ON THE PRIME GRAPH OF A FINITE GROUP WITH UNIQUE NONABELIAN COMPOSITION FACTOR

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Abstract. We say that finite groups are isospectral if they have the same sets of orders of elements. It is known that every nonsolvable finite group G isospectral to a finite simple group has a unique nonabelian composition factor, that is, the quotient of G by the solvable radical of G is an almost simple group. The main goal of this paper is prove that this almost simple group is a cyclic extension of its socle.

To this end, we consider a general situation when G is an arbitrary group with unique nonabelian composition factor, not necessarily isospectral to a simple group, and study the prime graph of G, where the prime graph of G is the graph whose vertices are the prime numbers dividing the order of G and two such numbers r and s are adjacent if and only if $r \neq s$ and G has an element of order rs. Namely, we establish some sufficient conditions for the prime graph of such a group to have a vertex adjacent to all other vertices. Besides proving the main result, this allows us to refine a recent result by P. Cameron and N. Maslova concerning finite groups almost recognizable by prime graph.

Keywords: almost simple group, group of Lie type, order of an element, recognition by spectrum, prime graph

1. Introduction

Given a finite group G, we denote the set of prime divisors of the order of G by $\pi(G)$. The set of element orders of G is called the spectrum of G and denoted by $\omega(G)$. If $\omega(G) = \omega(H)$, then G and H are said to be isospectral.

Suppose that G is a finite group isospectral to a finite nonabelian simple group L. Then G is either solvable, in which case L is one of $L_3(3)$, $U_3(3)$, $S_4(3)$, or has exactly

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one nonabelian composition factor (see [5, Theorem 2]). In what follows, we assume that G is not solvable, and so G has a normal series

$$1 \leqslant K < H \leqslant G, \tag{1.1}$$

where K is the solvable radical of G, H/K is a nonabelian simple group and G/K is an almost simple group with socle H/K. Denoting H/K by S, we may identify G/K with a subgroup of $\operatorname{Aut} S$, and then G/H with a subgroup of $\operatorname{Out} S = \operatorname{Aut} S/S$. Observe that G/H is solvable.

If L is sufficiently 'large', more precisely, if L is a classical group of dimension at least 38 or a non-classical group other than Alt_6, Alt_{10}, J_2, ${}^3D_4(2)$, then $K = 1$ and $H \simeq L$ (see [9,13]). Furthermore, it follows that G/H is cyclic (see [8] and the references therein). In general case, K is not always trivial and H/K is not always isomorphic to L but in all known examples, G/H is cyclic. This observation suggests us to conjecture that G/H is always cyclic and the main goal of this paper is to prove this conjecture.

Theorem 1. *Let L be a finite nonabelian simple group and let G be a nonsolvable finite group with $\omega(G) = \omega(L)$. Suppose that $1 \leqslant K < H \leqslant G$ is the normal series of G as in (1.1). Then G/H is cyclic. Furthermore, if H/K is a simple group of Lie type other than $L_2(q)$, then G/H does not contain diagonal automorphisms.*

If L is sporadic or alternating, Theorem 1 is a direct consequence of the known description of groups isospectral to L. If L is a group of Lie type, the proof has several ingredients. The first is the well-known property of spectra of groups of Lie type stated in Lemma 2.1 in Section 2. The second is the nilpotency of the solvable radical of G established in [18]. The third is the following Theorem 2 which concerns all finite groups of some specific structure, not only those isospectral to simple groups.

Theorem 2. *Suppose that a finite group G has a normal series $1 \leqslant K < H \leqslant G$, where K is the solvable radical of G and $S = H/K$ is a finite simple group of Lie type. Suppose also that K is nilpotent.*

(i) *If $S \neq L_2(q)$ and G/H contains a diagonal automorphism of S of prime order r, then $rs \in \omega(G)$ for all $s \in \pi(G) \setminus \{r\}$.*
(ii) *If G/H is not cyclic, then there is $r \in \pi(G/H)$ such that $rs \in \omega(G)$ for all $s \in \pi(G) \setminus \{r\}$.*

