting all permutations of $\mathrm{Sym}(N)$.

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