The set $\omega(G)$ defines the prime graph of G as follows: the vertex set of this is $\pi(G)$ and two primes $r, s \in \pi(G)$ are adjacent if and only if $r \neq s$ and $rs \in \omega(G)$. The prime graph is also known as the Gruenberg–Kegel graph and we denote it by $GK(G)$. It is not hard to see that Theorem 2 states a property of the graph $GK(G)$ rather than of the whole set $\omega(G)$. This allows us to apply this theorem to the problem of recognition of simple groups by prime graph. Recently, P. Cameron and N. Maslova [1] proved several new results relating to this problem. In Theorem 3, we slightly refine Theorem 1.4 of [1].

Theorem 3. *There exists a function $F(x) = O(x^5)$ such that for each labeled graph Γ, the following conditions are equivalent:*

(i) *there exist infinitely many groups H such that $GK(H) = \Gamma$;*
(ii) *there exist more than $F(|V(\Gamma)|)$ groups H such that $GK(H) = \Gamma$, where $V(\Gamma)$ is the set of the vertices of Γ.*

In fact, Theorem 1.4 of [1] states exactly the same as Theorem 3 but with x^7 in place of x^5.

2. Proofs of Theorems 1 and 2

We begin this section with notation and preliminary results. We write $L_n^\varepsilon(q)$ and $E_6^\varepsilon(q)$ assuming that $\varepsilon \in \{+,-\}$, $L_n^+(q) = L_n(q)$, $L_n^-(q) = U_n(q)$, $E_6^+(q) = E_6(q)$, and $E_6^-(q) = {}^2E_6(q)$. If r is a prime and a is an integer, then $(a)_r$ is the highest power of r dividing a. If S is a group of Lie type, then $\operatorname{Inndiag} S$ is the subgroup of $\operatorname{Aut} S$ generated by inner and diagonal automorphisms, and $\operatorname{Outdiag} S$ is the image of $\operatorname{Inndiag} S$ in $\operatorname{Out} S$. Also we use the terms 'field automorphism' and 'graph automorphism' of S according to [4, Definition 2.5.13].

Lemma 2.1. *If S is a finite simple group of Lie type, then for every $r \in \pi(S)$ there is $s \in \pi(S)$ such that $r \neq s$ and $rs \notin \omega(S)$.*

Proof. This follows from [16, 17] (see, for example, [6, Lemma 2.2]). □

Lemma 2.2. *Let S be a finite simple group of Lie type in characteristic p. If r divides $|\operatorname{Outdiag} S|$ and $rp \notin \omega(S)$, then either $S = L_2(q)$, or $S = L_3^\varepsilon(q)$ and $(q-\varepsilon)_3 = 3$.*

Proof. This follows, for example, from [16, Propositions 3.1 and 3.2]. □

Lemma 2.3. *Let S be a finite simple group of Lie type in characteristic p. If $r \in \pi(S)$, r is odd and $2r \notin \omega(S)$, then either a Sylow r-subgroup of S is cyclic, or $S = L_2(q)$ and $r = p$, or $S = L_3^\varepsilon(q)$, $p = 2$, $r = 3$ and $(q-\varepsilon)_3 = 3$.*

Proof. This follows from the results of [16, Sections 3 and 4] and the cross-characteristic Sylow structure of groups of Lie type [3, (10-2)]. □

Lemma 2.4. *Let $S = {}^t\Sigma(q)$ be a finite simple group of Lie type, not a Suzuki–Ree group, and let φ be a field automorphism of S of prime order r. Then $r\cdot\omega({}^t\Sigma(q^{1/r})) \subseteq \omega(S \rtimes \langle\varphi\rangle)$.*

Proof. This follows from the Lang–Steinberg theorem [14, Section 10] (see, for example, [7, Lemma 2.8]). □

Lemma 2.5. *Suppose that G is a finite group, K is a normal subgroup of G and every $g \in G \setminus K$ acts fixed-point-freely on K. Then every odd order Sylow subgroup of G/K is cyclic and a Sylow 2-subgroup of G/K is cyclic or generalized quaternion.*

Proof. This is a well-known property of fixed-point-free automorphisms (see, for example, [10, Satz 8.7]). □

Proof of Theorem 2. Denote the defining characteristic of S by p, G/K by $\overline{G}$ and G/H by $\widehat{G}$. As we remarked in the introduction, $\widehat{G}$ can be regarded as a subgroup of $\operatorname{Out} S$.

Clearly, we may assume that either $\operatorname{Outdiag} S \neq 1$ or $\operatorname{Out} S$ is not cyclic, in particular, we may assume that S is not a Suzuki–Ree group and so $3 \in \pi(S)$.

(i) Suppose that $r \in \pi(\widehat{G} \cap \operatorname{Outdiag} S)$. Observe that $r \in \pi(S)$ and $r \neq p$. By Lemma 2.2, it follows that $rp \in \omega(S)$ unless $S = L_3^\varepsilon(q)$, $r = 3$ and $(q-\varepsilon)_3 = 3$. In this case $PGL_3^\varepsilon(q) \leqslant \overline{G}$, and since $PGL_3^\varepsilon(q)$ has an element of order $p(q-\varepsilon)$, we see that $rp \in \omega(\overline{G})$.

Suppose that $s \in \pi(S)$ and $s \neq p$. If $s \in \pi(\operatorname{Outdiag} S)$, then $rs \in \omega(S)$ since $\operatorname{Outdiag} S$ is abelian. So we may assume that $s \notin \pi(\operatorname{Outdiag} S)$. The maximal tori of $\operatorname{Inndiag} S$ are isomorphic to those of the universal version $\tilde{S}_u$ of $\tilde{S}$, where $\tilde{S} = S$ if S is not of type B_n or C_n, and $\tilde{B}_n(q) = C_n(q)$, $\tilde{C}_n(q) = B_n(q)$ (see [2, Section 4.4]). Since every maximal torus of $\tilde{S}_u$ contains the center $Z(\tilde{S}_u)$ of $\tilde{S}_u$ and $|Z(\tilde{S}_u)| = |\operatorname{Outdiag} S|$, we see that $\operatorname{Inndiag} S$ includes a maximal torus whose order is divisible by $s|\operatorname{Outdiag} S|$. So $\overline{G}$ contains an abelian subgroup of order sr.

Let $s \in \pi(\overline{G}) \setminus \pi(S)$. Since $s \neq 2,3$ and $s \notin \pi(\mathrm{Outdiag}\, S)$, it follows that G/K contains a field automorphism of S of order s. By Lemma 2.4, we have $s \cdot \omega(S_0) \subseteq \omega(G/K)$, where S_0 is a group of the same Lie type as S. If $r = 2,3$, then it is clear that $r \in \pi(S_0)$. If $r \neq 2,3$, then $S = L_n^{\varepsilon}(q)$, r divides $(n, q-\varepsilon)$ and $S_0 = L_n^{\varepsilon}(q^{1/s})$. Since r divides $p^{r-1} - \varepsilon^{r-1}$ and $r - 1 \leqslant n - 1$, we see that $r \in \pi(S_0)$.

Let $s \in \pi(K) \setminus \pi(\overline{G})$. If $r = 2$, then s is adjacent to r in $GK(G)$ by [15, Proposition 2]. So we may assume that r is odd. If $S = E_6^{\varepsilon}(q)$ or $S = L_n^{\varepsilon}(q)$ with $n \geqslant 4$, then S includes a torus of the form $\mathbb{Z}_{q-\varepsilon} \times \mathbb{Z}_{q-\varepsilon}$, and hence S includes an elementary abelian group of order r^2. If $L = L_3^{\varepsilon}(q)$, then $PGL_3^{\varepsilon}(q) \leqslant \overline{G}$ and so $\overline{G}$ includes an elementary abelian group of order r^2. Now we apply Lemma 2.5 to conclude that $rs \in \omega(G)$.

(ii) Let $S \neq L_2(q)$. By (i), we may assume that $\widehat{G} \cap \mathrm{Outdiag}\, S = 1$. Then either $\widehat{G}$ includes an elementary abelian group of order 2^2, or $S = O_8^+(q)$ and, up to conjugation in $\mathrm{Out}\, S$, $\widehat{G}$ contains the image of the graph automorphism γ of S induced by the symmetry of the Dynkin diagram of order 3.

In the first case, $S = L_n(q)$, $O_{2n}^+(q)$, or $E_6(q)$, and we claim that 2 is adjacent to all odd primes in $GK(G)$. By [15, Proposition 2], every $s \in \pi(K) \cup \pi(\widehat{G})$ is adjacent to 2. Now let $t \in \pi(S)$ and suppose that $2t \notin \omega(S)$. Excluding for a while the case when $t = 3$, $S = L_3(q)$, $p = 2$, $(q-1)_3 = 3$ and applying Lemma 2.3, we conclude that a Sylow t-subgroup T of S is cyclic, and hence $N_{\overline{G}}(T)/C_{\overline{G}}(T)$ is cyclic. On the other hand, by the Frattini argument, $N_{\overline{G}}(T)/(N_{\overline{G}}(T) \cap S) \simeq \widehat{G}$, and so a Sylow 2-subgroup of $N_{\overline{G}}(T)$ is not cyclic. Thus $2 \in C_{\overline{G}}(T)$, and $2t \in \omega(\overline{G})$.

Suppose that $t = 3$, $S = L_3(q)$, $p = 2$, and $(q-1)_3 = 3$. Since $\widehat{G}$ includes an elementary abelian group of order 2^2, it follows that $\overline{G}$ contains a field automorphism of S of order 2, and so $6 \in \omega(\overline{G})$.

Now suppose that $S = O_8^+(q)$ and $\overline{G}$ contains the graph automorphism γ. The centralizer of γ in S is isomorphic to $G_2(q)$ [3, (9-1)] and so $3s \in \omega(G)$ for all $3 \neq s \in G_2(q)$. Since S includes an elementary abelian group of order 9, we conclude that $3s \in \omega(G)$ for all $s \in \pi(K) \setminus \{3\}$. Also a $2'$-Hall subgroup of $\mathrm{Out}\, S$ is abelian, and hence 3 is adjacent to every $s \in \pi(\widehat{G}) \setminus \{2,3\}$ in $GK(\widehat{G})$.. Let $s \in \pi(S) \setminus \{3\}$ and $3s \notin \omega(S)$. Then s divides q^2+q+1 or q^2-q+1, therefore, $s \in \pi(G_2(q))$ and, as we remarked, $3s \in \omega(G)$. Thus 3 is adjacent to all vertices in $GK(G)$.

Let $S = L_2(q)$, where $q = p^l$. We claim that 2 is adjacent to all odd primes in $GK(G)$. Since $\mathrm{Out}\, S$ is a direct product of cyclic groups of orders $(2, q-1)$ and l, it follows that p is odd, l is even and $\overline{G} = PGL_2(q) \rtimes \langle \varphi \rangle$, where φ is a field automorphism of S of even order. Since $PGL_2(q)$ contains elements of orders $q \pm 1$ and $2p \in \omega(\overline{G})$ by Lemma 2.4, we see that 2 is adjacent to every odd $s \in \pi(\overline{G})$. Let $s \in \pi(K)$ be odd. A Sylow 2-subgroup of $PGL_2(q)$ is dihedral, and so it cannot act fixed-point-freely on a Sylow s-subgroup of K by Lemma 2.5. Hence $2s \in \pi(G)$, and the proof of Theorem 2 is complete. □

Now we are able to prove Theorem 1. Let $S = H/K$. Clearly, we may assume that $\mathrm{Out}\, S$ is not cyclic. In particular, we may assume that S is neither sporadic nor alternating with the following convention: if $S = Alt_6 \simeq L_2(9)$, we regard S as a group of Lie type.

If L is sporadic and $L \neq J_2$, or if $L = Alt_n$ and $n \neq 6, 10$, then $G \simeq L$ (see [11] and [5] respectively). If $L = J_2$, then $G \simeq L$ or $S = Alt_8$ by [11]. If $L = Alt_{10}$, then $G \simeq L$ or $S = Alt_5$ by [12]. If $L = Alt_6$, we regard L as a group of Lie type.

Let L be a group of Lie type. By [18, Theorem 1], it follows that K is nilpotent, and so G satisfies the hypothesis of Theorem 2. If G/H is not cyclic or if $S \neq L_2(q)$ and

G/H contains a diagonal automorphism of S, then there is $r \in \pi(G)$ adjacent to all other vertices in $GK(G)$. But this is impossible by Lemma 2.1 since $GK(G) = GK(L)$. This contradiction completes the proof of Theorem 1.

3. Groups almost recognizable by prime graph

Given a positive integer k, a finite group G is said to be k-recognizable by prime graph if there are exactly k pairwise nonisomorphic finite groups H with $GK(H) = GK(G)$ and almost recognizable by prime graph if it is k-recognizable for some k.

By [1, Theorem 1.3], if G is almost recognizable by prime graph, then G is almost simple and each group H with $GK(H) = GK(G)$ is almost simple. So if G is a k-recognizable group, then k is at most the number of almost simple groups H such that $\pi(H) = \pi(G)$. By [1, Proposition 4.2], this number is at most $O(|\pi(G)|^7)$. A direct corollary of this discussion is the following theorem.

Theorem A [1, Theorem 1.4]. *There exists a function $F(x) = O(x^7)$ such that for each labeled graph Γ, the following conditions are equivalent:*

(i) *there exist infinitely many groups H such that $GK(H) = \Gamma$;*
(ii) *there exist more then $F(|V(\Gamma)|)$ groups H such that $GK(H) = \Gamma$, where $V(\Gamma)$ is the set of the vertices of Γ.*

It is clear that estimating k we do not need to calculate all almost simple groups H such that $\pi(H) = \pi(G)$. It is sufficient to calculate those H whose prime graph satisfies some necessary conditions for H to be almost recognizable by prime graph. One of these conditions is stated in [1, Theorem 1.3]: 2 is nonadjacent to at least one odd prime in $GK(H)$. But in fact this condition can be strengthened: every $r \in \pi(H)$ is nonadjacent to at least one prime $s \neq r$ in $GK(H)$. Indeed, otherwise $GK(H) = GK(H \times \mathbb{Z}_r^k)$ for all positive integers k. Applying Theorem 2, we see that it sufficient to calculate H such that H/S is cyclic, where S is the socle of H.

Lemma 3.1. *There is a function $F(x) = O(x^2)$ such that if S is a finite simple group of Lie type, then there are at most $F(|\pi(S)|)$ almost simple groups H with socle S such that H/S is cyclic.*

Proof. Let n be the Lie rank of S and $q = p^l$ the order of the base field of S. Denote the number of divisors of l by $d(l)$. By [1, Lemma 2.7], we have $n \leqslant 2|\pi(S)| + 3$ and $d(l) \leqslant |\pi(S)| + 1$.

Steinberg's theorem [4, Theorem 2.5.12] states that $\operatorname{Out} S = \operatorname{Outdiag} S \rtimes \Phi_S\Gamma_S$, where $|\operatorname{Outdiag} S| \leqslant n + 1$ and $\Phi_S\Gamma_S$ is either a subgroup in $\mathbb{Z}_l \times Sym_3$ or a cyclic group of order $2l$ or $3l$. In any case the number of cyclic subgroups of $\Phi_S\Gamma_S$ is at most $6d(l)$. Thus the number of cyclic subgroups of $\operatorname{Out} S$ is at most $6(n+1)d(l)$, which is $O(|\pi(S)|^2)$ by the preceding paragraph. □

Now we are ready to prove Theorem 3 (in fact we follow the lines of the proof of [1, Theorem 1.4] but Theorem 2 allows us to use the bound of Lemma 3.1 instead of that of [1, Proposition 4.1]). It is sufficient to show that there exists a function $F(x) = O(x^5)$ such that for every finite group G, if G is almost recognizable by prime graph, then there are at most $F(|\pi(G)|)$ pairwise nonisomorphic groups H with $GK(H) = GK(G)$.

Assume that G is k-recognizable by prime graph. By [1, Theorem 1.3], each group H with $GK(H) = GK(G)$ is almost simple. Furthermore, as we remarked, every $r \in \pi(H)$ is nonadjacent to at least one prime $s \neq r$ in $GK(H)$. By Theorem 2, it follows that H is a cyclic extension of its socle. By [1, Proposition 3.1], the number of nonabelian simple groups S such that $\pi(S) \subseteq \pi(G)$ is bounded by $F_1(|\pi(G)|)$ with $F_1(x) = O(x^3)$. Applying Lemma 3.1, we see that the number of almost simple groups H with socle S such that H/S is cyclic is at most $F_2(|\pi(S)|)$, where $F_2(x) = O(x^3)$. Thus $k \leqslant F_1(|\pi(G)|)F_2(|\pi(G)|) = O(|\pi(G)|^5)$, and this completes the proof of Theorem 3.

References

[1] P. J. Cameron and N. V. Maslova, Criterion of unrecognizability of a finite group by its Gruenberg-Kegel graph, 2021, arXiv:2012.01482 [math.GR].

[2] R. W. Carter, Finite groups of Lie type. Conjugacy classes and complex characters, John Wiley & Sons, New York, 1985.

[3] D. Gorenstein and R. Lyons, Local structure of finite groups of characteristic 2 type, Memoirs Amer. Math. Soc. **276**, 1983.

[4] D. Gorenstein, R. Lyons, and R. Solomon, The classification of the finite simple groups. Number 3, Amer. Math. Soc. Surveys and Monographs **40.3**, 1998.

[5] I. B. Gorshkov, Recognizability of alternating groups by spectrum, *Algebra Logic* **52** (2013), no. 1, 41–45.

[6] M. A. Grechkoseeva, On element orders in covers of finite simple groups of Lie type, *J. Algebra Appl.* **14** (2015), 1550056 [16 pages].

[7] M. A. Grechkoseeva, On spectra of almost simple groups with symplectic or orthogonal socle, *Siberian Math. J.* **57** (2016), no. 4, 582–588.

[8] M. A. Grechkoseeva, On spectra of almost simple extensions of even-dimensional orthogonal groups, *Siberian Math. J.* **59** (2018), no. 4, 623–640.

[9] M. A. Grechkoseeva and A. V. Vasil'ev, On the structure of finite groups isospectral to finite simple groups, *J. Group Theory* **18** (2015), no. 5, 741–759.

[10] B. Huppert, Endliche Gruppen. I, Springer-Verlag, Berlin, 1967.

[11] V. D. Mazurov and W. J. Shi, A note to the characterization of sporadic simple groups, *Algebra Colloq.* **5** (1998), no. 3, 285–288.

[12] A. M. Staroletov, Groups isospectral to the degree 10 alternating group, *Siberian Math. J.* **51** (2010), no. 3, 507–514.

[13] A. Staroletov, On almost recognizability by spectrum of simple classical groups, *Int. J. Group Theory* **6** (2017), no. 4, 7–33.

[14] R. Steinberg, Endomorphisms of linear algebraic groups, Memoirs Amer. Math. Soc. **80**, 1968.

[15] A. V. Vasil'ev, On connection between the structure of a finite group and the properties of its prime graph, *Siberian Math. J.* **46** (2005), no. 3, 396–404.

[16] A. V. Vasil'ev and E. P. Vdovin, An adjacency criterion for the prime graph of a finite simple group, *Algebra Logic* **44** (2005), no. 6, 381–406.

[17] A. V. Vasil'ev and E. P. Vdovin, Cocliques of maximal size in the prime graph of a finite simple group, *Algebra Logic* **50** (2011), no. 4, 291–322.

[18] N. Yang, M. A. Grechkoseeva, and A. V. Vasil'ev, On the nilpotency of the solvable radical of a finite group isospectral to a simple group, *J. Group Theory* **23** (2020), no. 3, 447–470.